On Multicolor Ramsey Numbers for Even Cycles in Graphs *

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Abstract

The multicolor Ramsey number $R_r(H)$ is defined to be the smallest integer $n = n(r)$ with the property that any r-coloring of the edges of complete graph K_n must result in a monochromatic subgraph of K_n isomorphic to H. In this paper, we study the case that H is a cycle of length $2k$. If $2k \geq r+1$ and r is a prime power, we show that $R_r(C_{2k}) > r^2 + 2k - r - 1$.

Keywords: *multicolor Ramsey number; cycle; Galois field; latin square*

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph G with vertex set $V(G)$ and edge set $E(G)$, we denote the order and the size of G by $p(G) = |V(G)|$ and $q(G) = |E(G)|$, respectively.

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If there exists a r-coloring of the edges of a graph G such that there is no monochromatic subgraph of G isomorphic to H, we say that H is *r-avoidable in G*. The multicolor Ramsey number $R_r(H)$ is the smallest integer n such that H is not r-avoidable in K_n.

A *cutpoint* of a graph is one whose removal increases the number of components. A *nonseparable* graph is connected, nontrivial, and has no cutpoints. A *block* of a graph is a maximal nonseparable subgraph. Let K_n be a complete graph with order n, $K_{n,m}$ be a complete n by m bipartite graph. C_m is a cycle of length m. $\langle G; S\rangle$ denotes the induced subgraph of vertex set $S \subseteq V(G)$.

Chung [2] showed that

$$R_r(K_3) \geq 3R_{r-1}(K_3) + R_{r-3}(K_3) - 3.$$

Chung, Graham[3] and Ivring[8] independently proved that

$$R_r(C_4) \geq r^2 - r + 2$$

for $r-1$ being a prime power, and

$$R_r(C_4) \leq r^2 + r + 1.$$

Lazebnik and Woldar[9] gave that

$$R_r(C_4) \geq r^2 + 2$$

for odd prime power r, and the result was extended to any prime power r in [10, 11].

Bondy and Erdös[1] obtained that

$$R_2(C_{2k+1}) = 4k + 1,$$

and conjectured that

$$R_3(C_{2k+1}) = 8k + 1, \quad k \geq 2.$$

Faudree and Schelp [6] determined that

$$R_2(C_{2k}) = 3k - 1.$$

Graham, Rothschild and Spencer[7] gave that

$$2^r k < R_r(C_{2k+1}) < 2(r+2)!\, k,$$

$$R_r(C_{2k}) > (r-1)(k-1),$$
$$R_r(C_{2k}) \leq 201rk, \quad r \leq 10^k/201k.$$

Dzido, Nowik and Szuca[5] proved that

$$R_r(C_{2k}) > (r+1)k + (r \bmod 2) - 2. \tag{1.1}$$

In [13], it was shown that

$$R_r(C_{2k}) > 2(r-1)(k-1)+1. \tag{1.2}$$

For the literature on small Ramsey numbers we refer to [12] and the relevant references given in it.

In this paper, we study the case that H is an even cycle C_{2k} such that $2k \geq r+1$ and r is a prime power, and prove that

$$R_r(C_{2k}) > r^2 + 2k - r - 1. \tag{1.3}$$

In order to accomplish this, we first describe a special C_m-free graph F_r with order r^2 and size $r(r^2-1)/2$, where r is a prime power and $m = r+1$. We show how to color the edges of the complete graph K_{r^2} in r colors such that each monochromatic subgraph is isomorphic to F_r. Such colorings can be viewed as edge decompositions of K_{r^2} into isomorphic copies of F_r. Clearly, the existence of such colorings implies that $R_r(C_m) > r^2$. Then we show that the edge coloring of K_{r^2} can be extended to r-coloring of the edges of $K_{r^2+m-r-1}$ such that each monochromatic subgraph is C_m-free when $m > r+1$. Taking $m = 2k$, this will prove inequality (1.3).

2 Construction and Proofs

The graph F_r with order r^2 and size $r(r^2-1)/2$ is defined as follows:

$$\begin{aligned} V(F_r) &= \{v_{ij} : 1 \leq i, j \leq r\}, \\ E(F_r) &= \{(v_{ij}, v_{i'j}) : 1 \leq i < i' \leq r, 1 \leq j \leq r\} \cup \\ &\quad \{(v_{1j}, v_{1j'}) : 1 \leq j < j' \leq r\}. \end{aligned}$$

Then F_r consists of $r+1$ blocks with order r, hence, for any cycle C_m for $m \geq r+1$, we have $C_m \nsubseteq F_r$.(see Fig. 1-2, where $F_3 = G_{3,1}$ and $F_4 = G_{4,1}$ respectively).

Lemma 1. Let r be a prime power. If $m = r+1$, then C_m is r-avoidable in K_{r^2}.

Proof. Let

$$V(K_{r^2}) = \{v_{ij} : 1 \leq i, j \leq r\}.$$

Since r is a prime power, there must exist a complete set of mutually orthogonal latin squares $\{L_2, L_3, \ldots, L_r\}$ with elements $1, 2, \ldots, r$ (see [4, p. 167, Theorem 5.2.4]). Let

$$L_t = \begin{bmatrix} L^t_{11} & L^t_{12} & \cdots & L^t_{1r} \\ L^t_{21} & L^t_{22} & \cdots & L^t_{2r} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ L^t_{r1} & L^t_{r2} & \cdots & L^t_{rr} \end{bmatrix}, \quad t = 1, 2, \ldots, r,$$

where $L^t_{1j} = j$ for $1 \le j \le r$ and $L^1_{ij} = j$ for $2 \le i \le r$, $1 \le j \le r$, namely,

$$L_1 = \begin{bmatrix} 1 & 2 & \cdots & r \\ 1 & 2 & \cdots & r \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & 2 & \cdots & r \end{bmatrix}.$$

Then the sets $S_{i,j}$ of vertices are defined according to the following rule:

$$\begin{aligned} S_{i,j} &= \{v_{1j}, v_{2L^i_{2j}}, \ldots, v_{rL^i_{rj}}\}, \quad && 1 \le i \le r,\ 1 \le j \le r, \\ S_{i,r+1} &= \{v_{i1}, v_{i2}, \ldots, v_{ir}\}, && 1 \le i \le r. \end{aligned} \tag{2.1}$$

Let

$$\begin{aligned} B_{i,j} &= \langle K_{r^2};\ S_{i,j} \rangle, \quad && 1 \le i \le r,\ 1 \le j \le r+1, \\ E_1 &= \textstyle\bigcup_{1 \le i,j \le r} E(B_{i,j}), \\ E_2 &= \textstyle\bigcup_{1 \le i \le r} E(B_{i,r+1}). \end{aligned}$$

Since $L_2, L_3, \ldots, L_r$ are mutually orthogonal latin squares, it follows that

$$\begin{aligned} E(K_{r^2}) &= E_1 \cup E_2, \\ E_1 \cap E_2 &= \emptyset, \\ E(B_{i,j}) \cap E(B_{i',j'}) &= \emptyset, \quad && 1 \le i, i' \le r,\ 1 \le j, j' \le r, \\ & && (i,j) \ne (i',j'), \\ E(B_{i,r+1}) \cap E(B_{i',r+1}) &= \emptyset, && 1 \le i < i' \le r. \end{aligned}$$

Hence

$$\Big(\bigcup_{j=1}^{r+1} E(B_{i,j})\Big) \cap \Big(\bigcup_{j=1}^{r+1} E(B_{i',j})\Big) = \emptyset, \qquad 1 \le i < i' \le r.$$

Let

$$E(G_{r,i}) = \bigcup_{j=1}^{r+1} E(B_{i,j}), \tag{2.2}$$

then

$$\begin{aligned} G_{r,i} &\cong F_r, \quad 1 \le i \le r, \\ E(G_{r,i}) \cap E(G_{r,i'}) &= \emptyset, \quad 1 \le i < i' \le r. \end{aligned}$$

Since $q(G_{r,i}) = r(r^2-1)/2$ for $1 \le i \le r$, it follows that

$$E(K_{r^2}) = \bigcup_{i=1}^{r} E(G_{r,i}).$$

Hence, the edges of K_{r^2} can be decomposed into r isomorphic copies of F_r. We can color the edges of $E(K_{r^2})$ in r colors as follows: all the edges of $G_{r,i}$ will be in the i-th color for $1 \le i \le r$. So, C_m is r-avoidable in K_{r^2}. □

We use two examples, taking $r = 3$ and 4 in Lemma 1, to illustrate the construction. Assume that $r = 3$, by Lemma 1, we have

$$L_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}.$$

Then the sets $S_{i,j}$ for $1 \le i \le 3$ and $1 \le j \le 4$ are given as follows:

$$\begin{aligned} S_{1,1} &= \{v_{11}, v_{21}, v_{31}\},\ S_{1,2} = \{v_{12}, v_{22}, v_{32}\}, \\ S_{1,3} &= \{v_{13}, v_{23}, v_{33}\},\ S_{1,4} = \{v_{11}, v_{12}, v_{13}\}. \end{aligned}$$

$$\begin{aligned} S_{2,1} &= \{v_{11}, v_{22}, v_{33}\},\ S_{2,2} = \{v_{12}, v_{23}, v_{31}\}, \\ S_{2,3} &= \{v_{13}, v_{21}, v_{32}\},\ S_{2,4} = \{v_{21}, v_{22}, v_{23}\}. \end{aligned}$$

$$\begin{aligned} S_{3,1} &= \{v_{11}, v_{23}, v_{32}\},\ S_{3,2} = \{v_{12}, v_{21}, v_{33}\}, \\ S_{3,3} &= \{v_{13}, v_{22}, v_{31}\},\ S_{3,4} = \{v_{31}, v_{32}, v_{33}\}. \end{aligned}$$

Hence, the 3-coloring of the edges of K_9 is shown in Fig. 1, where $G_{3,i}$ denotes the subgraph of K_9 whose edges are all in the i-th color for $1 \le i \le 3$.

Assume that $r = 4$, by Lemma 1, we have

$$L_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}, L_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix}, L_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}.$$

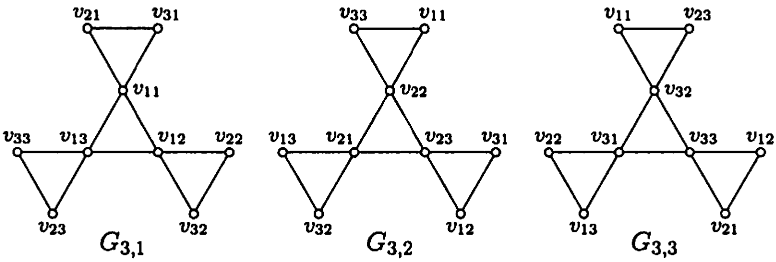

Fig. 1. A 3-coloring of the edges of K_9

Then the sets $S_{i,j}$ for $1 \leq i \leq 4$ and $1 \leq j \leq 5$ are given as follows:

$$S_{1,1} = \{v_{11}, v_{21}, v_{31}, v_{41}\}, \quad S_{1,2} = \{v_{12}, v_{22}, v_{32}, v_{42}\},$$
$$S_{1,3} = \{v_{13}, v_{23}, v_{33}, v_{43}\}, \quad S_{1,4} = \{v_{14}, v_{24}, v_{34}, v_{44}\},$$
$$S_{1,5} = \{v_{11}, v_{12}, v_{13}, v_{14}\}.$$

$$S_{2,1} = \{v_{11}, v_{22}, v_{33}, v_{44}\}, \quad S_{2,2} = \{v_{12}, v_{21}, v_{34}, v_{43}\},$$
$$S_{2,3} = \{v_{13}, v_{24}, v_{31}, v_{42}\}, \quad S_{2,4} = \{v_{14}, v_{23}, v_{32}, v_{41}\},$$
$$S_{2,5} = \{v_{21}, v_{22}, v_{23}, v_{24}\}.$$

$$S_{3,1} = \{v_{11}, v_{23}, v_{34}, v_{42}\}, \quad S_{3,2} = \{v_{12}, v_{24}, v_{33}, v_{41}\},$$
$$S_{3,3} = \{v_{13}, v_{21}, v_{32}, v_{44}\}, \quad S_{3,4} = \{v_{14}, v_{22}, v_{31}, v_{43}\},$$
$$S_{3,5} = \{v_{31}, v_{32}, v_{33}, v_{34}\}.$$

$$S_{4,1} = \{v_{11}, v_{24}, v_{32}, v_{43}\}, \quad S_{4,2} = \{v_{12}, v_{23}, v_{31}, v_{44}\},$$
$$S_{4,3} = \{v_{13}, v_{22}, v_{34}, v_{41}\}, \quad S_{4,4} = \{v_{14}, v_{21}, v_{33}, v_{42}\},$$
$$S_{4,5} = \{v_{41}, v_{42}, v_{43}, v_{44}\}.$$

Hence, the 4-coloring of the edges of K_{16} is shown in Fig. 2, where $G_{4,i}$ denotes the subgraph of K_{16} whose edges are all in the i-th color for $1 \leq i \leq 4$.

The above coloring way can be extended to r-coloring of the edges of $K_{r^2+m-r-1}$ by the following lemma.

Lemma 2. Let r be a prime power. If $m > r+1$, then C_m is r-avoidable in $K_{r^2+m-r-1}$.

Proof. Suppose that the vertices of $K_{r^2+m-r-1}$ are ordered: $u_1, u_2, \ldots, u_{m-r-1}, v_{11}, v_{12}, \ldots, v_{1r}, \ldots, v_{r1}, v_{r2}, \ldots, v_{rr}$. Let G_X be the induced subgraph of the first $m-r-1$ vertices of $K_{r^2+m-r-1}$, G_Y be the induced subgraph of the remaining r^2 vertices. And let G_Y be the edge-disjoint

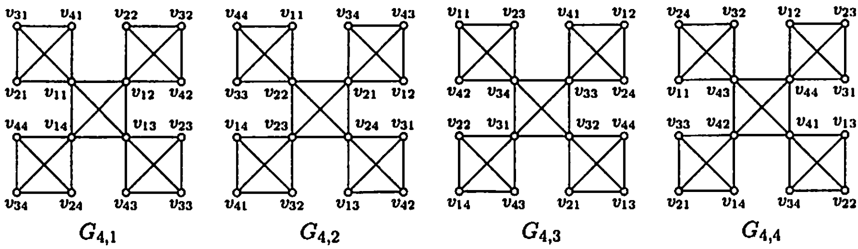

Fig. 2. A 4-coloring of the edges of K_{16}

union of $G_{Y_1}, G_{Y_2}, \ldots, G_{Y_r}$, where $G_{Y_i} = G_{r,i}$ defined as (2.2), i.e.,

$$\begin{aligned} V(G_X) &= \{u_i : 1 \le i \le m - r - 1\}, \\ V(G_Y) &= \{v_{ij} : 1 \le i, j \le r\}, \\ E(G_X) &= \{u_i u_j : 1 \le i < j \le m - r - 1\}, \\ E(G_{Y_i}) &= E(G_{r,i}), \qquad 1 \le i \le r, \\ E(G_Y) &= \textstyle\bigcup_{i=1}^{r} E(G_{Y_i}). \end{aligned}$$

Then

$$\begin{aligned} E(G_X) \cap E(G_Y) &= \emptyset, \\ E(G_{Y_i}) \cap E(G_{Y_j}) &= \emptyset, \qquad 1 \le i < j \le r. \end{aligned}$$

Let G_{XY} be the complete bipartite graph with order $r^2 + m - r - 1$, $V(G_X)$ and $V(G_Y)$ be its two parts respectively. And let G_{XY} be the edge-disjoint union of $G_{XY_1}, G_{XY_2}, \ldots, G_{XY_r}$, where G_{XY_i} is isomorphic to $K_{m-r-1,r}$, i.e.,

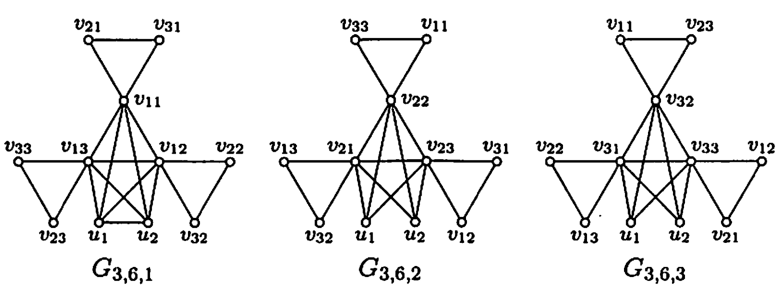

Fig. 3. A 3-coloring of the edges of K_{11}

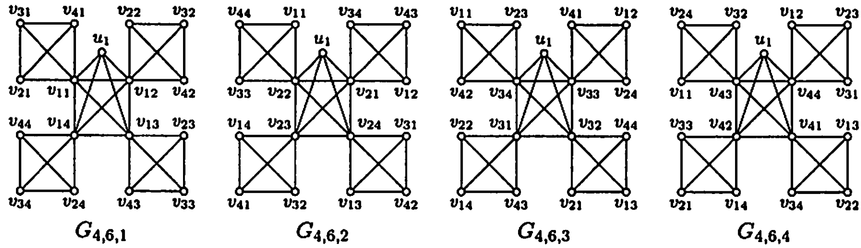

Fig. 4. A 4-coloring of the edges of K_{17}

$$\begin{aligned} V(G_{XY}) &= V(G_X) \cup V(G_Y) \\ &= \{u_i:\ 1 \le i \le m-r-1\} \cup \{v_{jk}:\ 1 \le j,\ k \le r\}, \\ E(G_{XY_i}) &= \{u_j v_{ik}:\ v_{ik} \in S_{i,r+1},\ 1 \le j \le m-r-1,\ 1 \le k \le r\}, \\ & \quad 1 \le i \le r, \\ E(G_{XY}) &= \bigcup_{i=1}^{r} E(G_{XY_i}), \end{aligned}$$

where $S_{i,r+1}$ are defined as (2.1). Then we have

$$\begin{aligned} E(K_{r^2+m-r-1}) &= E(G_X) \cup E(G_{XY}) \cup E(G_Y), \\ E(G_{XY_i}) \cap E(G_{XY_j}) &= \emptyset, \qquad 1 \le i < j \le r, \\ E(G_X) \cap E(G_{XY}) &= \emptyset, \\ E(G_Y) \cap E(G_{XY}) &= \emptyset. \end{aligned}$$

Let

$$E(G_{r,m,i}) = \begin{cases} E(G_X) \cup E(G_{XY_i}) \cup E(G_{Y_i}), & i = 1, \\ E(G_{XY_i}) \cup E(G_{Y_i}), & 2 \le i \le r. \end{cases}$$

Then

$$\begin{aligned} E(G_{r,m,i}) \cap E(G_{r,m,j}) &= \emptyset, \qquad 1 \le i < j \le r, \\ E(K_{r^2+m-r-1}) &= \bigcup_{i=1}^{r} E(G_{r,m,i}). \end{aligned}$$

We can color the edges of $E(K_{r^2+m-r-1})$ in r colors as follows: all the edges of $G_{r,m,i}$ will be in the i-th color for $1 \le i \le r$. Now we consider the graph $G_{r,m,i}$. Every $G_{r,m,i}$ is composed of $r+1$ blocks. Notice that the center block has $m-1$ vertices and the other blocks have r vertices. Since

$m > r+1$, it follows that $G_{r,m,i}$ is composed of $r+1$ blocks which have the order at most $m-1$ for $1 \leq i \leq r$. Hence,

$$C_m \nsubseteq G_{r,m,i}, \quad 1 \leq i \leq r.$$

We conclude that C_m is r-avoidable in $K_{r^2+m-r-1}$ for $m > r+1$. □

Taking $r = 3$ and $m = 6$ in Lemma 2, we may have C_6 is 3-avoidable in K_{11}. And taking $r = 4$ and $m = 6$ in Lemma 2, we may have C_6 is 4-avoidable in K_{17}. Their r-colorings are shown in Fig. 3 and Fig. 4 respectively, where $G_{r,m,i}$ denotes the subgraph of $K_{r^2+m-r-1}$ whose edges are all in the i-th color for $1 \leq i \leq r$.

3 Conclusion

Taking $m = 2k$ in Lemma 1 and 2, we have

Theorem 1. For a prime power r, if $2k \geq r+1$, then $R_r(C_{2k}) > r^2 + 2k - r - 1$.

So, for a prime power r, when $(r+1)/2 \leq k < r + (1-(r \bmod 2))/(r-1)$, the results of inequality (1.3) are better than the ones of inequality (1.1), and are better than the ones of inequality (1.2) when $r+1 \leq 2k < r + 2/(r-2)+3$. The comparison between the inequalities (1.1) and (1.3) is shown in Tab. 1, and the comparison between the inequalities (1.2) and (1.3) is shown in Tab. 2, where r is a prime power and $2k \geq r+1$.

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Tab. 1. The comparison between the inequalities (1.1) and (1.3)

r	$2k$	$R_r(C_{2k}) >$ $(r+1)k + (r \bmod 2) - 2$	$R_r(C_{2k}) > r^2 + 2k - r - 1$
2	4	$R_2(C_4) > 4$	$R_2(C_4) > 5$
2	6	$R_2(C_6) > 7$	$R_2(C_6) > 7$
2	8	$R_2(C_8) > 10$	$R_2(C_8) > 9$
2	10	$R_2(C_{10}) > 13$	$R_2(C_{10}) > 11$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	4	$R_3(C_4) > 7$	$R_3(C_4) > 9$
3	6	$R_3(C_6) > 11$	$R_3(C_6) > 11$
3	8	$R_3(C_8) > 15$	$R_3(C_8) > 13$
3	10	$R_3(C_{10}) > 19$	$R_3(C_{10}) > 15$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	6	$R_4(C_6) > 13$	$R_4(C_6) > 17$
4	8	$R_4(C_8) > 18$	$R_4(C_8) > 19$
4	10	$R_4(C_{10}) > 23$	$R_4(C_{10}) > 21$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	6	$R_5(C_6) > 17$	$R_5(C_6) > 25$
5	8	$R_5(C_8) > 23$	$R_5(C_8) > 27$
5	10	$R_5(C_{10}) > 29$	$R_5(C_{10}) > 29$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

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Tab. 2. The comparison between the inequalities (1.2) and (1.3)

r	$2k$	$R_r(C_{2k}) >$ $2(r-1)(k-1)+1$	$R_r(C_{2k}) > r^2 + 2k - r - 1$
2	4	$R_2(C_4) > 3$	$R_2(C_4) > 5$
2	6	$R_2(C_6) > 5$	$R_2(C_6) > 7$
2	8	$R_2(C_8) > 7$	$R_2(C_8) > 9$
2	10	$R_2(C_{10}) > 9$	$R_2(C_{10}) > 11$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	4	$R_3(C_4) > 5$	$R_3(C_4) > 9$
3	6	$R_3(C_6) > 9$	$R_3(C_6) > 11$
3	8	$R_3(C_8) > 13$	$R_3(C_8) > 13$
3	10	$R_3(C_{10}) > 17$	$R_3(C_{10}) > 15$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	6	$R_4(C_6) > 13$	$R_4(C_6) > 17$
4	8	$R_4(C_8) > 19$	$R_4(C_8) > 19$
4	10	$R_4(C_{10}) > 25$	$R_4(C_{10}) > 21$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	6	$R_5(C_6) > 17$	$R_5(C_6) > 25$
5	8	$R_5(C_8) > 25$	$R_5(C_8) > 27$
5	10	$R_5(C_{10}) > 33$	$R_5(C_{10}) > 29$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

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