On the partition dimension and connected partition dimension of wheels *

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Abstract

Let G be a connected graph. For a vertex $v \in V(G)$ and an ordered k-partition $\Pi = \{S_1, S_2, ..., S_k\}$ of $V(G)$, the representation of v with respect to Π is the k-vector $r(v|\Pi) = (d(v, S_1), d(v, S_2), ..., d(v, S_k))$. The k-partition Π is said to be resolving if the k-vectors $r(v|\Pi)$, $v \in V(G)$, are distinct. The minimum k for which there is a resolving k-partition of $V(G)$ is called the partition dimension of G, denoted by $pd(G)$. A resolving k-partition $\Pi = \{S_1, S_2, ..., S_k\}$ of $V(G)$ is said to be connected if each subgraph $< S_i >$ induced by S_i $(1 \leq i \leq k)$ is connected in G. The minimum k for which there is a connected resolving k-partition of $V(G)$ is called the connected partition dimension of G, denoted by $cpd(G)$. In this paper, the partition dimension as well as the connected partition dimension of the wheel W_n with n spokes are considered, by showing that $\lceil (2n)^{1/3} \rceil \leq pd(W_n) \leq 2\lceil n^{1/2} \rceil + 1$ and $cpd(W_n) = \lceil (n+2)/3 \rceil$ for

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$n \geq 4$.

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1 Introduction

If G is a connected graph, the distance $d(u,v)$ between two vertices u and v in G is the length of a shortest path between them. The diameter of G is the largest distance between two vertices in $V(G)$. For a vertex v of a graph G and a subset S of $V(G)$, the distance between v and S is $d(v,S) = \min\{d(v,x)|x \in S\}$. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be an ordered k-partition of vertices of G and let v be a vertex of G. The representation $r(v|\Pi)$ of v with respect to Π is the k-tuple $(d(v,S_1), d(v,S_2), \ldots, d(v,S_k))$. If distinct vertices of G have distinct representations with respect to Π, then Π is called a resolving partition for G. The cardinality of a minimal resolving partition is called the partition dimension of G, denoted by $pd(G)$ [2],[3]. A resolving partition $\Pi = \{S_1, S_2, ..., S_k\}$ of $V(G)$ is called connected if each subgraph $< S_i >$ induced by S_i $(1 \leq i \leq k)$ is connected in G. The minimum k for which there is a connected resolving k-partition of $V(G)$ is called the connected partition dimension of G, denoted by $cpd(G)$ [9].

The concepts of resolvability have previously appeared in the literature (see [2]–[4], [6]–[9]). These concepts have some applications in chemistry for representing chemical compounds [4] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [6].

If $d(x,S) \neq d(y,S)$ we shall say that the class S separates vertices x and y. If a class S of Π separates vertices x and y we shall also say that Π separates x and y. From these definitions it can be observed that the property of a given partition Π of the vertices of a graph G to be a resolving partition of G can be verified by investigating the pairs of vertices in the same class. Indeed, every vertex $x \in S_i$ $(1 \leq i \leq k)$ is at distance 0 from S_i, but is at a distance different from zero from any other class S_j with $j \neq i$. It follows that $x \in S_i$ and $y \in S_j$ are separated either by S_i or by S_j for every $i \neq j$.

The wheel W_n for $n \geq 3$ is the graph $C_n + K_1$ obtained by joining all vertices of a cycle $C_n = v_0, v_1, \ldots, v_{n-1}$ to a further vertex c called the center. Thus W_n contains $n+1$ vertices, the center and n rim vertices and has diameter 2. In this paper, we consider the partition dimension as well as the connected partition dimension of W_n for any integer $n \geq 4$.

2 The partition dimension of the wheel

A frequent question in graph theory concerns how the value of a parameter is changed by making a small change in the graph. In this section, we consider how the partition dimension of a connected graph G is affected by the addition of a single vertex. Consider the partition dimension of the wheel W_n for $n \geq 3$. Clearly $pd(C_n) = 3$, while $pd(W_3) = 4$ as it is K_4, $pd(W_n) = 3$ when $4 \leq n \leq 7$, and $pd(W_n) = 4$ when $8 \leq n \leq 19$. In this section we determine some bounds for $pd(W_n)$, but the question of determining the exact value of this parameter for wheels remains unsettled.

It is clear that each rim vertex has two neighboring vertices apart from the center. Thus each of rim vertices has distance 1 to at most two classes other than the class that contains the center. It follows that if $\Pi = \{S_1, S_2, \ldots, S_k\}$ is a resolving k-partition of $V(W_n)$, $(x_1, x_2, ..., x_k)$ is the k-vector representation of any rim vertex and $c \in S_1$, then there are at most two x_i such that $x_i = 1$ for every $2 \leq i \leq k$.

We present two lemmas regarding the cardinality of the classes of a resolving partition of $V(W_n)$ that contain or not the center.

Lemma 2.1 *Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a resolving k-partition of $V(W_n)$. If $c \in S_1$, then $|S_1| \leq 1 + \binom{k-1}{2} + \binom{k-1}{1} + \binom{k-1}{0}$.*

Proof: We deduce that $r(c|\Pi) = (0, 1, 1, \ldots, 1)$ and $r(v|\Pi) = (0, \ldots)$ for $v \in S_1 \backslash \{c\}$. Since the diameter of W_n is 2, the elements of the k-vector representation $r(v|\Pi)$ of each rim vertex $v \in S_1 \backslash \{c\}$ other than the first element can be 1 or 2. But there can be at most two elements equal to 1 in the vector representation apart from the first position of the vector. This implies that there are $k - 1$ positions in the vector representation of the rim vertices that can be filled by at most two 1's and the rest can be filled by 2's. Thus there are at most $\binom{k-1}{2} + \binom{k-1}{1} + \binom{k-1}{0}$ distinct vector representations for all vertices $v \in S_1 \backslash \{c\}$. Together with the vector representation of the center, we have at most $1 + \binom{k-1}{2} + \binom{k-1}{1} + \binom{k-1}{0}$ distinct representations. Therefore $|S_1| \leq 1 + \binom{k-1}{2} + \binom{k-1}{1} + \binom{k-1}{0}$. □

Lemma 2.2 *Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a resolving k-partition of $V(W_n)$. If $c \in S_1$, then $|S_i| \leq \binom{k-2}{2} + \binom{k-2}{1} + \binom{k-2}{0}$ for each $2 \leq i \leq k$.*

Proof: Consider a class other than S_1, without loss of generality say S_2, not containing the center. Then the vector representation for $w \in S_2$ is $r(w|\Pi) = (1, 0, \ldots)$. There are $k - 2$ positions in the vector representation of the rim vertices that can be filled by at most two 1's and the rest can be filled by 2's. Thus there are at most $\binom{k-2}{2} + \binom{k-2}{1} + \binom{k-2}{0}$ distinct representations for all vertices $w \in S_2$. Therefore $|S_i| \leq \binom{k-2}{2} + \binom{k-2}{1} + \binom{k-2}{0}$ for each $2 \leq i \leq k$. □

With the two Lemmas above, we are now in the position to prove the following theorem.

Theorem 2.1 *For every $n \geq 4$ we have:*

$$\lceil (2n)^{1/3} \rceil \leq pd(W_n) \leq p+1,$$

where p is the smallest prime number such that $p(p-1) \geq n$.

Proof: Lower bound. Let $pd(W_n) = k$ and $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a resolving k-partition of $V(W_n)$. Let $c \in S_1$. By Lemma 2.1 we have $|S_1| \leq 1 + \binom{k-1}{2} + \binom{k-1}{1} + \binom{k-1}{0}$ and by Lemma 2.2 we have $|S_i| \leq \binom{k-2}{2} + \binom{k-2}{1} + \binom{k-2}{0}$ for $2 \leq i \leq k$. We get
$|V(W_n)| = n + 1 = \sum_{i=1}^{k} |S_i| \leq 1 + \sum_{i=0}^{2} \binom{k-1}{i} + (k-1)\sum_{i=0}^{2} \binom{k-2}{i}$, which implies $n \leq (k^3 - 3k^2 + 6k - 2)/2 < k^3/2$ for every $k \geq 2$. It follows that $k \geq \lceil (2n)^{1/3} \rceil$.

Upper bound. Let p be the smallest prime number such that $p(p-1) \geq n$. Since p is prime, the sequence $0, i, 2i, 3i, \ldots, (p-1)i$, where $1 \leq i \leq p-1$ and all numbers are reduced modulo p, is a permutation of the set $\{0, 1, \ldots, p-1\}$. Consider the sequence $(x_j)_{j=1,\ldots,p(p-1)} = X_1, X_2, \ldots, X_{(p-1)/2}$, where for each $1 \leq i \leq (p-1)/2$ the subsequence

$$X_i = 0, 0, i, i, 2i, 2i, 3i, 3i, \ldots, (p-1)i, (p-1)i$$

contains $2p$ terms and each pair of equal elements different from $0, 0$ is obtained from the previous one by adding i modulo p to each component. The resolving partition $\Pi = \{S_1, \ldots, S_{p+1}\}$ of $V(W_n)$ is defined as follows:
a) if $n = p(p-1)$ then $S_{p+1} = \{c\}$ and each element v_i $(0 \leq i \leq n-1)$ is assigned to the class $S_{x_{i+1}+1}$;
b) if $n < p(p-1)$ then $S_{p+1} = \{c, v_{n-1}\}$ and each element v_i $(0 \leq i \leq n-2)$ is assigned to the class $S_{x_{i+1}+1}$.
From the construction it can be observed that for any two vertices v_i, v_{i+1} in the same class, vertices v_{i-1} and v_{i+2} belong to different classes. Also, if v_i and v_j belong to the same class S_p and $i < j$, $j \neq i+1$, then at least one pair of vertices from $\{v_{i-1}, v_{j-1}\}$, $\{v_{i-1}, v_{j+1}\}$, $\{v_{i+1}, v_{j-1}\}$, $\{v_{i+1}, v_{j+1}\}$ consists of vertices that belong to two classes S_q, S_r such that $q, r \neq p$ and $q \neq r$. In the case b vertices c and v_{n-1} can be separated by a class of Π. It follows that Π is a resolving connected partition of $V(W_n)$ having $p+1$ classes, which implies $pd(W_n) \leq p+1$.

□

Corollary 2.2 *For every $n \geq 4$, $pd(W_n)$ verifies*

$$\lceil (2n)^{1/3} \rceil \leq pd(W_n) \leq 2\lceil n^{1/2} \rceil + 1$$

Proof: Since prime number p must satisfy $p(p-1) \geq n$ we can take $p \geq \lceil n^{1/2} \rceil + 1$. We shall apply Bertrand's postulate, proved for the first time by Chebyshev, which asserts that for every $n \geq 1$, there is some prime number p with $n < p \leq 2n$ (see [1]). We deduce that there exists a prime number p such that $\lceil n^{1/2} \rceil < p \leq 2\lceil n^{1/2} \rceil$, hence $pd(W_n) \leq p + 1 \leq 2\lceil n^{1/2} \rceil + 1$. □

3 The connected partition dimension of the wheel

It is clear the $cpd(C_n) = 3$ and $cpd(W_3) = 4$ as it is K_4. In this section we consider the connected partition dimension of W_n for every $n \geq 4$. Let $\Pi = \{S_1, \ldots, S_k\}$ be a connected resolving k-partition of $V(W_n)$ such that the center $c \in S_1$. Every class of Π different from S_1 induces a subpath consisting of consecutive vertices of C_n; also vertices of S_1 belonging to C_n induce $r \geq 0$ disjoint subpaths $L_1, \ldots, L_r$ consisting each of consecutive vertices of C_n. A sequence of consecutive vertices $v_i, v_{i+1}, \ldots, v_j$ on C_n (indices are considered modulo n) will be called a window if these vertices do not belong to S_1 but $v_{i-1}, v_{j+1} \in S_1$. It is clear that each window includes some classes of Π different from S_1. Each class of Π containing vertex v_i or v_j will be called a boundary class.

Lemma 3.1 *Let $\Pi = \{S_1, \ldots, S_k\}$ be a connected resolving k-partition of $V(W_n)$ such that the center $c \in S_1$. Then:*
i) $|S_j| \leq 3$ for every $2 \leq j \leq k$ and $|S_j| \leq 2$ if S_j is a boundary class;
ii) $|L_i| \leq 3$ for every $1 \leq i \leq r$ and at most one L_i contains three vertices.

Proof: If L_i or S_j contains at least four distinct vertices $v_l, v_{l+1}, \ldots, v_{m-1}, v_m$, then v_{l+1} and v_{m-1} cannot be separated by any other class. If S_j is a boundary class containing at least three vertices, then there exist the vertices $v_l, v_{l-1}, v_{l-2}, v_{l-3}$ such that $v_l \in S_1$ and $v_{l-1}, v_{l-2}, v_{l-3} \in S_j$ or $v_l, v_{l+1}, v_{l+2}, v_{l+3}$ such that $v_l \in S_1$ and $v_{l+1}, v_{l+2}, v_{l+3} \in S_j$. We deduce that the vertices v_{l-1}, v_{l-2} and v_{l+1}, v_{l+2}, respectively cannot be separated by Π.
If there exists L_p and L_q, $1 \leq p < q \leq r$ containing three vertices each, i.e., $L_p = v_l, v_{l+1}, v_{l+2}$ and $L_q = v_m, v_{m+1}, v_{m+2}$, then the vertices v_{l+1} and v_{m+1} cannot be separated by Π, a contradiction.

□

Theorem 3.1 *The following equality holds:*

$$cpd(W_n) = \begin{cases} 3 & \text{for } n = 4. \\ \lceil \frac{n+2}{3} \rceil & \text{for } n \geq 5. \end{cases}$$

Proof: Let $V(W_n) = \{c, v_0, v_1, ..., v_{n-1}\}$. We first show that $cpd(W_n) = 3$ for $4 \le n \le 7$. As a consequence of Proposition 2.2 in [9], showing that $cpd(G) = 2$ if and only if $G \cong P_n$, we have $cpd(W_n) \ge 3$ for any $n \ge 4$. To show that $cpd(W_n) \le 3$ for $4 \le n \le 7$, we give a connected resolving partition $\Pi = \{S_1, S_2, S_3\}$ for each $n = 4, 5, 6, 7$ as follows.

- For $n = 4$, $S_1 = \{c, v_0, v_1\}$, $S_2 = \{v_2\}$, and $S_3 = \{v_3\}$.
- For $n = 5$, $S_1 = \{c, v_0, v_1, v_2\}$, $S_2 = \{v_3\}$, and $S_3 = \{v_4\}$.
- For $n = 6$, $S_1 = \{c, v_0, v_1, v_2\}$, $S_2 = \{v_3, v_4\}$, and $S_3 = \{v_5\}$.
- For $n = 7$, $S_1 = \{c, v_0, v_1, v_2\}$, $S_2 = \{v_3, v_4\}$, and $S_3 = \{v_5, v_6\}$.

Thus $cpd(W_n) = 3$ for $4 \le n \le 7$.

Now we show that $cpd(W_n) = \lceil \frac{n+2}{3} \rceil$, for $n \ge 8$. Let $\Pi = \{S_1, S_2, \ldots, S_{\lceil \frac{n+2}{3} \rceil}\}$, where $S_1 = \{c, v_0, v_1, v_2\}$ and $S_2 = \{v_3, v_4\}$, $S_i = \{v_{3(i-1)-1}, v_{3(i-1)}, v_{3(i-1)+1}\}$ for $3 \le i \le \lceil \frac{n+2}{3} \rceil - 2$ and $S_{\lceil \frac{n+2}{3} \rceil - 1} = \{v_{n-3}\}$ when $n \equiv 2 \bmod 3$ or $S_{\lceil \frac{n+2}{3} \rceil - 1} = \{v_{n-4}, v_{n-3}\}$ when $n \equiv 0 \bmod 3$, or $S_{\lceil \frac{n+2}{3} \rceil - 1} = \{v_{n-5}, v_{n-4}, v_{n-3}\}$ when $n \equiv 1 \bmod 3$, and $S_{\lceil \frac{n+2}{3} \rceil} = \{v_{n-2}, v_{n-1}\}$. It can be easily verified that any two elements in the same class have distinct representations and all these classes induce connected subgraphs, so $cpd(W_n) \le \lceil \frac{n+2}{3} \rceil$.

In order to show that $cpd(W_n) \ge \lceil \frac{n+2}{3} \rceil$, let $\Pi = \{S_1, \ldots, S_k\}$ be a connected resolving k-partition of $V(W_n)$ and suppose that $c \in S_1$. Let $\varphi(l)$ denote the minimum number of classes of Π different from S_1 included in a window with l vertices. From Lemma 3.1 we deduce easily that for every $l \ge 1$,

$$\varphi(l) = \begin{cases} (l+3)/3 & \text{if } l \equiv 0 \bmod 3, \\ (l+2)/3 & \text{if } l \equiv 1 \bmod 3, \\ (l+4)/3 & \text{if } l \equiv 2 \bmod 3. \end{cases}$$

Suppose first that $S_1 \ne \{c\}$ and the vertices of S_1 belonging to C_n induce $r \ge 1$ disjoint subpaths $L_1, \ldots, L_r$ of C_n containing $n_1, \ldots, n_r$ vertices and r windows containing $l_1, \ldots, l_r$ vertices such that $\sum_{i=1}^{r} l_i + \sum_{i=1}^{r} n_i = n$. It follows that $k \ge \sum_{i=1}^{r} \varphi(l_i) + 1 \ge \sum_{i=1}^{r} (l_i + 2)/3 + 1 = \frac{1}{3}(\sum_{i=1}^{r} l_i + 2r) + 1 = \frac{1}{3}(n - \sum_{i=1}^{r}(n_i - 2)) + 1 \ge (n-1)/3 + 1 = (n+2)/3$ since $n_i \le 2$ and at most a single value $n_i = 3$. It follows that $k \ge \lceil (n+2)/3 \rceil$.

If $S_1 = \{c\}$ then $r = 0$ and denote by $\psi(n)$ the minimum number of classes of Π different from S_1, which consists of subpaths with at most three vertices of C_n. We get

$$\psi(n) = \begin{cases} n/3 & \text{if } n \equiv 0 \bmod 3, \\ (n+2)/3 & \text{if } n \equiv 1 \bmod 3, \\ (n+1)/3 & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

In this case we deduce $k \geq 1 + \psi(n) \geq \lceil (n+2)/3 \rceil$, which concludes the proof. $\square$

References

[1] M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, Springer-Verlag, 1999.

[2] G. Chartrand, E. Salehi, P. Zhang, On the partition dimension of a graph, *Congr. Numer.* 130(1998) 157-168.

[3] G. Chartrand, E. Salehi, P. Zhang, The partition dimension of a graph, *Aequationes Math.* 59(2000) 45-54.

[4] G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.* 105(2000) 99-113.

[5] F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combin.* 2(1976) 191-195.

[6] R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Computer Vision, Graphics, and Image Processing* 25(1984) 113-121.

[7] P. J. Slater, Leaves of trees, *Congr. Numer.* 14(1975) 549-559.

[8] P. J. Slater, Dominating and reference sets in a graph, *J. Math. Phys. Sci.* 22(1988) 445-455.

[9] V. Saenpholphat, P. Zhang, Connected partition dimension of graphs, *Discussiones Mathematicae Graph Theory* 22 (2002) 305-323.