P/d-Graphs of Tournaments

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Abstract

Vertices x and y are called *paired* in tournament T if there exists a vertex z in the vertex set of T such that either x and y beat z or z beats x and y . Vertices x and y are said to be *distinguished* in T if there exists a vertex z in T such that either x beats z and z beats y , or y beats z and z beats x . Two vertices are *strictly paired (distinguished)* in T if all vertices of T pair (distinguish) the two vertices in question. The p/d-graph of a tournament T is a graph which depicts strictly paired or strictly distinguished pairs of vertices in T. P/d-graphs are useful in obtaining the characterization of such graphs as domination and domination-compliance graphs of tournaments. We shall see that p/d-graphs of tournaments have an interestingly limited structure as we characterize them in this paper. In so doing, we find a method of constructing a tournament with a given p/d-graph using adjacency matrices of tournaments.

1 Introduction

A *tournament* is an oriented complete graph. There has been an assortment of graphs defined on the vertex set of a tournament which reflect some structural characteristic of the tournament. The *domination graph* of a tournament is one such graph and has a edge between vertices of the tournament whenever the vertices form a dominant pair, i.e., whenever the union of their outsets includes all other vertices of the tournament. Domination graphs of tournaments were characterized in a series of papers (see [9], [8], [10], and [7]). Recently, the idea of the domination graph of a tournament has been extended to "k-domination" in [12]. Other recent studies include domination in tournaments with ties, and domination graphs of proper subgraphs of tournaments in [5], [6], and [1]. This paper will characterize the *p/d-graph* of a tournament, a graph that was instrumental in the characterization of *domination-compliance* graphs of tournaments (see [11], [2], [3], and [4].)

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The p/d-graph of a tournament T is the graph defined on the vertex set of T which describes specific structural characteristics of certain pairs of vertices in T. The following definitions originally appeared in [7].

Definition 1.1 *Two vertices x and y of a tournament T are* paired *if there exists a third vertex w in T such that $x \to w$ and $y \to w$ or $w \to x$ and y.*

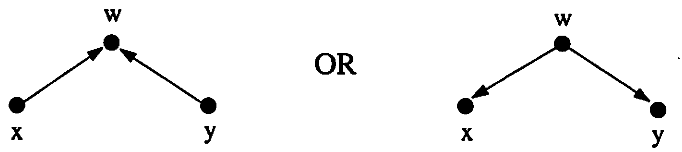


Figure 1: x and y are Paired by w.

Definition 1.2 *Two vertices x and y of a tournament T are* distinguished *if there exists a third vertex w in T such that $x \to w \to y$ or $y \to w \to x$.*

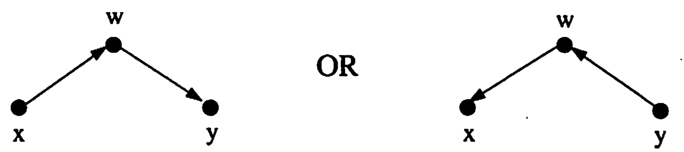


Figure 2: x and y are Distinguished by w.

There are tournaments with the property that every pair of vertices is both paired and distinguished by some vertex in the tournament. These tournaments have been very useful in characterizing domination graphs of tournaments.

Definition 1.3 *A tournament T is* well-covered *if every pair of vertices in T are both paired and distinguished.*

It has been proven that well-covered tournaments only exist on 4 or more than 5 vertices (see [7]).

Tournaments which are not well-covered have the property that some pair of vertices are only paired or only distiguished by all other vertices. The following two definitions describe this situation.

Definition 1.4 Strictly paired: *two vertices in a tournament T are strictly paired if no vertex in T distinguishes them.*

Definition 1.5 Strictly distinguished: *Two vertices in a tournament T are strictly distinguished if no vertex in T pairs them.*

Vertices which have the property of being strictly paired or strictly distinguished in a tournament are important in the study of the domination-compliance graph of a tournament. To aid in the detection of such pairs of vertices in a tournament, we have the *p/d-graph* of a tournament.

Definition 1.6 *Given a tournament T, let $p/d(T)$ be the graph on the vertex set of T with edges between vertices which are strictly paired or strictly distinguished in T.*

We say that an edge xy in p/d(T) is a p edge if vertices x and y are strictly paired. Edge xy is a d edge if vertices x and y are strictly distinguished. Figure 3 is an example of the p/d-graph of a 5-tournament.

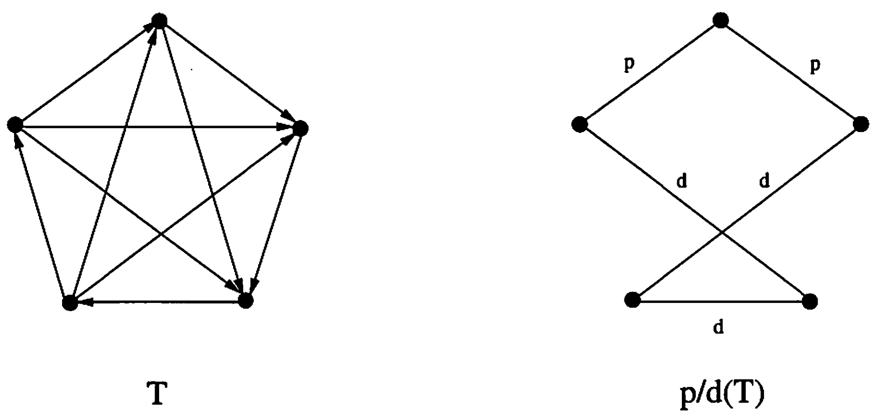


Figure 3: Example of p/d(T)

2 The Characterization of P/d-graphs

The first step towards the characterization of p/d-graphs of tournaments is the following theorem, which is proven in [3].

Theorem 2.1 (Doherty and Lundgren [3]) *The p/d-graph of any tournament has maximum degree 2.*

Theorem 2.1 provides us with the most important characterization theorem. Using this theorem, we know that p/d-graphs are restricted to unions of isolated vertices, paths and cycles. The question addressed here is exactly which unions are realizable as the p/d-graph of a tournament. We will find the trickiest cases are smaller. In addition to a characterization of p/d-graphs, we shall demonstrate a method of constructing a tournament with specific p/d-graph. We construct the adjacency matrix of a tournament in such a way that we control the pairs of vertices that are strictly paired or strictly distinguished in the tournament. Throughout the paper, T is an n-tournament where $n \geq 3$. When it is useful, we will show the orientation of edges in p/d(T) according to the arcs in T.

2.1 Cycles

The following proposition is a simple result about paths in p/d graphs, and the direction of arcs to a path from a vertex not on the path related to the number d edges on the path. This result will be used throughout the paper.

Proposition 2.1 *Consider any path in the p/d graph of tournament T with endpoints u and v. Let q be a vertex not on this path. Then q pairs u and v if and only if there exists an even number of d edges on path uv.*

Proof: Note that on any subpath of uv that has only p edges, q has only outarcs or only inarcs to all vertices on the subpath. Also, q will necessarily distinguish two vertices adjacent to a d edge, thus each time we come upon a d edge in a path, our pairings from that point until we reach another d edge will be in the opposite direction as the pairings prior to that d edge. Thus, q pairs u and v if and only if all edges between u and v are p edges, or if there are an even number of d edges on path uv. ■

This proposition can be stated the equivalent way as q distinguishes u and v if and only if there are an odd number of d edges on path uv. Note that if we want to relate the number of p edges to the arcs from q to u and v, then we must know the number of edges in the path to use Proposition 2.1. Let us call a path with an even number of vertices (and thus an odd number of edges) an even path, and similarly for odd paths. Then we can write the following Lemmas which easily follow from Proposition 2.1.

Lemma 2.1 *Let uv be an odd path, and let q be a vertex not on uv in $p/d(T)$. Then the number of p edges on the path uv is even if and only if q pairs u and v.*

Lemma 2.2 *Let uv be an even path, and let q be a vertex not on uv in p/d(T). Then the number of p edges on the path uv is even if and only if q distinguishes u and v.*

The following theorem describes the only possible combinations of p and d edges that can make up a three-path in p/d(T). It is important to both this section, and the next.

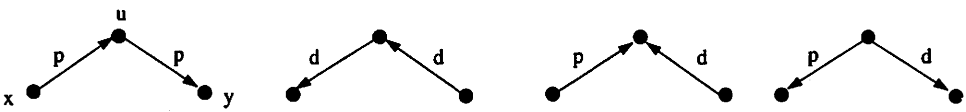


Figure 4: Possible 3-Paths in p/d(T).

Theorem 2.2 Suppose that the 3-path $x - u - y$ is a subgraph of p/d(T). If xu and uy are both paired edges or both distinguished edges, then u distinguishes x and y. If only one of xu and uy is a paired edge, then u pairs x and y.

Proof: Let $x - u - y$ be a 3-path in p/d(T). If xu and uy are paired edges and $x \rightarrow u$, then $x \rightarrow y$ and since x and u are strictly paired, $u \rightarrow y$. If $u \rightarrow x$, then $y \rightarrow x$ and thus $y \rightarrow u$.

If xu and uy are distinguished edges and $x \rightarrow u$, then $y \rightarrow x$. Since x and u are strictly distinguished, then $u \rightarrow y$. If $u \rightarrow x$, then $x \rightarrow y$, and thus $y \rightarrow u$. In all case, u distinguishes x and y.

Without loss of generality, suppose that xu is a distinguished edge and uy is a paired edge. If $x \rightarrow u$, then $x \rightarrow y$ and $y \rightarrow u$. If $u \rightarrow x$, then $y \rightarrow x$ and $u \rightarrow y$. So u pairs x and y. ■

The following eliminates one type of cycle.

Theorem 2.3 There cannot be a cycle in p/d(T) of all paired edges.

Proof: (See Figure 5). Suppose that there is a cycle, C, of all paired edges in p/d(T). Let u be a vertex in C. Let v be adjacent to u on C. If $u \rightarrow v$, then u beats all vertices on C, since adjacent vertices are strictly paired. Similarly, if u is beaten by v, then all vertices on C beat u. Thus, if w is also adjacent to u on C, u must pair v and w. This is a contradiction to Theorem 2.2. ■

We shall refer to a cycle as "even" or "odd" depending on the parity of the number of vertices in the cycle. We shall see that the parity of the number of p or d edges in the cycle along with the parity of the number of

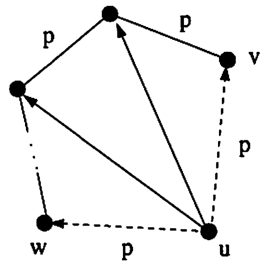


Figure 5: Example of Cycle if edges are all p Edges (Dotted arcs represent both edges in $p/d(\mathrm{T})$, and arcs in T).

cycle vertices will determine whether or not a cycle can be a p/d graph of some tournament.

Theorem 2.4 Suppose that a cycle C is a proper subgraph of $p/d(\mathrm{T})$. Then C is even if and only if the number of paired edges in C is even.

Proof: Suppose that cycle C is a proper subgraph of $p/d(\mathrm{T})$. Let q be a vertex not on C. We will consider the direction of arcs in T from vertices of C to q.

Suppose that there are no paired edges on C. Let u and v be two adjacent vertices on C. Let $C - uv$ denote the path from u to v on C excluding the edge uv. Consider the arcs of T in $C - uv$. Suppose that C is odd. Then $C - uv$ is an odd path from u to v, and by Lemma 2.1, q pairs u and v, which is a contradiction.

Suppose that there is a p edge on C. Let u and v be the endpoints of this edge. Consider the direction of arcs in T from $C - uv$ to the vertex q as above. Note that q must pair u and v . If C is odd, this will happen if and only if the number of p edges in $C - uv$ is even by Lemma 2.1, and thus the number of p edges in C is odd. If C is even, there are an even number of d edges in C if and only if the number of p edges in $C - uv$ is odd (Proposition 2.1), and thus the number of p edges in C is even. This proves the above statement. ■

Theorem 2.5 Suppose $p/d(\mathrm{T})$ contains an m-cycle, C. Then m is even if and only if the number of p edges in C is odd.

Proof: (See figure 6). Suppose that C is an m-cycle in $p/d(\mathrm{T})$. Then C must have a d edge by Theorem 2.3. Let the endpoints of this d edge be

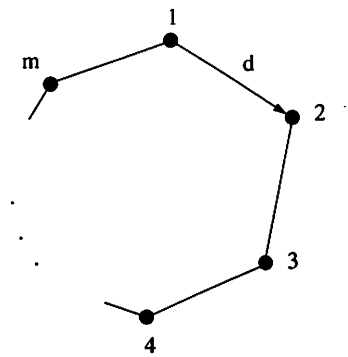


Figure 6: The m-Cycle, Theorem 2.5

u and v where $u \to v$. Label vertex u "1" and label vertex v "2". Label the other vertices in order from v around C, $3, 4, \ldots, m$. Now consider the following cases.

Suppose that $2 \to 3$.

Note that 23 is a d edge since 2 distinguishes 1 and 3, and 12 is a d edge (Theorem 2.2). Suppose that there are an odd number of d edges from 2 to m, and thus an even number of d edges from 3 to m. Then by Proposition 2.1, 2 pairs 3 and m, thus $2 \to m$. Since there is an edge from 1 to m in p/d(T), and 2 distinguishes them, 1 and m must be strictly distinguished. Since we assumed that there are an odd number of d edges from 2 to m, and we add edges 12 and $m1$ as d edges, we get an odd number of d edges on C in this case. Now if there are an even number of d edges from 2 to m, then Proposition 2.1 gives us that $1 \to m$. By Theorem 2.2, we have that $1m$ is a p edge. Counting d edges on C again, we have an even number from 2 to m plus edge 12, or an odd number of d edges on C in this case. For both cases, C has an odd number of d edges, thus m is even if and only if there are an odd number of p edges on C.

So suppose that $3 \to 2$. Note that 23 is a p edge by Theorem 2.2, since 2 pairs 1 and 3, and 12 is a d edge. Suppose that there are an odd number of d edges on C between 2 and m. By Proposition 2.1, $m \to 1$ and by Theorem 2.2, $m1$ is a d edge. Counting d edges, we have an odd number from 2 to m plus 12 and $m1$ give an odd number of d edges on C in this case. If we assume that there are an even number of d edges between 2 and m, Proposition 2.1 gives us that $1 \to m$, and Theorem 2.2 gives us that $1m$ is a p edge. counting d edges again gives an odd number of d edges on C, and we have the same conclusion as above, that is, there are an odd number of d edges on C, and m is even if and only if there are an odd number of p edges on C. ∎

These two results yield the following.

Theorem 2.6 If T is an n-tournament and $p/d(\mathrm{T})$ contains a cycle, then $p/d(\mathrm{T}) = C_n$.

Proof: If C is a cycle in $p/d(\mathrm{T})$, it cannot be a proper subgraph of $p/d(\mathrm{T})$ by Theorems 2.4 and 2.5. ■

2.2 Paths and Isolated Vertices

We now know that if there is a cycle in the p/d-graph of a tournament, then it must include all vertices of T. We now look to p/d-graphs which are the unions of paths. We see that our Proposition 2.1 and Lemmas 2.1 and 2.2 combine to give us the following Theorem.

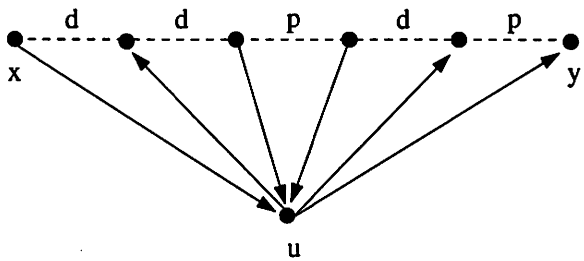


Figure 7: Example of Path and Vertex not on Path. (Dotted edges are in $p/d(\mathrm{T})$, arcs are in T).

Theorem 2.7 If there is a path P in $p/d(\mathrm{T})$ between two vertices x and y , then all vertices not on P either pair x and y or all distinguish x and y .

Proof: By Proposition 2.1 and Lemmas 2.1 and 2.2 we see that a vertex not on a path must pair or distinguish the endpoints of the path according to the number of d edges on the path, or the parity of the path and number of p edges on the path. These results are proven for an arbitrary vertex not on the path. ■

Corollary 2.1 If there is a path P in $p/d(\mathrm{T})$ between vertices x and y, then if x pairs (distinguishes) two vertices not on P, so does y.

Proof: Suppose that x pairs two vertices, u and v , not on path P. By Theorem 2.7, u and v both pair or both distinguish x and y . It is then impossible for y to distinguish u and v . The reverse is true if x distinguishes u and v . ■

Corollary 2.2 Suppose that P is a path which is not the subgraph of a cycle in $p/d(\mathrm{T})$. Let x and y be the endpoints of P. Then all vertices on $P\backslash\{x,y\}$ either pair x and y , or all vertices on $P\backslash\{x,y\}$ distinguish x and y .

Proof: Suppose P is a path in $p/d(\mathrm{T})$ as above. Choose any two vertices (say u and v) other than x and y and note that P contains a unique path between them which does not contain x or y . By Theorem 2.7, x and y treat u and v the same, so x and y both pair u and v or both distinguish u and v. Note that when x and y both pair (distinguish) u and v in the same direction, u and v both pair x and y. If x and y both pair (distinguish) u and v with arcs in opposite directions, u and v both distinguish x and y. Thus, u and v either both pair or both distinguish x and y. ■

Note that vertices along the path $P\backslash\{x,y\}$ from x to y may distinguish (pair) x and y , while vertices not on $P\backslash\{x,y\}$ may pair (distinguish) x and y . This must happen if P is not a cycle, since endpoints of the path must be both paired and distinguished. If all vertices are on a single path, the following is a simple consequence of Corollary 2.2.

Corollary 2.3 If $p/d(\mathrm{T})$ contains an n-path then $p/d(\mathrm{T})$ is an n-cycle.

Proof: Suppose that P is an n-path in $p/d(\mathrm{T})$with endpoints x and y . By Corollary 2.2, since P contains every vertex, x and y must be strictly paired or strictly distinguished. Thus there is an edge between x and y in $p/d(\mathrm{T})$, so $p/d(\mathrm{T})$ is a cycle. ■

Theorem 2.8 $p/d(\mathrm{T})$ cannot be the union of an $n-1$-path, and an isolated vertex.

Proof: Suppose that $p/d(\mathrm{T})$ consists of a path P from x to y , and a vertex u . Note that edge xu is not in the edge set of $p/d(\mathrm{T})$, so there is a vertex v which pairs x and u , and a vertex w which distinguishes x and u . But v and w must be on path P, so there is a unique path from v to w in $p/d(\mathrm{T})$. By Corollary 2.1, v and w must both pair or both distinguish x and u , a contradiction. ■

If there is a path P in the p/d-graph of some tournament between two vertices x and y , then we know that vertices not on that path will either all pair or all distinguish x and y . Whether they are paired or distinguished will depend on the parity of the number of d edges on P. If there is an odd number of d edges on P, then x and y must be distinguished by all vertices not on P. If there is an even number of d edges on P, then all vertices not on P pair x and y . If we are concerned only with how vertices not on P interact with x and y , we can model the situation accurately by shortening P to one edge, a d edge if there is an odd number of d edges on P, and a p edge if there is an even number of d edges on P. If useful, we can orient the edges in our shortened model using the direction of the arc between x and y in T. If there are several important segments of a path to examine, we can shorten the paths on each segment and use the orientation of the arcs in T between endpoints of each segment (see Figure 8). Note that in the shortened model, the direction of the arcs follows from Theorem 2.2. This will be useful in the next theorem.

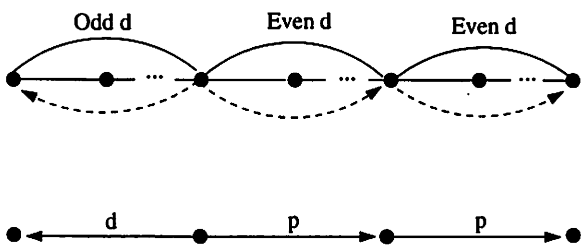


Figure 8: Example of Shortened Path Model, Dotted Arcs are in T.

Theorem 2.9 $p/d(\mathrm{T})$ cannot be the union of two paths.

Proof: Suppose that $p/d(\mathrm{T})$ is the union of two paths, P_1 and P_2. Let v_1 and u_1 be the end points of P_1, while v_2 and u_2 are the endpoints of P_2. Without loss of generality, let $v_1 \to v_2$. Since v_1 and v_2 are not strictly paired or distinguished in T, there must exist a vertex which pairs them, and a vertex which distinguishes them. These vertices cannot be in the same path by Corollary 2.2. So suppose there is a vertex in P_1, x, which pairs v_1 and v_2, and a vertex in P_2, y, which distinguishes v_1 and v_2. Since there is a path in $p/d(\mathrm{T})$ from x and y to the end vertices, u_1 and u_2 respectively, we have by Corollary 2.1 that u_1 pairs v_1 and v_2, and u_2 distinguishes v_1 and v_2. We would like to show that there is an edge in $p/d(\mathrm{T})$ from one of the four endpoints (v_1, u_1, v_2, or u_2) to another endpoint not on the same path to obtain a contradiction. Suppose that we examine v_1 and u_2, for example, and we find that v_2 and u_1 both pair them. Since all other vertices of T are on paths containing either v_2 or u_1, we have that every vertex in T must pair v_1 and u_2. Thus, we can use the shortened path model and examine the 4-tournaments on the vertices v_1, v_2, u_1, and u_2 that can arise. Given our conditions, there are two possible tournaments. One tournament occurs when u_1 beats v_1 and v_2, while $v_1 \rightarrow u_2 \rightarrow v_2$. The other occurs when u_1 beats v_1 and v_2, while $v_2 \rightarrow u_2 \rightarrow v_1$. It can be easily checked that the other two tournaments that can arise when u_1 pairs and u_2 distinguishes v_1 and v_2 are isomorphic to these two.

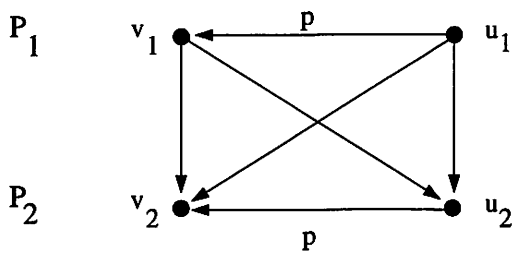


Figure 9: The 4-Tournament, T_1.

The first possible 4-tournament, say T_1 is in Figure 9. By our assumption we have that u_1 beats v_1 and v_2, while $v_1 \rightarrow u_2 \rightarrow v_2$. It follows by Theorem 2.7 that $u_1 \rightarrow u_2$ since v_1 pairs v_2 and u_2 and $u_1 \rightarrow v_2$. We can easily compute $p/d(T_1)$ and see that the vertices v_1 and u_2 are strictly paired - yielding an edge between them in $p/d(T_1)$, and consequently an edge between them in the p/d-graph of the resulting tournament T. This is the contradiction we are looking for.

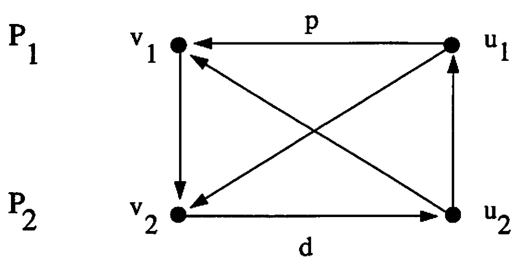


Figure 10: The 4-Tournament, T_2.

The second possible 4-tournament, say T_2, is in Figure 10. Note that in $p/d(T_2)$, v_2 and u_1 are strictly distinguished in p/d(T), a contradiction.

If we look at the four tournaments that arise between v_1, v_2, u_1, and u_2, and if we assume that there is a vertex in P_1 which distinguishes v_1 and v_2

and a vertex in P_2 which pairs v_1 and v_2, we get the previous two tournaments again. Thus, there will be some edge in p/d(T) between endpoints of P_1 and P_2, proving that there cannot be a p/d-graph which is the union of two paths. ■

Recall that a tournament T is well-covered if every pair of vertices is both paired and distinguished in T. This means that no pair of vertices is strictly paired or strictly distinguished in T. The two-path case above needed separate attention, because the well-covered property is not defined for tournaments with less than three vertices. We may now prove a more general result.

Theorem 2.10 If there is a tournament T with p/d-graph the union of k paths and m isolated vertices, $k + m > 2$, then there is a well-covered tournament on $k + m$ vertices.

It would be helpful to give an outline of the proof for this theorem before we procede. We first start with a tournament T whose p/d-graph is given as k paths and m isolated vertices. We then look at the subtournament of T, called T_E, consisting of an endpoint of each path and each isolated vertex. If we consider the p/d-graph of this subtournament, $p/d(T_E)$, we know that it either contains an edge, or it does not. If it does not, then T_E is well-covered, and thus there must exist a well-covered tournament on $k + m$ vertices. So we suppose that $p/d(T_E)$ does contain an edge, which of course means that in T_E two vertices are either strictly paired or strictly distinguished. Now since T_E is a subtournament of T, the arcs of T_E that are necessary for this to occur are also in T. But we know the p/d-graph of T, and armed with Corollary 2.1 and Theorem 2.7 we can determine quite a bit about arcs in T. The assumption that $p/d(T_E)$ is not well-covered and the necessary arcs in T_E (thus T) that this assumption forces will provide a contradiction to the arc structure of T determined by it's p/d-graph, and application of the above proposition and theorem. Several cases are necessary, but each case gives an edge in p/d(T) which should not be there, providing the contradiction.

Proof: Suppose that T is a tournament whose p/d-graph is the union of k paths and m isolated vertices, where $k + m > 2$. Also, suppose that there is no well-covered tournament on $k+m$ vertices. Choose an endpoint of each of the k paths and look at the $(k + m)$-subtournament on the k endpoints and the m isolated vertices, say T_E. Note that some pair of these vertices is strictly paired or strictly distinguished in T_E, since T_E is not well-covered. Let the vertices of T_E be labeled $v_1, v_2, \ldots, v_{k+m}$, and let v_i be an endpoint of path P_i. For now, consider an isolated vertex as a trivial path. Suppose that there is an edge in $p/d(T_E)$ between v_i and v_j. Without loss of generality, let $v_j \rightarrow v_i$.

Suppose that v_i and v_j are strictly distinguished in T_E. By an application of Corollary 2.1, all vertices in paths other than P_i and P_j must distinguish v_i and v_j in T, since an endpoint of each path distinguishes them in T. Since there is no edge in p/d(T) between v_i and v_j, we see that some vertex on path P_i or P_j must pair them, thus they cannot both be trivial paths in p/d(T). Suppose that there exists a vertex, x, in P_i that pairs v_i and v_j. Note that since there is a path in p/d(T) from x to the other endpoint of path P_i, say u_i, Corollary 2.1 states that u_i must also pair v_i and v_j. Suppose that u_i beats v_i and v_j.

Since $v_j \rightarrow v_i$, we have that there are an odd number of d edges on path P_i, by Proposition 2.1. Thus, all vertices of T not on P_i must distinguish v_i and u_i by Theorem 2.7. Since all vertices not on P_i or P_j also distinguish v_i and v_j(by our assumption), we have that all vertices not on P_i or P_j must pair v_j and u_i (note that if q is such a vertex, the direction of the arc between q and v_i is opposite of the direction of the arcs between q and both v_j and u_i. Thus, q must pair v_j and u_i). Also, since v_i pairs v_j and u_i, all vertices on path $P_i \backslash \{u_i\}$ must pair v_j and u_i by Corollary 2.1. Thus, if there is no vertex on path P_j which distinguishes v_j and u_i, then v_j and u_i are strictly paired in T, a contradiction. So suppose that there is a vertex, y, in P_j which distinguishes v_j and u_i. Note that v_j cannot be an isolated vertex in p/d(T). If we apply Corollary 2.1 to the path from y to the endpoint of P_j, say u_j, then u_j must also distinguish v_j and u_i. There are two possible ways for u_j to distinguish v_j and u_i. First, let $v_j \rightarrow u_j \rightarrow u_i$. Now consider u_j and u_i. Since $u_j \rightarrow u_i \rightarrow v_j$, we see that P_j has an odd number of d edges (Proposition 2.1). Then since $v_j \rightarrow v_i$, we must have that $v_i \rightarrow u_j$, also by Proposition 2.1. This means that both v_i and v_j distinguish u_i and u_j, so all vertices on $P_i \backslash \{u_i\}$ and $P_j \backslash \{u_j\}$ distinguish them. Recall that vertices not on P_i or P_j pair v_j and u_i. They must also distinguish v_j and u_j since there is an odd number of d edges on P_j. These two results yield that all vertices not on P_i or P_j must distinguish u_i and u_j. Thus, u_i and u_j are strictly distinguished in T, a contradiction. If we have that u_j distinguishes v_j and u_i as $u_i \rightarrow u_j \rightarrow v_j$, we follow the same process. We have that u_i and u_j are paired by all vertices in $P_i \backslash \{u_i\}$ and $P_j \backslash \{u_j\}$. Also, vertices not in P_i or P_j must pair v_j and u_i by the result above, and they pair v_j and u_j (since both beat v_i), thus all vertices not in P_i or P_j must pair u_i and u_j. So all vertices in $T \backslash \{u_i, u_j\}$ pair u_i and u_j, which is a contradiction. Now if we had assumed that u_j pairs v_i and v_j, but v_i and v_j beat u_j, we could follow the same process and obtain two cases as above, with the same conclusions, thus a contradiction. We assumed that the vertex x that paired v_i and v_j was in P_i. If it had been in P_j, the same conclusions would emerge, also leading to a contradiction. Now suppose that v_i and v_j are strictly paired in T_E. By a similar argument as above, there must be some vertex in P_i or P_j which distinguishes v_i and v_j. If we suppose that a vertex in P_i distinguishes them, then as above, we know that u_i distinguishes them. Suppose that $v_j \to u_i \to v_i$. Then note that there are an even number of d edges in P_i by Proposition 2.1, so all vertices not on P_i must pair u_i and v_i. This, along with the fact that all vertices not on P_i or P_j must pair v_i and v_j yields that all vertices not on P_i or P_j pair u_i and v_j. Since v_i also pairs them, so do all vertices on $P_i \backslash \{u_i\}$. So suppose that there is a vertex on P_j which distinguishes them. Then u_j must distinguish them, say as $u_i \to u_j \to v_j$. Now consider u_i and u_j once again. All vertices in $\mathrm{T}\backslash\{u_i, u_j\}$ must distinguish them, since vertices on P_i or P_j do, and vertices not on P_i or P_j pair u_i and v_j, but distinguish v_j and u_j. Thus, u_i and u_j are strictly distinguished in T, a contradiction.

If u_j distinguishes v_i and v_j as $v_j \to u_j \to u_i$, we achieve the same contradiction, with all vertices in $\mathrm{T}\backslash\{u_i, u_j\}$ pairing u_i and u_j. Also, if u_i distinguishes v_i and v_j as $v_i \to u_i \to v_j$, we again obtain two cases with the same conclusions as above.

Thus, if there is no well-covered tournament on $k + m$ vertices, then there is some edge in $p/d(\mathrm{T})$ which we assumed was not there. ■

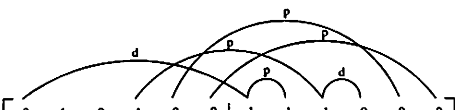
Recall that well-covered tournaments only exist on four and greater than 5 vertices. Thus, there is no p/d-graph of any tournament which is the union of k paths and m isolated vertices if $k + m = 3$ or $k + m = 5$.

2.3 The Main Theorem

We will show in this section that $p/d(\mathrm{T})$ can be any other union of paths and isolated vertices. In order to prove this, we will make use of the adjacency matrix of a tournament to construct tournaments whose p/d-graph is anything not listed above.

Suppose that $A(T)$ is the adjacency matrix of tournament T. Let the vertices of T be labeled $1, 2, \ldots, n$ in the order of the rows of $A(T)$. Note that if two vertices, i and j, are strictly paired in T, then columns i and j will be identical, except at rows i or j where either $i \to j$ or $j \to i$. Similarly, rows i and j will be identical, except at columns i or j. If vertices i and j are strictly distinguished in T, then columns i and j will be exactly the opposite (that is, where there is a 1 in column i, there will be a 0 in column j, and vice-versa) except at rows i or j. The same is true of the rows. This provides a simple way to construct paths in $p/d(\mathrm{T})$ as we construct the adjacency matrix of a tournament one dimension at a time, and thus the tournament itself.

Suppose that we want to construct a tournament T whose p/d-graph is



	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	0	1	0	0	1	1	1	0	0	0
2	0	0	1	0	1	0	1	1	0	1	1	0
3	1	0	0	1	0	0	0	0	1	0	0	0
4	0	1	0	0	1	0	1	1	0	1	1	0
5	1	0	1	0	0	0	0	0	0	1	0	0
6	1	1	1	1	1	0	0	0	1	0	1	0
7	0	0	1	0	1	1	0	0	0	1	1	1
8	0	0	1	0	1	1	1	0	0	1	1	1
9	0	1	0	1	1	0	1	1	0	1	1	0
10	1	0	1	0	0	1	0	0	0	0	0	1
11	1	0	1	0	1	0	0	0	0	1	0	0
12	1	1	1	1	1	1	0	0	1	0	1	0

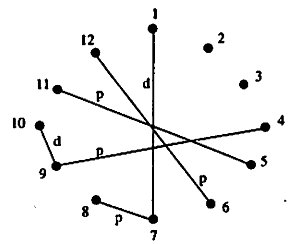


Figure 11: Building a Tournament with Specific p/d-Graph: $k = 4$, $m = 2$.

none of the forbidden unions above, but is composed of m isolated vertices, and k paths. Choose any well-covered tournament W on $m+k$ vertices (we are guaranteed that one exists for $m+k = 4$ or > 5), and place the adjacency matrix of this tournament, $A(W)$, in the upper left hand corner of the adjacency matrix, $A(T)$, that we shall construct (i.e., W is a subtournament of T). Label the vertices of W $1, 2, \ldots, m+k$ in the order of the rows. We will build an $n \times n$ adjacency matrix of a tournament T with desired p/d-graph one dimension at a time, starting with the $(m+k) \times (m+k)$ matrix $A(W)$. We may add columns (thus also rows, as is consistent with an adjacency matrix of a tournament) to the matrix $A(W)$ corresponding to vertices in each of the k paths in the following manner. Choose k columns out of the existing $m+k$ columns of $A(W)$. Look at column j for example. Add a column and its corresponding row to the matrix that either mirrors or is opposite to column j. Continue adding columns in this manner for each edge in the jth path of p/d(T). Do this for each of the k chosen columns. The resulting matrix will be the adjacency matrix of a tournament with p/d-graph having k paths and m isolated vertices.

Note that our method of construction ensures that $A(T)$ is the adjacency matrix of a tournament T with desired general p/d-graph, i.e., m isolated vertices and k paths. The addition of each vertex (dimension in

$A(T)$) does not affect the strictly paired or strictly distinguished relationship of existing vertices. The reason for this is the following. When we add a column/row (since we add both) to our matrix, we are adding them as mirror or opposite to an existing column/row. In tournament terms, we are adding a vertex, say i, to a tournament in a strictly paired or strictly distinguished relationship to another existing vertex, say j in the tournament. Thus, all arcs between the newly added vertex i and the existing vertices in T are predetermined by the direction of the arcs between j and the other vertices of T, and the relationship (strictly paired or strictly distinguished) between i and j. It is easy to see that when j pairs/distinguishes vertices in T, so does i.

Please refer to Figure 11 for an example and note that W is the 6 vertex subtournament in the upper left hand corner of the matrix. If one examines column 1 of the matrix, and compares it to column 7, one can see that column 7 is an opposite image (in terms of 1's and 0's) of column 1, except in row 7. This creates a d edge in the p/d-graph of the tournament, as noted in the p/d-graph below the matrix. Note that the direction of the arc from vertex 1 to vertex 7 is arbitrary with respect to the strictly distinguished property. The 0's or 1's in the 4th row, 7th column or 7th row, 4th column may not be opposite as the others are, but this will not affect the strictly distinguished property of the vertices 1 and 7 in the tournament. Note that column 4 and column 9 are mirror images of each other except in row 9, giving a p edge between vertices 4 and 9 in the resulting p/d(T).

This construction can be done for $m + k = 4$ or $m + k \geq 6$, since we are guaranteed to have a well-covered tournament for these number of vertices. This method of construction allows us to prove the necessity of Theorem 2.10.

Theorem 2.11 There is a tournament T whose p/d-graph is the union of k paths and m isolated vertices if and only if there is a well-covered tournament on $k + m$ vertices.

Proof: Sufficiency has been proved in Theorem 2.10. For necessity, note that if there is a well-covered tournament, we may use the construction technique explained above to construct the tournament T with said p/d-graph. ■

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