UNITARY ANALOGUES OF SOME FORMULAE OF INGHAM

by

V.SITARAMAIAH and M.V.SUBBARAO [1]

1 Introduction

In 1927, among other results, A.E.Ingham (cf.[3]) established the following asymptotic formulae : As $x \to \infty$,

$$\sum_{n\le x}\sigma(n)\sigma(n+k) \sim \frac{1}{3}\frac{\zeta^2(2)}{\zeta(4)}\sigma_{-3}(k)x^3, \tag{1.1}$$

and

$$\sum_{n\le x}\phi(n)\phi(n+k) \sim \frac{1}{3}x^3\prod_p\left(1-\frac{2}{p^2}\right)\prod_{p|k}\frac{p^3-2p+1}{p(p^2-2)}. \tag{1.2}$$

In the above k denotes a positive integer and as usual ϕ is the Euler-totient function ;$\sigma_s(n)$ the sum of the s-th powers of the divisors of n, $\sigma_1(n) = \sigma(n)$ and $\zeta(s)$ is the Riemann-Zeta function. In fact the formula(1.2) was stated by Ingham(cf.[3],eq.(18)) without proof.Later, L.Mirsky(cf.[4],eq.(30)) obtained the formula(1.2) with error term $o(x^2\log^2 x)$ which he deduced in [4] as a particular case of a general result.

The object of the present paper is to obtain asymptotic formulae for the unitary analogues of the sums in (1.1) and (1.2).

[1] Partially supported by an NSERC grant.
AMS subject Classification :11A25
Key words and Phrases :Euler totient,sum of divisor function, unitary divisors and their sum asymptotic estimates

A divisor d of n is called a unitary divisor (cf.[1]) if $(d, n/d) = 1$, where the symbol (a, b), as usual,denotes the greatest common divisor of a and b. The notation $d \| n$ means that d is a unitary divisor of n. Let $\sigma^*(n)$ denote the sum of the unitary divisors of n and ϕ^* denote the unitary analogue of the Euler-totient function ϕ. It is known (cf.[1]) that σ^* ,ϕ^* are multiplicative , $\sigma^*(p^\alpha) = p^\alpha + 1$ and $\phi^*(p^\alpha) = p^\alpha - 1$, where p is a prime and α a positive integer.Also,

$$\phi^*(n) = n \sum_{d \| n} \frac{\mu^*(d)}{d}, \tag{1.3}$$

where $\mu^*(n) = (-1)^{\omega(n)}$ is the unitary analogue of the M öbius function (cf.[1]) $\omega(n)$ being the number of distinct prime factors of n with $\omega(1) = 0$.

In the present paper, we show that (see Theorem 3.1 and (3.18)),

$$\sum_{n \leq x} \sigma^*(n)\sigma^*(n+k) = \frac{AB(k)x^3}{3} + O_k\left(x^2(\log x)^4\right), \tag{1.4}$$

and

$$\sum_{n \leq x} \phi^*(n)\phi^*(n+k) = \frac{A^*B^*(k)x^3}{3} + O_k\left(x^2(\log x)^4\right), \tag{1.5}$$

where $A, B(k)$, A^* and $B^*(k)$ are as given in (2.3),(2.10),(2.12) and (2.14).

We use Ingham's method (cf.[3]) in establishing (1.4) and we proceed as in Mirsky (cf.[4]) to prove (1.5). Incidentally we obtain an asymptotic formula for the sum $\sum_{\substack{n \leq x \\ n \equiv \ell (\bmod m)}} \frac{\phi^*(n)}{n}$ with uniform error term in ℓ and m (Lemma 2.8) , for any integer ℓ and a positive integer m , which is required in the proof of (1.5). We develop preliminaries in §2 and the main results are proved in §3.

2 Preliminaries

Throughout the folowing the letters, m ,$N, k, \alpha, \beta,$ and γ denote positive integers. The letter p is reserved for primes.We use the notation $p^\alpha \| k$ to mean that $p^\alpha | k$ and $p^{\alpha+1} \nmid k$.

Lemma 2.1. Let

$$G(m, N, k) = \sum_{\substack{r|m \\ (mr,N)|k}} \frac{\mu(r)(mr, N)}{r}, \tag{2.1}$$

where μ is the Möbius function. Then $G(m, N, k)$ is a multiplicative function of m and

$$G(p^\alpha, N, k) = \begin{cases} \left(p^\alpha, N)(1 - \frac{1}{p}\right), & \text{if } p^{\alpha+1} \nmid N, (p^{\alpha+1}, N)|k \\ 0, & \text{if " } p^{\alpha+1}|N, p^{\alpha+1}|k\text{" or if } (p^\alpha, N) \nmid k \\ p^\alpha, & \text{if } p^{\alpha+1}|N, p^\alpha \| k. \end{cases}$$

Proof : Let

$$G'(m, n) = \begin{cases} (m, n), & \text{if } (m, n)|k, \\ 0, & \text{otherwise}, \end{cases}$$

so that

$$G(p^\alpha, N, k) = \sum_{0 \le a \le 1} \frac{\mu(p^a)}{p^a} G'(p^{\alpha+a}, N)$$

$$= G'(p^\alpha, N) - \frac{1}{p} G'(p^{\alpha+1}, N)$$

$$
= \begin{cases} (p^\alpha, N) - \frac{1}{p}(p^{\alpha+1}, N), & \text{if } (p^{\alpha+1}, N) | k, \\ (p^\alpha, N), & \text{if } (p^{\alpha+1}, N) \not| k \text{ and } (p^\alpha, N) | k, \\ 0, & \text{if } (p^\alpha, N) \not| k. \end{cases} \tag{2.2}
$$

If $(p^{\alpha+1}, N) \neq p^{\alpha+1}$, clearly $(p^{\alpha+1}, N) = (p^\alpha, N)$. Thus either $(p^{\alpha+1}, N) = (p^\alpha, N)$ or "$(p^{\alpha+1}, N) = p^{\alpha+1}$ and $(p^\alpha, N) = p^\alpha$." Further, $(p^\alpha, N)|k$ and $(p^{\alpha+1}, N) \not| k$ if and only if $p^{\alpha+1}|N$ and $p^\alpha \| k$. Using this in (2.2),Lemma 2.1 follows.

Lemma 2.2. We have

$$
\sum = \sum_{m=1}^{\infty} \frac{G(m, N, k)}{m^2} = AH(N, k),
$$

where

$$
A = \prod_p \left(1 + \frac{1}{p(p+1)}\right) \tag{2.3}
$$

and

$$
H(N, k) = \prod_{p|N} \frac{p(p+1)}{p^2+p+1} \prod_{\substack{p^\beta \| N \\ p^\beta | k}} \left(1 + \frac{1}{p^{\beta-1}(p+1)}\right) \prod_{\substack{p^\beta \| N \\ p^\beta \not| k \\ p^\gamma \| k}} \left(1 + \frac{1}{p^\gamma}\right). \tag{2.4}
$$

Proof : From (2.1), it is clear that

$$
|G(m, N, k) \leq k \sum_{r|m} \frac{\mu^2(r)}{r} = k\frac{\psi(m)}{m} \leq k\tau(m), \tag{2.5}
$$

ψ being the Dedekind ψ-function and $\tau(m)$ the number of divisors of m. Hence the infinite series $\sum$ converges absolutely. Also, the general term of this series is a multiplicative function of m. Hence the series can be expanded as an Euler-Infinite product (cf.[2],Theorem 286). Writing $G(m) = G(m, N, k)$ and using Lemma 2.1, we obtain,

$$\sum = \prod_{p \nmid N} \left\{1 + \frac{1}{p(p+1)}\right\} \prod_{p^\beta \| N} \left\{1 + \sum_{\alpha=1}^{\infty} \frac{G(p^\alpha)}{p^{2\alpha}}\right\}. \tag{2.6}$$

Let $p^\beta \| N$ and $p^\gamma \| k$. We have

$$\sum_{\alpha=1}^{\infty} \frac{G(p^\alpha)}{p^{2\alpha}} = \frac{p-1}{p} \sum_{\substack{p^{\alpha+1} \nmid N \\ (p^{\alpha+1}, N) | k}} \frac{(p^\alpha, N)}{p^{2\alpha}} + \sum_{\substack{p^{\alpha+1} | N \\ p^\alpha \| k \\ (p^\alpha, N) | k}} \frac{1}{p^\alpha}$$

$$= \begin{cases} \frac{p-1}{p} . p^\beta \sum_{\alpha=\beta}^{\infty} \frac{1}{p^{2\alpha}}, & \text{if } p^\beta | k \\ \frac{1}{p^\gamma}, & \text{if } p^\beta \nmid k. \end{cases}$$

$$= \begin{cases} \frac{1}{p^{\beta-1}(p+1)}, & \text{if } p^\beta | k \\ \frac{1}{p^\gamma}, & \text{if } p^\beta \nmid k. \end{cases} \tag{2.7}$$

Substituting (2.7) into (2.6), we obtain Lemma 2.2.

Lemma 2.3. Let

$$I(m, k) = \sum_{t | m} \frac{\mu(t)}{t} H(mt, k), \tag{2.8}$$

where $H(N, k)$ is as given in (2.4).Then $I(m, k)$ is a multiplicative function of m and

$$\frac{p^2+p+1}{p(p+1)}.I(p^\alpha,k) = \begin{cases} \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p^\alpha}\right), & \text{if } p^{\alpha+1}|k, \\ 1-\frac{1}{p}+\frac{1}{p^{\alpha-1}(p+1)}-\frac{1}{p^{\alpha+1}}, & \text{if } p^\alpha\|k, \\ 1-\frac{1}{p}, & \text{if } p\not|k, \\ \left(1+\frac{1}{p^\gamma}\right)\left(1-\frac{1}{p}\right), & \text{if } p^\gamma\|k, p^\alpha \not|k. \end{cases} \tag{2.9}$$

Proof : $I(p^\alpha,k) = H(p^\alpha,k) - \frac{1}{p}H(p^{\alpha+1},k)$. Now (2.9) follows from the definition of $H(N,k)$ given in (2.4).

Lemma2.4. We have

$$\sum{}' = \sum_{m=1}^{\infty} \frac{I(m,k)}{m^2} = B(k),$$

where

$$B(k) = \prod_{p\not| k}\left(1+\frac{1}{p^2+p+1}\right)\prod_{p^\gamma\|k}\left\{1+\frac{p^{3\gamma+1}+2p^{3\gamma}+2p^{\gamma-1}-p^2-1}{p^{3\gamma-1}(p^2+p+1)^2}\right\}. \tag{2.10}$$

Proof : From (2.5) and Lemma 2.2,

$$|H(N,k)| \le \frac{k}{A}\sum_{m=1}^{\infty}\frac{\tau(m)}{m^2} = O(k).$$

Hence

$$|I(m,k)| = O\left(k\sum_{t|m}\frac{\mu^2(t)}{t}\right) = O\left(k\frac{\psi(m)}{m}\right) = O\left(k\tau(m)\right), \tag{2.11}$$

so that the series $\sum'$ converges absolutely . Expanding this series as an Euler Infinite product, we obtain Lemma 2.4, by using Lemma 2.3 and on simplification.

The following lemmas (Lemmas 2.5-2.7)can be obtained on lines similar to those of Lemmas 2.2-2.4. We suppress the details of proof.

Lemma 2.5 We have for any integer ℓ and a positive integer m,

$$\sum_{d=1}^{\infty} \frac{\mu^*(d)G(d,m,\ell)}{d^2} = A^* H^*(m,\ell),$$

where

$$A^* = \prod_p \left(1 - \frac{1}{p(p+1)}\right), \tag{2.12}$$

and

$$H^*(m,\ell) = \prod_{p|m} \frac{p(p+1)}{p^2+p-1} \prod_{\substack{p^\beta \| m \\ p^\beta | \ell}} \left(1 - \frac{1}{p^{\beta-1}(p+1)}\right) \prod_{\substack{p^\beta \| m \\ p^\beta \nmid \ell \\ p^\gamma \| \ell}} \left(1 - \frac{1}{p^\gamma}\right). \tag{2.13}$$

Lemma 2.6. If $H^*(m,\ell)$ is as given in (2.13), let

$$I^*(d,k) = \sum_{r|d} \frac{\mu(r)H^*(dr,k)}{r}.$$

Then $I^*(d,k)$ is a multiplicative function of d and

$$\frac{p^2+p-1}{p(p+1)} . I^*(p^\alpha,k) = \begin{cases} \left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^\alpha}\right), & \text{if } p^{\alpha+1}|k, \\ 1-\frac{1}{p}-\frac{1}{p^{\alpha-1}(p+1)}+\frac{1}{p^{\alpha+1}}, & \text{if } p^\alpha\|k, \\ 1-\frac{1}{p}, & \text{if } p \nmid k, \\ \left(1-\frac{1}{p^\gamma}\right)\left(1-\frac{1}{p}\right), & \text{if } p^\gamma\|k, p^\alpha \nmid k. \end{cases}$$

Lemma 2.7.

$$\sum_{d=1}^{\infty} \frac{\mu^*(d)I^*(d,k)}{d^2} = B^*(k),$$

where

$$B^*(k) = \prod_{p \nmid k} \left(1 - \frac{1}{p^2+p-1}\right) \prod_{p^\gamma \| k} \left\{1 - \frac{p^{3\gamma+1}+p^2+1}{p^{3\gamma-1}(p^2+p+1)(p^2+p-1)}\right\}. \tag{2.14}$$

Lemma 2.8. We have for any integer ℓ and any positive integer m,

$$\sum^* = \sum_{\substack{n \leq x \\ n \equiv \ell \ (\mathrm{mod}\ m)}} \frac{\phi^*(n)}{n} = \frac{x}{m} A^* H^*(m,\ell) + O\left(|\ell| \log^2 x\right), \tag{2.15}$$

where A^* and $H^*(m,\ell)$ are as given in Lemma 2.5, the $O-$constant in (2.15) is uniform in ℓ,m and x.

Proof : Since $\phi^*(n) = n \sum_{\substack{d\delta = n \\ (d,\delta)=1}} \frac{\mu^*(d)}{d}$, we have

$$\sum^* = \sum_{\substack{d\delta \leq x \\ d\delta \equiv \ell (\mathrm{mod}\ m) \\ (d,\delta)=1}} \frac{\mu^*(d)}{d} = \sum_{d \leq x} \frac{\mu^*(d)}{d} \sum_{\substack{\delta \leq x/d \\ (\delta,d)=1 \\ \delta d \equiv \ell (\mathrm{mod}\ m)}} 1$$

$$= \sum_{d \leq x} \frac{\mu^*(d)}{d} \sum_{t|d} \mu(t) \sum_{\substack{u \leq x/dt \\ dtu \equiv \ell (\mathrm{mod}\ m)}} 1$$

$$= \sum_{d \leq x} \frac{\mu^*(d)}{d} \sum_{t|d} \mu(t) \left\{\frac{x(dt,m)}{dtm} + O(1)\right\}$$

$$= \frac{x}{m} \sum_{d \leq x} \frac{\mu^*(d)}{d^2} \sum_{\substack{t|d \\ (dt,m)|\ell}} \frac{\mu(t)(dt,m)}{t} + O\left(\sum_{d \leq x} \frac{\tau(d)}{d}\right)$$

$$= \frac{x}{m} \sum_{d=1}^{\infty} \frac{\mu^*(d) G(d,m,\ell)}{d^2} + O\left(\frac{x}{m} \sum_{d > x} \frac{|G(d,m,\ell)|}{d^2}\right) + O\left(\log^2 x\right). \tag{2.16}$$

From (2.5),

$$\sum_{d>x} \frac{|G(d,m,\ell)|}{d^2} = O\left(|\ell| \sum_{d>x} \frac{\psi(d)/d}{d^2}\right) = O\left(\frac{|\ell|}{x}\right), \tag{2.17}$$

since $\sum_{d\leq x} \frac{\psi(d)}{d} = O(x)$ and partial summation. Substituting (2.17) into (2.16), we obtain Lemma 2.8 from Lemma 2.5.

Remark 2.1. From Lemma 2.8 and partial summation we obtain the formula

$$\sum_{\substack{n\leq x \\ n\equiv \ell (\bmod m)}} \phi^*(n) = \frac{x^2}{2m} A^* H^*(m,\ell) + O\left(|\ell| x \log^2 x\right). \tag{2.18}$$

Taking $m = 1$ in (2.18), we obtain the formula

$$\sum_{n\leq x} \phi^*(n) = \frac{x^2 A^*}{2} + O\left(x \log^2 x\right) \tag{2.19}$$

originally established by E.Cohen (cf.[1],Corollary 4.1.2). The error term in (2.19) was improved to $O\left(x \log^{5/3} x (\log\log x)^{4/3}\right)$ by D.Suryanarayana and R.Sita Rama Chandra Rao (cf.[5],(1.5)).

3 Main Results

Theorem 3.1. We have

$$T \equiv \sum_{n\leq x} \sigma^*(n)\sigma^*(n+k) = \frac{AB(k)x^3}{3} + O_k\left(x^2(\log x)^4\right), \tag{3.1}$$

where A and $B(k)$ are as given in (2.3) and (2.10) respectively.

Proof : We proceed as in A.E.Ingham [3]. Let $X^2 = x(x+k)$. Using $\sigma^*(n) = \sum_{\substack{d\delta=n \\ (d,\delta)=1}} d$, we can write

$$T = T_1 + T_2 - R, \tag{3.2}$$

where

$$T_1 = \sum_{\substack{d\delta \le x \\ (d,\delta)=(d',\delta')=1 \\ d\delta+k=d'\delta' \\ dd' \le X}} dd', \tag{3.3}$$

$$T_2 = \sum_{\substack{d\delta \le x \\ (d,\delta)=(d',\delta')=1 \\ d\delta+k=d'\delta' \\ \delta\delta' \le X}} dd', \tag{3.4}$$

$$R = \sum_{\substack{d\delta \le x \\ (d,\delta)=(d',\delta')=1 \\ d\delta+k=d'\delta' \\ dd' \le X \\ \delta\delta' \le X}} dd', \tag{3.5}$$

We now estimate T_1. We have

$$T_1 \le \sum_{\substack{d\delta \le x \\ d\delta+k=d'\delta' \\ dd' \le X}} dd' = \sum_{d \le x} d \sum_{d' \le \frac{X}{d}} d' \sum_{\substack{\delta \le x/d \\ d\delta \equiv -k (\bmod d')}} 1$$

$$= \sum_{d \le x} d \sum_{\substack{d' \le X/d \\ (d,d') | k}} d' \left\{ \frac{x(d,d')}{dd'} + O(1) \right\}$$

$$\le kx \sum_{d \le x} \sum_{d' \le X/d} 1 + O\left(\sum_{d \le x} d \sum_{d' \le X/d} d' \right)$$

$$= O_k(x^2 \log x) + O_k(x^2 \log x)$$

$$= O_k(x^2 \log x),$$

so that

$$T_1 = O_k(x^2 \log x). \tag{3.6}$$

We proceed to estimate R. We have

$$R = \sum_{d \le x} d \sum_{d' \le X/d} d' \sum_{\substack{\delta \le x/d \\ d\delta \equiv -k (\mathrm{mod}\ d') \\ \delta(d\delta + k) \le Xd'}} 1$$

$$\le \sum_{d \le x} d \sum_{d' \le X/d} d' \sum_{\substack{\delta \le x/d \\ d\delta \equiv -k (\mathrm{mod}\ d') \\ \delta^2 d \le Xd'}} 1$$

$$\le \sum_{d \le x} d \sum_{d' \le X/d} d' \sum_{\substack{\delta \le \sqrt{\frac{Xd'}{d}} \\ d\delta \equiv -k (\mathrm{mod})\ d')}} 1$$

$$\le \sum_{d \le x} d \sum_{\substack{d' \le X/d \\ (d', d) | k}} d' \left\{ \sqrt{\frac{Xd'}{d}} \frac{(d, d')}{d'} + O(1) \right\}$$

$$\le k\sqrt{X} \sum_{d \le x} \sqrt{d} \sum_{d' \le \frac{X}{d}} \sqrt{d'} + O\left(\sum_{d \le x} d \sum_{d' \le \frac{X}{d}} d' \right)$$

$$= O_k\left(\sqrt{X} X^{3/2} \sum_{d \le x} \frac{1}{d} \right) + O_k\left(x^2 \log x \right)$$

$$= O_k(x^2 \log x) + O_k(x^2 \log x) = O_k(x^2 \log x).$$

Thus

$$R = O_k(x^2 \log x). \tag{3.7}$$

The main term in (3.1) can be obtained from T_2. We have

$$T_2 = \sum_{\substack{d\delta \le x \\ \delta\delta' \le X \\ d\delta \equiv -k \pmod{\delta'} \\ (d,\delta) = (\frac{d\delta+k}{\delta'}, \delta') = 1}} d\left(\frac{d\delta + k}{\delta'}\right)$$

$$= \sum_{\delta\delta' \le X} \frac{\delta}{\delta'} \sum_{\substack{d \le x/\delta \\ (d,\delta)=1 \\ (\frac{d\delta+k}{\delta'}, \delta')=1 \\ d\delta \equiv -k \pmod{\delta'}}} d^2 + k \sum_{\substack{d\delta \le x \\ \delta\delta' \le X \\ d\delta \equiv -k \pmod{\delta'} \\ (d,\delta) = (\frac{d\delta+k}{\delta'}, \delta') = 1}} \frac{d}{\delta'}$$

$$= \sum_1 + k \sum_2, \tag{3.8}$$

say. We have

$$\sum_1 = \sum_{\delta\delta' \le X} \frac{\delta}{\delta'} \sum_{r|\delta'} \mu(r) \sum_{\substack{d \le x/\delta \\ (d,\delta)=1 \\ \delta' r | d\delta + k}} d^2$$

$$= \sum_{\delta\delta' \le X} \frac{\delta}{\delta'} \sum_{r|\delta'} \mu(r) \sum_{\substack{t|\delta \\ (\delta t, \delta' r)|k}} \mu(t) t^2 \sum_{\substack{u \le x/\delta t \\ \delta t u \equiv -k \pmod{\delta' r}}} u^2. \tag{3.9}$$

If $(a, m)|b$, we have the asymptotic formula

$$\sum_{\substack{n \le y \\ an \equiv b \pmod m}} 1 = \frac{y(a,m)}{m} + O(1),$$

so that by partial summation we obtain the formula

$$\sum_{\substack{n \le y \\ an \equiv b \pmod m}} n^2 = \frac{1}{3} \frac{y^3(a,m)}{m} + O(y^2). \tag{3.10}$$

Using (3.10) in (3.9), we obtain

$$\sum_1 = \sum_{\delta\delta' \le X} \frac{\delta}{\delta'} \sum_{\substack{r|\delta' \\ t|\delta \\ (\delta t, \delta' r)|k}} \mu(r)\mu(t)t^2 \left\{ \frac{1}{3} \frac{x^3}{\delta^3 t^3} \frac{(\delta t, \delta' r)}{\delta' r} + O\left(\frac{x^2}{\delta^2 t^2}\right) \right\}$$

$$= \frac{x^3}{3} \sum_{\delta\delta' \le X} \frac{1}{(\delta\delta')^2} \sum_{\substack{r|\delta' \\ t|\delta \\ (\delta t, \delta' r)|k}} \frac{\mu(r)\mu(t)}{rt} (\delta t, \delta' r) + O\left(x^2 \sum_{\delta\delta' \le X} \frac{\tau(\delta)\tau(\delta')}{\delta\delta'} \right)$$

$$= \frac{x^3}{3} \sum_3 + \sum_4, \tag{3.11}$$

say. Clearly,

$$\sum_4 = O\left(x^2 \log^4 x\right). \tag{3.12}$$

We have by Lemma 2.2,(2.17), Lemma 2.4 and (2.11),

$$\sum_3 = \sum_{\substack{arbt \le X \\ (ar^2, bt^2)|k}} \frac{1}{(abrt)^2} \frac{\mu(r)}{r} \frac{\mu(t)}{t} (ar^2, bt^2)$$

$$= \sum_{bt \le X} \frac{\mu(t)}{b^2 t^3} \sum_{m \le \frac{X}{bt}} \frac{1}{m^2} \sum_{\substack{ar = m \\ (mr, bt^2)|k}} \frac{\mu(r)}{r} (mr, bt^2)$$

$$= \sum_{bt \le X} \frac{\mu(t)}{b^2 t^3} \sum_{m \le \frac{X}{bt}} \frac{G(m, bt^2, k)}{m^2}$$

$$= \sum_{bt \le X} \frac{\mu(t)}{b^2 t^3} \left\{ AH(bt^2, k) + O_k\left(\frac{bt}{\lambda}\right) \right\}$$

$$= A \sum_{bt \le X} \frac{\mu(t)}{b^2 t^3} H(bt^2, k) + O_k\left(\frac{1}{X} \sum_{bt \le X} \frac{1}{bt^2} \right)$$

$$= A \sum_{m \le X} \frac{1}{m^2} \sum_{bt = m} \frac{\mu(t)}{t} H(mt, k) + O_k\left(\frac{\log X}{X}\right)$$

$$= A \sum_{m \le X} \frac{I(m, k)}{m^2} + O_k\left(\frac{\log X}{X}\right)$$

$$= AB(k) + O_k\left(\frac{1}{X}\right) + O_k\left(\frac{\log X}{X}\right) \tag{3.13}$$

Substituting (3.13) and (3.12) into (3.11), we obtain

$$\sum_1 = \frac{AB(k)x^3}{3} + O_k\left(x^2 \log^4 x\right). \tag{3.14}$$

Clearly from (3.8),

$$\sum_2 \leq \sum_{\delta\delta' \leq X} \frac{1}{\delta'} \sum_{d \leq x/\delta} d = O\left(x^2 \sum_{\delta\delta' \leq X} \frac{1}{\delta'\delta^2}\right) = O_k\left(x^2 \log x\right). \tag{3.15}$$

Putting (3.15) and (3.14) into (3.8), we obtain

$$T_2 = \frac{AB(k)x^3}{3} + O_k\left(x^2 \log^4 x\right). \tag{3.16}$$

Theorem 3.1 follows from (3.16),(3.7), (3.6) and (3.2).

Theorem 3.2. We have

$$\sum_{n \leq x} \frac{\phi^*(n)\phi^*(n+k)}{n(n+k)} = xA^*B^*(k) + O_k(\log^4 x), \tag{3.17}$$

where A^* and $B^*(k)$ are as given in (2.12) and (2.14).

Proof : We have by Lemmas 2.8 and 2.7,

$$\sum_{n \leq x} \frac{\phi^*(n)\phi^*(n+k)}{n(n+k)} = \sum_{n \leq x} \frac{\phi^*(n)}{n} \sum_{\substack{d \mid n+k \\ (d, \frac{n+k}{d})=1}} \frac{\mu^*(d)}{d}$$

$$= \sum_{d \leq x+k} \frac{\mu^*(d)}{d} \sum_{\substack{n \leq x \\ n \equiv -k (\text{mod } d) \\ (\frac{n+k}{d}, d)=1}} \frac{\phi^*(n)}{n}$$

$$= \sum_{d \leq x+k} \frac{\mu^*(d)}{d} \sum_{r \mid d} \mu(r) \sum_{\substack{n \leq x \\ n \equiv -k (\text{mod } dr)}} \frac{\phi^*(n)}{n}$$

$$= \sum_{d \le x+k} \frac{\mu^*(d)}{d} \sum_{r|d} \mu(r) \left\{ \frac{x}{dr} A^* H^*(dr, k) + O_k(\log^2 x) \right\}$$

$$= xA^* \sum_{d \le x+k} \frac{\mu^*(d)}{d^2} \sum_{r|d} \frac{\mu(r) H^*(dr, k)}{r} + O_k \left(\log^2 x \sum_{d \le x+k} \frac{\tau(d)}{d} \right)$$

$$= xA^* \sum_{d \le x+k} \frac{\mu^*(d) I^*(d, k)}{d^2} + O_k(\log^4 x)$$

$$= xA^* B^*(k) + O_k(1) + O_k(\log^4 x)$$

$$= xA^* B^*(k) + O_k(\log^4 x)$$

Hence Theorem 3.2 follows.

From Theorem 3.2 and partial summation we obtain the formula

$$\sum_{n \le x} \phi^*(n) \phi^*(n+k) = \frac{x^3 A^* B^*(k)}{3} + O_k(x^2 \log^4 x). \tag{3.18}$$

4 Concluding remarks

Apart from (1.1) and (1.2), A.E.Ingham (cf.[3])established asymptotic formulae for the sums

$$\sum_{n \le x} \tau(n) \tau(n+k), \qquad \sum_{m=1}^{n-1} \tau(m) \tau(n-m)$$

$$\sum_{m=1}^{n-1} \sigma(m) \sigma(n-m) \text{ and } \sum_{m=1}^{n-1} \phi(m) \phi(n-m). \tag{4.1}$$

At present, we are unable to establish asymptotic formulae for the corresponding unitary analogues, and hope to investigate these sums in a separate paper.

The sums (1.1),(1.2),(4.1) of Ingham are extensions of similar ones considered by Ramanujan for the case $k = 0$. Consequently,sums of this kind may be called Ramanujan-Ingham sums,following such usage by Yorchi Motohashi (see p.176 of his paper "On the distribution of the divisor function in arithmetic progressions, Acta Arithmetica XXII (1973),175-199).

It is curious to note that the order of $\sum_{n\leq x}\sigma^*(n)\sigma^*(n+k)$, for a fixed poistive integer k, is the same as the order of this sum for $k=0$, for it can be proved without much difficulty that

$$\sum_{n\leq x}(\sigma^*(n))^2 = \frac{\zeta(2)}{3\zeta(3)}x^3 + O(x^2(\log x)^{1/3}).$$

This is unlike the situation for the function $\tau(n)$ (denoting the number of divisors of n) for which Ramanujan showed the well known formula

$$\sum_{n\leq x}\tau^2(n) \sim \frac{1}{\pi^2}x(\log x)^3,$$

where as Ingham proved in [3] that for a positive integer k,

$$\sum_{n\leq x}\tau(n)\tau(n+k) \sim \frac{6}{\pi^2}\sigma_{-1}(k)x(\log x)^2,$$

so that the sum above on the left has a lower order (by a factor of $\log x$) than for the same sum for $k=0$.

Acknowledgement

The authors thank the referee for his useful suggestions.

References

[1] E.Cohen, *Arithmetical functions associated with the unitary divisors of an integer,* Math.Z.74(1960),66-80.

[2] G.H.Hardy and E.M.Wright, *An introduction to the Thoery of Numbers,* Fourth Edition,Oxfords Univewrsity Press,1960.

[3] A.E.Ingham, *Some Asymptotic formulae in the theory of numbers,* Jour.London Math,Soc.2(1927),178-82.

[4] L.Mirsky, *Summation formulae involving arithmetic functions,* Duke Math.J.16(1949),261-272.

[5] D.Suryanarayana and R.Sita Rama Chandra Rao, *On* $\sum_{n\leq x}\sigma^*(n)$ *and* $\sum_{n\leq x}\phi^*(n)$, Proc.Amer.Math.Soc.41(1973),61-66.

Department of Mathematics,
Pondicherry Engineering College,
Pondicherry, 605 014,
India.
e-mail : ramaiahpec@yahoo.co.in

Department of Mathematical Sciences,
University of Alberta,
Edmonton,
Alberta,
Canada T6G2G1.
e-mail : m.v.subbarao@ualberta.ca