A remark on Suzuki groups

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Abstract

The maximality of the Suzuki group $Sz(K,\sigma)$, K any commutative field of characteristic 2 admitting an automorphism σ such that $x^{\sigma^2} = x^2$, in the symplectic group $PSp_4(K)$, is proved.
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1 Introduction

In [1], Aschbacher defined eight "geometric" classes $\mathcal{C}_1 \ldots, \mathcal{C}_8$, of subgroups of the finite classical groups, and proved that a maximal subgroup either belongs to one of these classes or has a non-abelian simple group as its generalized Fitting subgroup. In the latter case, we say that the group belongs to the class $\mathcal{S}$.

At least seven of the eight Aschbacher's classes can be described as stabilizers of geometric configurations and hence one might prefer a direct approach to the classification of maximal subgroups, which is free of the classification of finite simple groups, using the natural representations of classical groups and their geometry. This is the approach adopted by R.H. Dye and O. H. King, and A. Cossidente and O. H. King, see for instance [2], [3], [4].

In [7, Section 7] Flesner proved that if M is a maximal subgroup of $PSp_4(2^t)$ which contains no central elations or noncentered skew elations, then either $q = 2$ and M is isomorphic to A_6, or M contains certain normal subgroups M_1 and M_2 such that $1 \leq M_2 < M_1 \leq M$, where M/M_1 and M_2 are of odd order, and M_1/M_2 is isomorphic to $PSL_2(q')$ or $Sz(q')$, the Suzuki group for some power q' of 2.

In this short note our aim is to provide a direct geometric proof of the maximality of the Suzuki group $Sz(2^t)$ in $PSp_4(2^t)$, $t \geq 3$ odd.

2 The maximality

From [10] we recall the following fundamental theorem

Theorem 2.1. *Let $\mathcal{O}$ be an ovoid in $PG(3,q)$, where q is a power of 2. Define the mapping ϕ from the set of planes of $PG(3,q)$ into the set of points as follows: if E is the tangent plane of $\mathcal{O}$ at P, then $\phi(E) = P$. If E is not a tangent plane, then $\phi(E)$ is the nucleus of $E \cap \mathcal{O}$ in E. Then ϕ is a bijection and $\pi = (\phi, \phi^{-1})$ is a symplectic polarity of $PG(3,q)$. Moreover, each collineation of $PG(3,q)$ which fixes $\mathcal{O}$ centralizes π.*

A consequence of this theorem is that the automorphism group of $\mathcal{O}$ turns out to be a subgroup of $PSp_4(q)$. If $\mathcal{O}$ is the Suzuki–Tits ovoid of $PG(3,q)$, $q = 2^{2m+1}$, $m \geq 1$

$$\mathcal{O} = \{(0,1,0,0), (1,x,y,z) | z = xy + x^2 x^\sigma + y^\sigma\},$$

where σ is the automorphism of $GF(q)$, given by $x \mapsto x^{2^{m+1}}$, then the automorphism group of $\mathcal{O}$ is the simple group $Sz(q)$ [11]. Thus we get the group embedding $Sz(q) \leq PSp_4(q)$.

In [8, Lemma 2.11] Liebeck proved that the group $Sz(q)$ has exactly two orbits on points of the symplectic polar space $W_3(q)$ defined by π, namely the ovoid $\mathcal{O}$ and its complement in $W_3(q)$. Geometrically, this depends upon the fact that a plane E of $PG(3,q)$ is either a tangent plane of $\mathcal{O}$ or $E \cap \mathcal{O}$ consists of the $q+1$ points of an oval of E.

If K is any commutative field of characteristic 2 with $|K| > 2$ admitting an automorphism σ such that $x^{\sigma^2} = x^2$, for all $x \in K$, then it is still true that the Suzuki group $S_z(K,\sigma)$ has two orbits on the symplectic space $PG(3,K)$ (one of them being the Suzuki–Tits ovoid). Indeed one might argue as follows.

From [13], the group $Sz(K,\sigma)$ acts 2-transitively on $\mathcal{O}$. Fix a point $P \in \mathcal{O}$. Then we have also fixed the corresponding line ℓ_P of the corresponding Lüneburg spread [9, Chapter IV]. Take two points $Q', R' \in \ell_P \setminus \{P\}$ and take two points Q, R of $\mathcal{O}$ collinear with Q' and R', respectively. By the 2–transitivity of $Sz(K,\sigma)$ on $\mathcal{O}$, we can fix P and map Q to R. Hence Q' is mapped to R'. We have proved that $Sz(K,\sigma)$ acts transitively on each spread line minus the ovoid points. Since $Sz(K,\sigma)$ acts transitively on the spread lines, the result follows.

Theorem 2.2. *The group $Sz(K,\sigma)$ is maximal in $PSp_4(K)$.*

Proof. Suppose that $Sz(K,\sigma) < H \leq PSp_4(K)$. Since $Sz(K,\sigma)$ is the full stabilizer of $\mathcal{O}$ in $PSp_4(K)$ [11], it follows that H will act transitively on points of the symplectic space $W_3(K)$. Let A be the non-degenerate alternating form associated to π. Let $X \in PG(3,K) \setminus \mathcal{O}$. The map $\alpha : Y \to Y + A(Y,X)X$ is an elation of $PSp_4(K)$. Let $M \in \mathcal{O}$. By the transitivity of H there is an element $f \in H$ such that $f^{-1}(X) = M$. Hence

$$\begin{aligned}(f^{-1}\alpha f)Y &= f^{-1}(fY + A(fY,X)X)\\ &= Y + A(Y,f^{-1}X)f^{-1}X\\ &= Y + A(Y,M)M.\end{aligned}$$

Thus $f^{-1}\alpha f$ is an elation of $PSp_4(K)$ centered on M and fixes $\mathcal{O}$. Now, $f^{-1}\alpha f$ is in H and so all symplectic elations centered on M are in H. Since H is transitive on the isotropic points, it follows that H contains all elations of $PSp_4(K)$. But $PSp_4(K)$ is generated by its elations [5], [6], [12] and hence $PSp_4(K) \leq H$ and $Sz(K,\sigma)$ is maximal in $PSp_4(K)$. □

Remark 2.3. Notice that when $K = GF(2)$, the group $Sz(2)$ is not maximal in $PSp_4(2)$. It turns out that $PSp_4(2) \simeq S_6$ and $Sz(2) \simeq AGL(1,5)$ which is a Frobenius subgroup of order 20 of $P\Omega_4^-(2) \cdot 2 < PSp_4(2)$.

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