Minimal Universal Bipartite Graphs

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Abstract

A graph U is ***(induced)-universal*** for a class of graphs X if every member of X is contained in U as an induced subgraph. We study the problem of finding universal graphs with minimum number of vertices for various classes of bipartite graphs: exponential classes, bipartite chain graphs, bipartite permutation graphs, and general bipartite graphs. For exponential classes and general bipartite graphs we present a construction which is asymptotically optimal, while for the other classes our solutions are optimal in order.

Keywords: Universal graph; Hereditary class of graphs

1 Introduction

Denote by Γ_n the class of all simple (undirected, without loops and multiple edges) graphs with vertex set $\{1, 2, \ldots, n\}$ and let $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$. Given a graph $G \in \Gamma$, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. Also, $|G| = |V(G)|$ is the order of G. If W is a subset of vertices of G, then $G[W]$ is the subgraph of G induced by W, i.e. the subgraph of G with the vertex set W and two vertices being adjacent if and only if they are adjacent in G. If a graph H is isomorphic to an induced subgraph of G, we say that H is embeddable into G. As usual, we denote by K_n the complete graph on n vertices and by $K_{n,m}$ the complete bipartite graph with parts of size n and m.

By $\mathcal{E}_{i,j}$ we denote the class of graphs whose vertices can be partitioned into at most i independent sets and j cliques. In particular, $\mathcal{E}_{2,0}$ is the class of bipartite graphs, $\mathcal{E}_{0,2}$ is the class of co-bipartite graphs, and $\mathcal{E}_{1,1}$ is the class of split graphs [12].

A class of graphs $X \subseteq \Gamma$ is called *hereditary* if $G \in X$ implies $H \in X$ for every graph H isomorphic to an induced subgraph of G. Let us denote $X_n = X \cap \Gamma_n$.

For an arbitrary hereditary class X, a graph UX_n is called an n-universal X-graph if every graph in X_n is isomorphic to an induced subgraph of UX_n. From obvious cardinality arguments, we have

$$\log_2 |X_n| \leq n \log_2 |UX_n|.$$

Also, we trivially have

$$n \log_2 n \leq n \log_2 |UX_n|.$$

A sequence of universal X-graphs $\{UX_n,\ n = 1, 2, \ldots\}$ will be called *asymptotically optimal* if

$$\lim_{n\to\infty} \frac{n \log_2 |UX_n|}{\max(\log_2 |X_n|, n \log_2 n)} = 1.$$

and *optimal in order* (*order-optimal*) if there is a constant c such that for any $n \geq 1$,

$$\frac{n \log_2 |UX_n|}{\max(\log_2 |X_n|, n \log_2 n)} \leq c.$$

In the present paper we construct optimal universal graphs for several families of bipartite graphs, such as general bipartite graphs, bipartite permutation graphs as well as some specific families defined in the next section.

2 Preliminaries

It has been proven in [2] that for any infinite hereditary class X different from the class of all graphs,

$$\lim_{n\to\infty} \frac{\log_2 |X_n|}{\binom{n}{2}} = 1 - \frac{1}{k(X)}, \tag{1}$$

where $k(X)$ is a natural number called *index* of the class X (this result can also be found in [7]). The index $k(X)$ of a class X is the maximum k such that X contains a class $\mathcal{E}_{i,j}$ with $i + j = k$. Let us extend this definition by assuming that the index of any finite hereditary class is 0, and the index of the class of all graphs is infinity. With this extension, the family of all hereditary classes is partitioned into countable number of stratums, each of which consists of classes with the same index. Moreover, the classes $\mathcal{E}_{i,j}$ with the same value of $i + j$ are the only minimal classes in the respective stratum. In particular, for $k = 2$ there are exactly three minimal classes: bipartite graphs, complements of bipartite graphs, and split graphs. Therefore, an infinite hereditary class of graphs has index 1 if and only if it contains none of the three listed classes. The classes of index 1 and the respective stratum has been called in [3] *unitary*. The unitary stratum is of particular interest for several reasons. First, the universal algorithm proposed in [1] for asymptotically optimal representation of graphs in any non-unitary class X does not work for unitary classes, since equality (1) does not provide the asymptotic behavior of $\log_2 |X_n|$ when $k(X) = 1$. Second, the unitary stratum contains many classes of theoretical and practical importance, such as forests, planar, interval, permutation, chordal bipartite, line, threshold graphs, cographs, etc. In order to provide a differentiation of the unitary classes in accordance with their size, let us introduce the following definition: two graph classes X and Y will be called *isometric* if there are positive constants c_1, c_2 and n_0 such that $|Y_n|^{c_1} \leq |X_n| \leq |Y_n|^{c_2}$ for any $n > n_0$.

Clearly this isometry is an equivalence relation. The equivalence classes of this relation will be called *layers*.

All finite classes of graphs constitute a single layer, and all classes of index greater than 1 also constitute a single layer. Between these two extremes lies the unitary stratum, and it consists of infinitely many layers. To see this, consider the class Z^p of bipartite graphs containing no $K_{p,p}$ as an induced subgraph. From the well-known results on the maximum number of edges in graphs in Z^p (see e.g. [6, 10]), we have:

$$c_1 n^{2-\frac{2}{p+1}} < \log_2 |Z_n^p| < c_2 n^{2-\frac{1}{p}} \log_2 n. \tag{2}$$

This implies, in particular, that Z^p and Z^{2p} are non-isometric.

The first four lower layers in the unitary stratum have been distinguished in [17]:

- *constant* layer contains classes X with $\log_2 |X_n| = O(1)$,
- *polynomial* layer contains classes X with $\log_2 |X_n| = \Theta(\log_2 n)$,
- *exponential* layer contains classes X with $\log_2 |X_n| = \Theta(n)$,
- *factorial* layer contains classes X with $\log_2 |X_n| = \Theta(n \log_2 n)$.

Independently, the same result has been obtained by Alekseev in [3]. Moreover, Alekseev provided the first four layers with the description of all minimal classes. In particular, the factorial layer has 9 minimal classes, three of which are subclasses of bipartite graphs, another three are subclasses of co-bipartite graphs and the remaining three are subclasses of split graphs. The three minimal factorial classes of bipartite graphs are:

$\mathcal{P}_1$: the class of $2K_2$-free bipartite graphs, also known as *chain graphs* [19], *difference graphs* [15] or *bisplit graphs* [13]. A "typical" graph in this class is described in Section 4, where we study universal chain graphs.

$\mathcal{P}_2$: the class of graphs with vertex degree at most 1;

$\mathcal{P}_3$: the class of bipartite complements of graphs in $\mathcal{P}_2$, also known as almost complete bipartite graphs.

Along with the description of minimal classes, [3] proposes a structural characterization of the classes in the first three layers (some more involved results can be found in [5]). In particular, the structure of exponential classes of graphs can be characterized as follows.

Theorem 1 *For each exponential class X, there is a constant k such that every graph $G \in X$ can be partitioned into at most k subsets each of which is either an independent set or a clique and between any two subsets there are either all possible edges or none of them.*

This characterization shows that all exponential classes have a rather simple structure, which leads, in particular, to a simple construction of order-optimal universal graphs for the classes in this layer (Section 3).

The factorial layer is substantially richer. In fact, most of the unitary classes mentioned above are factorial (the unique exception in the above list is the class of chordal bipartite graphs, which is superfactorial [18]) and most of the works on induced-universal graphs relate to factorial classes, such as threshold graphs [14], trees (forests), planar graphs, or graphs of bounded arboricity [4, 8, 16]. In the present paper we supplement this list with two new results: universal graphs for minimal factorial classes of bipartite graphs (Section 4) and bipartite permutation graphs (Section 5).

Very little is known about universal graphs for non-unitary classes. In Section 6 we describe asymptotically optimal universal graphs for the class of general bipartite graphs, which is one of the three minimal non-unitary classes.

3 Exponential classes of graphs

Let X be an exponential class of graphs and k a constant associated to it. Denote $[n] = \{1, 2, \ldots, n\}$. The n-universal X-graph UX_n is defined as follows: Let $\bar{\Gamma}_k$ contain exactly one graph from each isomorphism class of Γ_k.

(a) The vertex set of UX_n is

$$V(UX_n) = \{ (\bar{G}, i, j, \delta) \mid \bar{G} \in \bar{\Gamma}_k,\ i \in [k],\ j \in [n],\ \delta \in \{0,1\} \}.$$

(b) Two distinct vertices $(\bar{G}_1, i_1, j_1, \delta_1)$ and $(\bar{G}_2, i_2, j_2, \delta_2)$ are adjacent in UX_n if and only if $\bar{G}_1 = \bar{G}_2$ and either $i_1 i_2 \in E(\bar{G}_1)$ or $i_1 = i_2$, $\delta_1 = \delta_2 = 1$.

First let us show that the constructed graph is indeed n-universal for the class X.

Theorem 2 *Every n-vertex graph in X is embeddable into UX_n.*

Proof. Let G be a graph with n vertices in X. Since X is an exponential class, the vertices of G can be partitioned into independent sets $V_1, \dots, V_r$ and cliques $V_{r+1}, \dots, V_p$ with $p \leq k$ such that if two vertices u and v belong to the same subset V_i then $N_{G \setminus V_i}(u) = N_{G \setminus V_i}(v)$. For each subset let us define a bijection $\phi_i : V_i \to [|V_i|] \subset [n]$. By contracting each subset V_j into a single vertex v_j we obtain a new graph H with at most k vertices. Then H is isomorphic to an induced subgraph of some $\bar{H} \in \bar{\Gamma}_k$; let the isomorphism be given by $\psi : V(H) \to V(\bar{H}) = [k]$. It is easily verified that mapping a vertex $v \in V_i$ to $(\bar{H}, \psi(v_i), \phi_i(v), \delta_i)$, where $\delta_i = \begin{cases} 0 & \text{if } i \leq r \\ 1 & \text{if } i > r \end{cases}$, provides us with an embedding of G into UX_n. ■

Since $\log_2 |X_n| = O(n)$ for any exponential class X we now conclude that

Theorem 3 *The graph UX_n defined by (a) and (b) is asymptotically optimal for the class X.*

4 Minimal factorial classes of bipartite graphs

In this section we show that for each of the three minimal factorial classes of bipartite graphs $\mathcal{P}_j$ ($j = 1, 2, 3$) there is an n-universal $\mathcal{P}_j$-graph with $2n$ vertices. For $j = 2$ and $j = 3$, the statement is trivial. Now we prove it for $j = 1$, i.e. for the class of chain graphs. To this end, let us introduce the following definitions and notations.

A bipartite graph will be called *prime* if it is connected and any two distinct vertices of the graph have different neighborhoods. It is known (see e.g. [11]) that in a prime chain graph G with parts V_1 and V_2, the cardinality of V_1 equals the cardinality of V_2. Moreover, for each $i = 1, 2$ and each $j = 1, \dots, |V_i|$, V_i contains exactly one vertex of degree j, and the vertices of V_i can be ordered under inclusion of their neighborhoods (i.e.

the neighborhoods of the vertices form a chain, which explains the name of these graphs). We shall call a vertex ordering "<" *increasing* if $x < y$ implies $N(x) \subseteq N(y)$, and *decreasing* if $x < y$ implies $N(y) \subseteq N(x)$.

Denote by $H_{n,m}$ the graph with nm vertices which can be partitioned into n independent sets $V_1 = \{v_{1,1}, \ldots, v_{1,m}\}$, ..., $V_n = \{v_{n,1}, \ldots, v_{n,m}\}$ so that for each $i = 1, \ldots, n-1$ and for each $j = 1, \ldots, m$, vertex $v_{i,j}$ is adjacent to vertices $v_{i+1,1}, v_{i+1,2}, \ldots, v_{i+1,j}$ and there are no other edges in the graph. In other words, every two consecutive independent set induce in $H_{n,m}$ a prime chain graph. The graph $H_{n,m}$ will be called *canonical*. An example of a canonical graph is given in Figure 1.

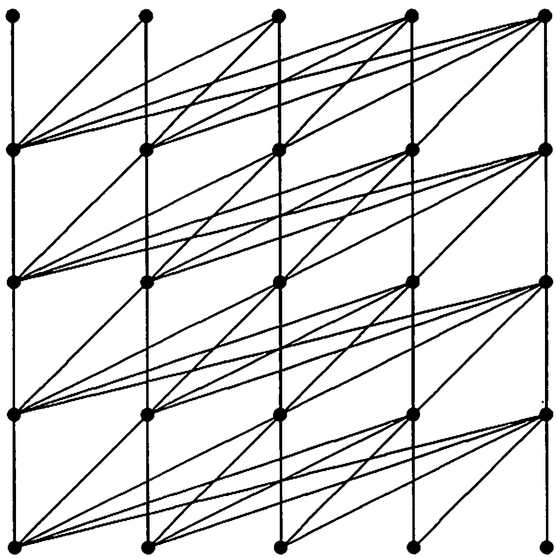

Figure 1: Canonical graph $H_{5,5}$

Theorem 4 *The graph $H_{2,n}$ is an n-universal chain graph.*

Proof. Let G be an n-vertex chain graph with parts V_1 and V_2. We shall assume that the vertices of V_1 are ordered decreasingly, while the vertices of V_2 are ordered increasingly. The graph $H_{2,n}$ containing G will be created by adding to G some new vertices and edges. To this end, we partition V_1 and V_2 into modules (i.e. subset of vertices with the same neighborhood) and denote the modules of V_1 by $V_1^1, V_1^3, V_1^5, \ldots, V_1^{2p-1}$, and the modules of V_2 by $V_2^2, V_2^4, V_2^6, \ldots, V_2^{2p}$ (observe that the number of modules in V_1 and V_2 must be equal). Now, for each odd $i = 1, \ldots, 2p-1$, we create a set of new vertices V_2^i of size $|V_1^i|$, and for each even $j = 2, \ldots, 2p$, we create a set of new vertices V_1^j of size $|V_2^j|$. The desired graph $H_{2,n}$ will contain two parts of vertices $V_1' = V_1^1 \cup V_1^2 \cup \ldots \cup V_1^{2k-1} \cup V_1^{2k}$ and $V_2' = V_2^1 \cup V_2^2 \cup \ldots \cup V_2^{2k-1} \cup V_2^{2k}$ of the same size. To complete the construction, we first re-index the vertices in V_1' and V_2' consecutively, following the order of subsets, and then for each $j = 1, 2, \ldots, |V_2'|$, we connect by an edge the j-th vertex of V_2' to the (not yet adjacent) i-th vertex of V_1' for each $i = 1, 2, \ldots, j$. According to the construction, the obtained graph $H_{2,m}$ is clearly a prime chain graph. Moreover, it contains G as an induced subgraph since no new edge connects two old vertices. ■

From definition of factorial classes and Theorem 4 we conclude that

Corollary 1 *Graph $H_{2,n}$ is an asymptotically optimal universal chain graph.*

5 Bipartite permutation graphs

In this section we extend the result of the previous one to the class of bipartite permutation graphs, i.e. the intersection of classes of bipartite graphs and permutation graphs. The result presented here is based on the following theorem proved in [9].

Theorem 5 *A connected graph G is bipartite permutation if and only if the vertex set of G can be partitioned into independent sets $V_1, \ldots, V_q$ so that*

(a) any two vertices in non-consecutive sets are non-adjacent,

(b) any two consecutive sets V_j and V_{j+1} induce a chain graph, denoted G_j,

(c) for each $j = 2, \ldots, q-1$, there is an ordering of vertices in the set V_j, which is decreasing for G_{j-1} and increasing for G_j.

With every connected bipartite permutation graph G we shall associate a partition as in Theorem 5 and the respective independent sets $V_1, \ldots, V_q$ will be called the layers of G. Observe that the graph $H_{n,m}$ defined in the previous section is a bipartite permutation graph. We will show now that $H_{n,n}$ contains all n-vertex bipartite permutation graphs as induced subgraphs.

Theorem 6 *Graph $H_{n,n}$ is an n-universal bipartite permutation graph.*

Proof. Let G be an n-vertex bipartite permutation graph. The proof will be given by induction on the number of connected components of G.

Assume first that G is connected. We will show by induction on the number of layers in G that $H_{n,n}$ contains G as an induced subgraph, moreover, the i-th layer of G belongs to the i-th layer of $H_{n,n}$. The basis of the induction is established in Theorem 4. Now assume that the proposition is valid for any connected bipartite permutation graph with $k \geq 2$ layers, and let G contain $k+1 \leq n$ layers. For $j = 1, \ldots, k+1$, let V_j denote the set of vertices in the j-th layer of G, $n_j = |V_j|$ and also let $m = n_1 + \ldots + n_k$.

Let $H_{k,m}$ be a canonical graph containing the first k layers of G as an induced subgraph. We denote the layers of $H_{k,m}$ by $W_1, \ldots, W_k$. Now we create an auxiliary graph H' out of $H_{k,m}$ by

(1) adding to $H_{k,m}$ the set of vertices V_{k+1},

(2) connecting the vertices of V_k (belonging to W_k) to the vertices of V_{k+1} as in G,

(3) connecting the vertices of $W_k - V_k$ to the vertices of V_{k+1} so as to make the existing ordering of vertices in W_k decreasing in the subgraph induced by W_k and V_{k+1}. More formally, whenever vertex $w_{k,i}$ in $W_k - V_k$ is connected to a vertex v in V_{k+1}, every vertex $w_{k,j}$ with $j < i$ must be connected to v too.

According to (2) and (3) the subgraph of H' induced by W_k and V_{k+1} is a chain graph. We denote this subgraph by G'. Clearly H' contains G as an induced subgraph. To extend H' to a canonical graph containing G we apply the induction hypothesis twice. First, we extend G' to a canonical chain graph as described in Theorem 4. This will add m new vertices to the $k+1$-th and n_k new vertices to k-th layer of the graph. Then we extend the first k layers to a canonical form. The resulting graph has $k+1 \leq n$ layers with n vertices in each layer. This completes the proof for the case when G is connected.

Now assume that G is disconnected. Denote by G_1 a connected component of G and by G_2 the rest of the graph. Also let $k_1 = |V(G_1)|$ and $k_2 = |V(G_2)|$. The first k_1 vertices in the first k_1 layers of $H_{n,n}$ induce the graph H_{k_1,k_1}, which, according to the above discussion, contains G_1 as an induced subgraph. The last k_2 vertices in the last k_2 layers of $H_{n,n}$ induce the graph H_{k_2,k_2}, which contains G_2 according to the inductive hypothesis. Therefore, $H_{n,n}$ contains G and the proof is complete. ■

Corollary 2 *Graph $H_{n,n}$ is an order-optimal universal bipartite permutation graph.*

In the rest of this section, we show that some similar results hold for unit interval graphs, i.e. intersection graphs of unit intervals of the real line. Indeed, between bipartite permutation graphs and unit interval graphs there is a close relation, which can be described as follows. Given a bipartite permutation graph G with layers $V_0, V_1, \ldots, V_q$, replace each independent set V_j with a clique (in other words, connect every two vertices in V_j). In this way, we obtain a unit interval graph. On the other hand, every connected unit interval graph can be partitioned into layers each of which is a clique. More formally,

Theorem 7 *A connected graph G is unit interval if and only if the vertex set of G can be partitioned into cliques $V_0, V_1, \ldots, V_q$ so that*

(a) any two vertices in non-consecutive cliques are non-adjacent,

(b) any two consecutive cliques V_{j-1} and V_j induce the complement of a chain graph, denoted G_j,

(c) for each $j = 1, 2, \ldots, q-1$, there is an ordering of vertices in V_j, which is decreasing for G_j and increasing for G_{j+1}.

This theorem can be proved by analogy with Theorem 5, for the prove of which the intersection model of bipartite permutation graphs has been used. We advise the reader to use the intersection model of unit interval graphs and leave the proof of Theorem 7 as an exercise. The relation between bipartite permutation and unit interval graphs suggests a similar construction of universal unit interval graphs.

6 General Bipartite Graphs

Let D_{n_1,n_2} denote the set of all bipartite graphs $G = (V_1, V_2, E)$ with parts of size $|V_1| = n_1$ and $|V_2| = n_2$. Also,

$$D_n = \bigcup_{n_1+n_2=n} D_{n_1,n_2}, \qquad D = \bigcup_{n=1}^{\infty} D_n.$$

We will construct an n-universal bipartite graph UD_n in the following way. With each partition $n = n_1 + n_2$ we associate a connected component UD_{n_1,n_2} of the graph UD_n which contains all graphs from D_{n_1,n_2} as induced subgraphs.

Lemma 1 *For a complete bipartite graph $K_{n_1,n_2} = (V_1, V_2, E_K)$ ($|V_1| = n_1$, $|V_2| = n_2$) there exists a partition $E_K = E_1 \cup^* E_2$ such that for all $v \in V_i$, $\deg_{E_i}(v) \le \lceil \frac{n_1+n_2}{4} \rceil$ holds ($i = 1, 2$).*

Proof. Let us assume $n_1 \le n_2$. Then there exists a set of edges $E_1 \subset E_K$ such that

- $\deg_{E_1}(v) = \lceil \frac{n_1+n_2}{4} \rceil$ for all $v \in V_1$
- $\left|\deg_{E_1}(w) - \deg_{E_1}(z)\right| \leq 1$ for all $w, z \in V_2$.

Let $E_2 = E_K \setminus E_1$; then for $v \in V_2$ we have $\deg_{E_2}(v) \leq n_1 - \left\lfloor \frac{n_1 \lceil \frac{n_1+n_2}{4} \rceil}{n_2} \right\rfloor$, therefore it suffices to show $n_1 - \left\lfloor \frac{n_1 \lceil \frac{n_1+n_2}{4} \rceil}{n_2} \right\rfloor \leq \lceil \frac{n_1+n_2}{4} \rceil$, or equivalently, $n_1 - \lceil \frac{n_1+n_2}{4} \rceil \leq \frac{n_1 \lceil \frac{n_1+n_2}{4} \rceil}{n_2}$. By rearranging this we get $n_1 n_2 \leq (n_1 + n_2) \lceil \frac{n_1+n_2}{4} \rceil$, which is true since by the inequality between arithmetic and geometric means we have $n_1 n_2 \leq \left(\frac{n_1+n_2}{2}\right)^2 \leq (n_1 + n_2) \lceil \frac{n_1+n_2}{4} \rceil$. ■

Now, using the notations from the above theorem and denoting the neighborhood of a vertex v with respect to an edge set E by $N_E(v) = \{u \mid uv \in E\}$, we define UD_{n_1,n_2} as follows:

- The vertex set of UD_{n_1,n_2} is $U_1 \cup U_2$, where

$$U_i = \{(v, F) \mid v \in V_i,\ F \subset N_{E_i}(v)\}$$

- Two vertices $(v_1, F_1) \in U_1$ and $(v_2, F_2) \in U_2$ are adjacent in UD_{n_1,n_2} if and only if either $v_1 \in F_2$ or $v_2 \in F_1$.

Let us consider an arbitrary bipartite graph $G = (V_1, V_2, E)$ in D_{n_1,n_2}. Then (still using our previous notations) it is easy to verify the following:

Proposition 1 *Mapping a vertex $v \in V_i$ to $(v, N_E(v) \cap N_{E_i}(v)) \in U_i$ $(i = 1, 2)$ provides us an embedding of G into UD_{n_1,n_2}.*

Proof. Let us first observe that the mapping is injective and the image of V_i lies inside the independent set U_i $(i = 1, 2)$. Therefore the proposition follows from the next chain of equivalences for a pair of vertices $v_1 \in V_1$ and $v_2 \in V_2$:

$$v_1 v_2 \in E \Leftrightarrow v_1 v_2 \in E \cap E_K \Leftrightarrow v_1 v_2 \in E \cap E_1 \text{ or } v_1 v_2 \in E \cap E_2$$

$$\Leftrightarrow v_2 \in N_E(v_1) \cap N_{E_1}(v_1) \text{ or } v_1 \in N_E(v_2) \cap N_{E_2}(v_2)$$

$$\Leftrightarrow (v_1, N_E(v_1) \cap N_{E_1}(v_1)) \in U_1 \text{ and } (v_2, N_E(v_2) \cap N_{E_2}(v_2)) \in U_2$$

$$\text{are adjacent in } UD_{n_1,n_2}.$$

■

As an immediate corollary we obtain

Theorem 8 *The graph UD_n constructed above is an asymptotically optimal n-universal bipartite graph.*

Proof. The universality of UD_n follows from the previous proposition. According to our construction

$$|UD_n| = \sum_{n_1+n_2=n} |UD_{n_1,n_2}| \le \sum_{n_1+n_2=n} 2^{\lceil \frac{n}{4} \rceil} n \le (n^2+n)2^{\frac{n}{4}+1}.$$

It is known (see e.g. [7]) that $|D_n| = 2^{n^2/4+o(n^2)}$, which implies asymptotic optimality. ■

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