Bondage number in oriented graphs*

Erfang Shan† Liying Kang
Department of Mathematics, Shanghai University, Shanghai 200436, P.R. China

Abstract

The bondage number $b(D)$ of a digraph D is the cardinality of a smallest set of arcs whose removal from D results in a digraph with domination number greater than the domination number of D. In this paper, we present some upper bounds on bondage number for oriented graphs including tournaments, and symmetric planar digraphs.

Key Words: digraph; domination number; bondage number

AMS subject classification: 05C20; 05C69

1 Introduction

Domination and other related concepts in undirected graphs are extensively studied, the same concepts are presented for digraphs. In terms of applications, digraphs come up more naturally in modelling real world problems, the questions of Facility Location, Assignment Problems etc. are very much related to the idea of domination or independent domination on digraphs. A comprehensive survey of domination in digraphs is given in [6].

*This research was partly supported by the National Natural Science Foundation of China, the Special Funds for Major Specialties of Shanghai Education Committee and the Young Science Foundation of Shanghai Education Committee under grant 01QN6262.

†Corresponding author. Email address: efshan@staff.shu.edu.cn (E.F. Shan)

A digraph $D = (V, A)$ consists of a finite vertex set V and an arc set $A \subseteq P$, where P is the set of all ordered pairs of distinct vertices of V. That is , D has no multiple loops and no multiple arcs (but pairs of opposite arcs are allowed). If $(u, v) \in A$ then the arc is directed from u to v and is denoted by $u \rightarrow v$. The vertex u is called a *predecessor* of v and v is called a *successor* of u. Moreover, u is said to be *adjacent to* v and v is *adjacent from* u. A digraph D is called an *oriented graph* or an *asymmetric digraph* if whenever (u, v) is an arc of D, then (v, u) is not an arc of D. A *symmetric digraph* is a digraph for which $(u, v) \in A$ implies $(v, u) \in A$. The sets $O(u) = \{v : (u, v) \in A\}$ and $I(u) = \{v : (v, u) \in A\}$ are called the *outset* and *inset* of the vertex u. More generally, for $S \subseteq V(D)$, we write $O(S) = \cup_{u \in S} O(u)$ and $I(S) = \cup_{u \in S} I(u)$. The *indegree* of a vertex u is given by $d^-(u) = |I(u)|$ and the *outdegree* of a vertex is $d^+(u) = |O(u)|$. We denote the minimum and maximum indegree and outdegree in D by $\delta^-(D), \Delta^-(D), \delta^+(D)$ and $\Delta^+(D)$, respectively. A set $S \subseteq V$ is *independent* if for all $u, v \in S, (u, v) \notin A$. The independent number $\beta_0(D)$ is the maximum cardinality among all independent sets of vertices of D. If for some k we have a sequence $\sigma = u_0, u_1, \ldots, u_k$ of vertices such that every u_{i+1} is a successor of u_i, then σ is a *directed walk* from u_0 to u_k of length k. If all the u_i's are different then σ is a *directed path.* For vertices u and v in a digraph D containing a u-v path , the *distance* $d(u, v)$ from u to v is the length of a shortest u-v directed path in D. The *underlying graph* of a digraph D is the graph obtained by replacing each arc (u, v) or symmetric pairs $(u, v), (v, u)$ of arcs by the edge of uv. We always assume that the underlying graph of the digraph D is connected.

A graph H is a minor of a graph G, if H can be obtained from G by the following three operations: delete a vertex, delete an edge, or contract an edge.

We define a set $S \subseteq V$ of a digraph D to be a *dominating set* of D if for all $v \notin S$, v is a successor of some vertex $u \in S$. The minimum cardinality among all dominating sets of D is called the *domination number* of D and is denoted by $\gamma(D)$. The *bondage number* $b(D)$ of a digraph D is the cardinality of a smallest set of arcs whose removal from D results in a digraph with domination number greater than $\gamma(D)$. A set $S \subseteq V$

is defined to be a *2-distance dominating set* of a digraph D if for every vertex $u \in V - S$ there exists a vertex $v \in S$ such that $d(v,u) \leq 2$. The minimum cardinality among all 2-distance dominating sets of D is called the *2-distance domination number* of D and is denoted by $\gamma_2(D)$. A set S of vertices in digraph D is called a *2-distance independent dominating set* of D if S is both independent and 2-dominating. The *2-independent domination number* $i_2(D)$ of D is the minimum cardinality among all 2-independent dominating sets of D.

Chartrand et al. [2] established bounds on domination number for a digraph. The purpose of this paper is to study bondage number and 2-distance domination number of digraphs. We will establish upper bounds on bondage number for symmetric planar digraphs and oriented graphs. Moreover, we also give an upper bound on 2-distance domination number for a digraph in terms of its order. The concept of bondage number in undirected graphs is well studied in graph theory (see [4],[5],[7],[8],[9]).

2 Preliminary results

For every $u \in V(D)$, let $A_u^- = \{(w,u) \in A(D) | w \in V(D)\}$, $A_u^+ = \{(u,w) \in A(D) \mid w \in V(D)\}$. For notation convenience, in what follows, we write $A_u = A_u^- \cup A_u^+$, $d(u) = d^+(u) + d^-(u)$, $N(u) = O(u) \cup I(u)$, $\delta(D) = min\{d(u) : u \in V(D)\}$, and $\Delta(D) = max\{d(u) : u \in V(D)\}$. Also, for $S \subseteq V$, $N(S) = O(S) \cup I(S)$. We begin with some elementary lemmas about bondage number of digraphs.

Lemma 2.1 *If D is a digraph , then $b(D) \leq d^-(u) + d(v) - |I(u) \cap I(v)|$ for every pair vertices u and v with $(u,v) \in A(D)$ and this bound is sharp.*

Proof. Let $H = D - (A_u^- \cup A_v - A_1 \cup \{(u,v)\})$, where $A_1 = \{(w,u) \in A(D) \mid w \in I(u) \cap I(v)\}$. If $\gamma(H) > \gamma(D)$, then $b(D) < d^-(u) + d(v) - |I(u) \cap I(v)|$. If $\gamma(H) = \gamma(D)$, then for every γ-set S of H, $v \notin S$. Otherwise, $S - \{v\}$ is a dominating set of D, which is impossible. Hence, $\gamma(D - (A_u^- \cup A_v - A_1)) = \gamma(H - \{(u,v)\}) = \gamma(H) + 1 > \gamma(D)$, and so $b(D) \leq d^-(u) + d(v) - |I(u) \cap$

$I(v)|$.

That the bound is sharp, may be seen by considering the bondage number of an orientation $\overrightarrow{K}_{1,n}$ of graph $K_{1,n}$ with center vertex u and $d^-(u) = n$. For every vertex $v \in V(\overrightarrow{K}_{1,n}) - \{u\}$, $d^-(v) = 0$. It is easily seen that $b(\overrightarrow{K}_{1,n}) = n = d^-(v) + d(u)$. □

Corollary 2.1 *If D is a digraph, then $b(D) \leq d(u) + d(v) - 1$ for every pair vertices u and v with $(u, v) \in A(D)$.*

Lemma 2.2 *If D is a digraph, then $b(D) \leq d(u) + d(v) - 1$ for every pair u and v of vertices with $I(u) \cap I(v) \neq \emptyset$.*

Proof. For some $w \in I(u) \cap I(v)$, let $H = D - (A_u \cup A_v - \{(w, u), (w, v)\})$. If $\gamma(H) > \gamma(D)$, then $b(D) \leq d(u) + d(v) - 2$. If $\gamma(H) = \gamma(D)$, then clearly $\gamma(H - \{(w, v)\}) = \gamma(H) + 1$, this implies the desired result. □

We recall some useful results that we will need.

Lemma 2.3 *If G is a bipartite planar graph with $n \geq 3$ vertices, then $|E(G)| \leq 2n - 4$.*

Lemma 2.4 *Suppose that G is a planar graph with $n \geq 3$ vertices, then $\delta(G) \leq 5$ and $|E(G)| \leq 3n - 6$.*

Lemma 2.5 (Dirac [3]) *If the minimum degree of graph G is at least 3, then G has a minor isomorphic to K_4.*

3 Bondage number of a symmetric planar digraph

In this section, we give an upper bound on bondage number for a symmetric planar digraph.

Theorem 3.1 *If D is a symmetric planar digraph, then $b(D) \leq 15$.*

Proof. Suppose to the contrary that D is a symmetric planar digraph with $b(D) \geq 16$. Now we define

$$\begin{aligned} V_1 &= \{u \in V(G) \mid d(u) \leq 8\} \\ V_2 &= \{u \in V(G) \mid d(u) = 10\} \end{aligned}$$

Then we have

Claim 1. For two distinct vertices u, v of V_1, $d_D(u, v) \geq 3$.

Otherwise, $d(u, v) \leq 2$. By Lemma 2.2, $b(D) \leq d(u) + d(v) - 1 = 8 + 8 - 1 = 15$, a contradiction.

Claim 2. For each $u \in V_1$, $v \in N(u)$, $d(v) \geq 14$.

Otherwise, $d(v) \leq 13$, by the symmetric of D, $d(v) \leq 12$. But then by Lemma 2.2 $b(D) \leq d^-(v) + d(u) \leq 6 + 8 = 14$, a contradiction.

Claim 3. V_2 is a independent set in D. For two distinct vertices $u, v \in V_2$, $d(u, v) \geq 2$.

Otherwise, by Lemma 2.1, $b(D) \leq d^-(u) + d(v) \leq 5 + 10 = 15$.

Let $V_2 = \{u_1, u_2, \ldots, u_k\}$ and $D_1 = D - V_1$. Define

$$\begin{aligned} H_0 &= D_1 \\ H_i &= H_{i-1} + F_i, \quad 1 \leq i \leq k. \end{aligned}$$

where $F_i \subseteq F_{u_i} = \{(u, v), (v, u) \mid u, v \in N(u_i), u \neq v, (u, v), (v, u) \notin A(H_{i-1})\}$ such that $H_{i-1} + F_i = H_i$ is still symmetric planar digraph and the underlying graph $H_i^*[N(u_i)]$ of $H_i[N(u_i)]$ is 2-connected.

Claim 4. If $V_2 \neq \emptyset$, then for each vertex $v \in N(V_2)$, v is of degree at least 14 in H_k.

In fact, let $u \in V_2$ and $v \in N(u)$. If, in D, $I(v) \cap I(u) = \emptyset$, by Lemma 2.1, $5 + d(v) \geq 16$, we have $d(v) \geq 12$. By the 2-connectivity of $H_k^*[N(x)]$, v is degree at least 14 in H_k. If $|I(v) \cap I(u)| = 1$, by Lemma 2.1, $5 + d(v) - 1 \geq 16$, we have $d(v) \geq 12$. By the 2-connectivity of $H_k^*[N(u)]$, v is degree at least 14 in H_k. If $|I(v) \cap I(u)| \geq 2$, by Lemma 2.1, $5 + d(v) - 2 \geq 16$, then $d(v) \geq 13$. Since D is a symmetric digraph, $d(v)$ is even, so $d(v) \geq 14$.

Now we consider underlying graph H_k^* of digraph H_k, H_k^* is a planar graph with the following properties.

(a) The minimum degree of H_k^* is 5,

(b) $V_2 = \{u \in V(H_k^*) \mid d_{H_k^*}(u) = 5\}$ and V_2 is independent in H_K^*,

(c) For every vertex $v \in N_{H_k^*}(V_2)$, $d_{H_k^*}(v) \geq 7$,

(d) For every vertex $v \in V(H_k^*) - (V_2 \cup N(V_2)), d_{H_k^*}(v) \geq 6$.

Let $\partial(V_2) = \{uv \in E(H_k^*) \mid u \in V_2, v \in N(V_2)\}$. Then $(V_2, N(V_2); \partial(V_2)\}$ is a bipartite planar graph with $5|V_2|$ edges. By Lemma 2.3,

$$5|V_2| \leq 2|V_2| + 2|N(V_2)| - 4.$$

Hence, $|N(V_2)| \geq 3|V_2|/2 + 2$. But

$$\begin{aligned}
|E(H_k^*)| &= \frac{1}{2} \sum_{v \in V(H_k^*)} d_{H_k^*}(v) \\
&\geq (5|V_2| + 7|N(V_2)| + 6(|V(H_k^*)| - |V_2| - |N(V_2)|))/2 \\
&= 3|V(H_k^*)| + |N(V_2)|/2 - |V_2|/2 \\
&\geq 3|V(H_k^*)| + |V_2|/4 + 1 \\
&> 3|V(H_k^*)| - 6,
\end{aligned}$$

contrary to Lemma 2.4. This completes the proof of the theorem. □

4 Bondage number of oriented graphs

In this section, we begin to turn our attention to oriented graphs. Our aim is to establish upper bounds on bondage number for oriented graphs.

Theorem 4.1 *If D is an oriented graph of order n, then $b(D) \leq \frac{3}{2}\Delta(D)$, and this bound is sharp.*

Proof. Let v be a vertex with $d^-(v) = \delta^-(D)$. If $d(v) = d^-(v)$, we choose a vertex $w \in I(v)$, by Lemma 2.1, it follows that $b(D) \leq d^-(w) + d(v) \leq (\Delta(D) - 1) + \delta^-(D) < \delta^-(D) + \Delta(D)$. If $d(v) > d^-(v)$, then there exists a vertex $u \in O(v)$, by Lemma 2.1 , we have $b(D) \leq d^-(v) + d(u) \leq \delta^-(D) + \Delta(D)$.

For a digraph D, it is well known that $\sum_{v\in V} d^-(v) = \sum_{v\in V} d^+(v) = \frac{1}{2}\sum_{v\in V} d(v)$. So $n\delta^-(D) \leq \frac{1}{2}\Delta(D)n$, thus $\delta^-(D) \leq \frac{1}{2}\Delta(D)$. Combining with $b(D) \leq \delta^-(D) + \Delta(D)$, we have $b(D) \leq 3\Delta(D)/2$.

That the bound is sharp, may be seen by checking the bondage number of a strong connected orientation $\vec{C}_{2n+1}$ of graph C_{2n+1}. Clearly, $b(\vec{C}_{2n+1}) = 3 = 3\Delta(\vec{C}_{2n+1})/2$. □

For an oriented planar graph, we can establish the following upper bound.

Theorem 4.2 *If D is an oriented planar graph of order n, then $b(D) \leq \Delta(D) + 2$.*

Proof. Let D^* be the underlying graph of D. For an oriented planar graph D, there must exist at least a vertex v with $d^-(v) \leq 2$. Otherwise, for every vertex $v \in V$, $d^-(v) \geq 3$, then $|E(D^*)| = |A(D)| = \sum_{v\in V} d^-(v) \geq 3n$, which contradicts Lemma 2.4. So, there exists a vertex v with $d^-(v) \leq 2$. If $O(v) \neq \emptyset$, then there exists a vertex $u \in O(v)$, by Lemma 2.1, we have $b(D) \leq d^-(v) + d(u) \leq 2 + \Delta(D)$. If $O(v) = \emptyset$, then $d(v) = d^-(v) \leq 2$. We choose a vertex $w \in I(v)$, then $b(D) \leq d^-(w) + d(v) \leq (\Delta(D) - 1) + 2 = \Delta(D) + 1$.

The above bound can be attained. Let D be a digraph with vertex set $V(D) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$ and arc set $A(D) = \{v_1v_3, v_3v_2, v_2v_1, v_1u_1, v_2u_1, u_3v_1, u_3u_2, u_1u_2, u_1u_3, v_3u_3, u_2v_3, u_2v_2\}$. Clearly, $\gamma(D) = 3$, and we can check that $b(D) = 6 = \Delta(D) + 2$. □

Furthermore, for some special oriented planar graphs, we can somewhat improve the upper bounds.

Theorem 4.3 *Let D be an oriented planar graph, and $V_1 = \{v \in V(D) \mid d(v) \leq 4\}$. If for any $u \in V - V_1$, there exists at most a vertex $v \in V_1$ such that $v \in O(u) \cup I(u)$, then $b(D) \leq 10$.*

Proof. Suppose to the contrary that $b(D) \geq 11$. Now we define $V_2 = \{v \in V(D) \mid d(v) = 5\}$. Then we have

Claim 1. For each $u \in N(V_1)$, $d(u) \geq 8$. Also, for each $u \in N(V_2)$, $d(u) \geq 7$.

Otherwise, by Corollary 2.1, $b(D) \leq d(u) + d(v) - 1 \leq 10$, a contradiction.

Claim 2. V_2 is an independent set in D.

Otherwise, if there exist vertices $u, v \in V(D)$ such that $(u, v) \in A(D)$. Then $b(D) \leq d^-(u) + d(v) \leq 9$, a contradiction.

Now we consider underlying graph D_1^* of $D_1 = D - V_1$. Then D_1^* is a planar graph with the following properties.

(a) The minimum degree of D_1^* is 5,

(b) $V_2 = \{v \in V(D_1^*) \mid d_{D_1^*}(v) = 5\}$,

(c) V_2 is an independent set in D_1^*,

(d) For every vertex $v \in N_{D_1^*}(V_2), d_{D_1^*}(v) \geq 7$.

Let $\partial(V_2) = \{uv \in E(D_1^*) \mid u \in V_2, v \in N(V_2)\}$. Then $(V_2, N(V_2); \partial(V_2))$ is a bipartite planar graph with $5|V_2|$ edges. As discussed in Theorem 3.1, we have

$$|E(D_1^*)| > 3|V(D_1^*)| - 6,$$

contrary to Lemma 2.4. This completes the proof of the theorem. □

Theorem 4.4 *Let D be an oriented planar graph with $\Delta(D) \geq 5$. If for every vertex v with $d(v) \geq 4$, it has $d^-(v) \geq 3$, then $b(D) \leq \Delta(D) + 1$.*

Proof. Let D be an oriented planar graph satisfying the hypothesis, then there exists a vertex $v \in V(D)$ such that $\delta^-(D) = d^-(v) \leq 2$.

Case 1. $\delta^-(D) = d^-(v) \leq 1$. If $O(v) \neq \emptyset$, then there exists a vertex $u \in O(v)$. By Lemma 2.1, we have $b(D) \leq d^-(v) + d(u) \leq \Delta(D) + 1$. If $O(v) = \emptyset$, then $d(v) = d^-(v) \leq 1$. We choose a vertex $w \in I(v)$, then $b(D) \leq d^-(w) + d(v) \leq (\Delta(D) - 1) + 1 = \Delta(D)$.

Case 2. $\delta^-(D) = 2$, and $\delta(D) = 2$. Let u be a vertex with $d(u) = \delta(D) = 2$, then $d^-(u) = 2$. We choose a vertex $w \in I(u)$, then $b(D) \leq d^-(w) + d(u) \leq (\Delta(D) - 1) + 2 = \Delta(D) + 1$.

Case 3. $\delta^-(D) = 2$, $\delta(D) \geq 3$. For every vertex $v \in V(D)$ with $d^-(v) = 2$, it has $d(v) \geq 3$. By the hypothesis of theorem we know that $d(v) = 3$. Let $B = \{v \in V(D) \mid d^-(v) = 2, d(v) = 3\}$. Suppose to the contrary that $b(D) \geq \Delta(D) + 2$, then for every vertex $u \in I(B)$, $d^-(u) = \Delta(D) - 1$. Otherwise, $d^-(u) \leq \Delta(D) - 2$, then $b(D) \leq d^-(u) + d(v) \leq (\Delta(D) - 2) + 3 = \Delta(D) + 1$, a contradiction. So for each $u \in I(B)$, we have $d^-(u) = \Delta(D) - 1 \geq 4$. Let $B' = I(B)$. Observe that for any two distinct vertices $v_1, v_2 \in B$, there exist two distinct vertices $u_1, u_2 \in B'$ such that $(u_1, v_1), (u_2, v_2) \in A(D)$. This implies that $|B'| \geq |B|$. Notice that the indegree of each vertex in $V - (B \cup B')$ is no less than 3. Let D^* be the underlying graph of D. We immediately obtain

$$
\begin{aligned}
|E(D^*)| = |A(D)| &= \sum_{v \in V} d^-(v) \\
&= \sum_{v \in B} d^-(v) + \sum_{v \in B'} d^-(v) + \sum_{v \in V-(B \cup B')} d^-(v) \\
&\geq (2+4)|B| + 3(n - 2|B|) \\
&\geq 3n,
\end{aligned}
$$

contrary to Lemma 2.4. The theorem follows. □

For an oriented tree, we obtain the following result.

Theorem 4.5 *If T is an oriented tree, then $b(T) \leq \Delta(T)$, and this bound is sharp.*

Proof. Let T be an oriented tree. Then there exists a vertex u such that $d(u) = 1$. If $d^-(u) = 0$, there exists a vertex $v \in O(u)$, then $b(T) \leq d^-(u) + d(v) \leq \Delta(T)$. If $d^-(u) = d(u) = 1$, there exists a vertex $v \in I(u)$, then $b(T) \leq d^-(v) + d(u) = d^-(v) + 1 \leq d(v) \leq \Delta(T)$.

That the bound is sharp, may be seen by considering the orientation $\vec{K}_{1,n}$ of graph $K_{1,n}$ with center vertex u and $d^-(u) = n$. It is easily checking that $b(\vec{K}_{1,n}) = n = \Delta(\vec{K}_{1,n})$. □

Theorem 4.6 *Let D be an oriented graph with underlying graph D^*, and $V_1 = \{v \in V(D) \mid d(v) = 1\}, V_2 = \{v \in V(D) \mid d(v) = 2\}$. If D^* has no minor isomorphic to K_4 and for every $u \in V(D) - (V_1 \cup V_2)$, there exists at most a vertex $v \in V_1 \cup V_2$ such that $uv \in E(D^*)$, then $b(D) \leq 4$.*

Proof. Let D be an oriented graph satisfying the hypothesis. Assume to the contrary that $b(D) \geq 5$. We can deduce that if $V_1 \neq \emptyset$ then $d(u) \geq 5$ for every $u \in N(V_1)$. Otherwise, if there exists a vertex v with $d(v) = 1$, and $d(u) \leq 4$ for $u \in N(v)$. By Corollary 2.1, $b(D) \leq d(u) + d(v) - 1 \leq 4$, contradicting our assumption. Now if $V_2 = \emptyset$, then $D^* - V_1$ is a graph with minimum degree at least 3. By Lemma 2.5, D^* has a minor isomorphic to K_4, a contradiction. Hence, we can suppose that $V_2 \neq \emptyset$. Let $V_2 = \{u_1, u_2, \ldots, u_k\}$. Clearly, V_2 is an independent set in D. Suppose $N(u_i) = \{x_i, y_i\}$, $1 \leq i \leq k$. Define $D_0 = D^*$ and for $1 \leq i \leq k$,

$$D_i = \begin{cases} D_{i-1} - u_i + x_i y_i, & \text{if } x_i y_i \notin E(D^*) \\ D_{i-1} - u_i & \text{if } x_i y_i \in E(D^*) \end{cases}$$

Since D^* has no minor isomorphic to K_4, $D_k - V_1$ also has no minor isomorphic to K_4, and so $D_k - V_1$ has a vertex of degree at most 2. By the assumption of the theorem and the structure of D_k, there must exist $r \in \{1, 2, \ldots, k\}$ such that $x_r y_r \in E(D^*)$ and $\min\{d(x_r), d(y_r)\} = 3$. Suppose $d(x_r) = 3$, without loss generality we assume that $(u_r, x_r) \in A(D)$, then $b(D) \leq d^-(u_r) + d(x_r) \leq 1 + 3 = 4$, a contradiction. The theorem follows. □

Theorem 4.7 *Let D be an oriented digraph of order $n \geq 4$. If $\gamma(D) \leq 2$, then $b(D) \leq \Delta(D)$.*

Proof. If $\gamma(D) = 1$, then there exists a vertex v such that $d^+(v) = n - 1$ and for any vertex $w \in V - \{v\}$, it must be $d^+(w) \leq n - 2$. Hence $\gamma(D - (v, w)) \geq 2 > \gamma(D)$, and so $b(D) = 1$.

If $\gamma(D) = 2$, suppose to the contrary that $b(D) \geq \Delta(D) + 1$. Let u be the vertex such that $d^+(u) = \Delta^+(D)$, then $\gamma(D - A_u) = \gamma(D)$. This implies that there exists a vertex v such that $S_1 = \{u, v\}$ is a dominating set of $D - A_u$, and so $d^+(v) = n - 2$.

Case 1. $(u, v) \notin A(D)$. Since $d^+(u) \geq d^+(v)$, $I(u) = \emptyset$. We choose a vertex $z \in O(u) \cap O(v)$, then $\gamma(D - A_z) = \gamma(D)$. That is, there exists a vertex z_1 such that $S_2 = \{z, z_1\}$ is a dominating set of $D - A_z$, $z_1 \neq u, v$, and z_1 is adjacent to all vertices except z. But $I(u) = \emptyset$, this is a contradiction.

Case 2. $(u, v) \in A(D)$. Since $d^+(u) \geq d^+(v)$, then $d^-(u) \leq 1$. If $d^-(u) = 0$, as discussed in Case 1, a contradiction will be yielded. If $d^-(u) = 1$, there exists a unique vertex $w \in I(u)$. We choose a vertex $x \in O(u) \cap O(v)$, then $\gamma(D - A_x) = \gamma(D)$. That is , there exists a vertex $x_1 \neq u$ such that $S_3 = \{x, x_1\}$ is a dominating set of $D - A_x$, and x_1 is adjacent to all vertices except x. Since $u \notin S_3$, in order to dominate u, it must be the case that $x_1 = w$. But $(w, v), (x, v) \notin A(D)$, this implies that v can't be dominated by S_3, a contradiction. So, $b(D) \leq \Delta(D)$. □

When $\gamma(D) = 3$, $b(D) \leq \Delta(D)$ is not necessarily correct. This can be seen from the bondage number of strong connected orientation $\overrightarrow{C_5}$ of the 5-cycle C_5, $b(\overrightarrow{C_5}) = 3 = \Delta(\overrightarrow{C_5}) + 1$.

A digraph D is transitive if whenever (u, v) and (v, w) are arcs of D, then (u, w) is also an arc of D. The tournaments have been received the greatest attention in oriented graphs. The following result gives a bondage number of a transitive tournament.

Theorem 4.8 *If a tournament T is transitive, then $b(T) = 1$.*

Proof. Assume that T is a transitive tournament of order n. Let $u, v \in V(T)$ and assume that $(u, v) \in A(T)$. For each $w \in O(v)$, since $(v, w) \in A(T)$ and $(u, v) \in A(T)$, it follows that $(u, w) \in A(T)$. Thus $d^+(u) \geq$

$d^+(v)+1$. This implies that no two vertices of T have the same outerdegree, so $\Delta^+(T) = n-1$ and $\gamma(T) = 1$. Let z be the vertex with $d^+(z) = \Delta^+(T) = n-1$. For every other vertex $w \in V - \{z\}$, it must be $d^+(w) \leq n-2$. Hence, $\gamma(T - \{(z, w)\}) \geq 2 > \gamma(T) = 1$. So, $b(T) = 1$. □

5 An upper on 2-domination number of a digraph

Theorem 5.1 ([1]) *A loopless digraph D has an independent set S such that each vertex of D not in S is reachable from a vertex in S by a directed path of length at most two.*

Theorem 5.2 *If D is a digraph of order n and $\delta^-(D) \geq 1$, then $\gamma_2(D) \leq \frac{n}{2}$.*

Proof. Let S be a maximal independent set of vertices of D. Since $\delta^-(D) \geq 1$, then for every $v \in S$, there is a vertex $u \in V - S$ such that $(u, v) \in A(D)$. This means that $V - S$ is a dominating set of D. Hence, $\gamma_2(D) + \beta_0(D) \leq n$. By Theorem 5.2, there exists a 2-independent dominating set for digraph D, so $\gamma_2(D) \leq i_2(D) \leq \beta_0(D)$. Consequently, $\gamma_2(D) \leq n/2$. □

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Elsevier, Amsterdam, 1976.

[2] G. Chartrand, F. Harary and B. Quan Yue, On the out-domination and in-domination number of a digraph, *Discrete Math.* 197-198 (1999), 179-183.

[3] G.A. Dirac, In abstraken graphen vorhandene vollstandige 4-graphen und ihre untereilungen, *Math. Nath.* 22 (1960), 61-85.

[4] J.F. Fink, M.S. Jacobson, L.F. Kinch, J. Roberts, The bondage number of a graph, *Discrete Math.* 86 (1990), 47-57.

[5] B.L. Hartnell, D.F. Rall, Bounds on the bondage number of a graph, *Discrete Math.* 128 (1994), 173-177.

[6] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, 1998.

[7] L.Y. Kang and J.J. Yuan, Bondage number of planar graphs, *Discrete Math.* 222 (2000), 191-198.

[8] U. Tescher, A counterexample to a conjecture on the bondage number of a graph, *Discrete Math.* 122 (1993), 393-395.

[9] Y.L. Wang, On the bondage number of a graph, *Discrete Math.* 159 (1996), 291-294.