On the Domination Number of the Product of Two Cycles

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Abstract. Let G = (V, E) be a graph. A subset $D \subseteq V$ is called a dominating set for G if for every $v \in V-D$, v is adjacent to some vertex in D. The domination number $\gamma(G)$ is equal to min $\{|D| : D$ is a dominating numbers of G$\}$.

In this paper we calculate the domination numbers $\gamma(C_m \times C_n)$ of the product of two cycles C_m and C_n of lengths m and n for m = 5 and $n \equiv 3 \pmod 5$, also for m = 6, 7 and arbitrary n.

Keywords: Domination, Dominating sets, Graph products, Cycles.

1. Introduction.

Let G = (V,E) be a graph . A subset of vertices $D \subseteq V$ is called a dominating set for G if for every vertex v of G, either $v \in D$ or v is adjacent to some vertex in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G . For an extensive survey of domination problems and a comprehensive bibliography, we refer the reader to the survey volume [4] edited by Hedetniemi and Laskar.

The Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph with vertex $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, where two vertices $(v_1,v_2),(u_1,u_2) \in V(G_1 \times G_2)$ are adjacent if and only if either $v_1u_1 \in E(G_1)$ and $v_2 = u_2$ or $v_1 = u_1$ and $v_2u_2 \in E(G_2)$.

The determination of the domination number of the product of two graphs seems to be a difficult problem. For the product of two path P_m and P_n, the domination number $\gamma(P_m \times P_n)$ were calculated for m = 1, 2, 3, 4 by Jacobson and Kinch [2] and for m = 5, 6 by Chang and Clark [3] . Beyond m = 6, the problem seems to be getting more difficult.

For the product of two cycle, Klavzar and Seifter [1] proved the following Results:

$$\gamma(C_3 \times C_n) = \lceil 3n/4 \rceil \quad (1),$$
$$\gamma(C_4 \times C_n) = n \quad (2),$$
$$\gamma(C_5 \times C_n) = n \qquad n \equiv 0 \pmod 5 \quad (3),$$

$\gamma(C_5 \times C_n) = n + 1 \qquad n = 3 \text{ or } n \equiv 1,\ 2, 4 \pmod 5 \qquad (4),$

$\gamma(C_5 \times C_n) \le n + 2 \qquad n \equiv 3 \pmod 5 \qquad (5),$

The graph of the product of two cycles is a toroidal grid compared to planar grids Arising from products of paths. Naturally, the domination problem for such toroidal grids is expected to be easier than that for planar grids due to the extra symmetries. It even might break into finitely many cases.

In this paper, we calculate the domination number $\gamma(Cm\times C_n)$ for the remaining case of $m = 5$ and for $m = 6,7$ and arbitrary n.

2. Notations and terminology.

Let C_n denote an n –cycle with vertices 1, 2, …, n. Then
$C_m\times C_n = \{(i, j): 1 \le i \le m, 1 \le j \le n\}$. The jth column of $C_m\times C_n$ is
$K_j = \{(i, j) : i = 1, 2, \dots, m\}$.
Let D be a dominating set for $C_m\times C_n$. We put $W_j = D\cap K_j$. Let $sj = |W_j|$. The sequence $(s_1, s_2, \dots, s_n)$ is called the dominating sequence corresponding to D.

We define

$$X_i = |\{j: s_j = i\}|, \quad i = 0, 1, \dots, m.$$

Then we have

$$X_0 + X_1 + \dots + X_m = n.$$

$$|D| = X_1 + 2X_2 + \dots + mX_m.$$

Suppose that $s_j = 0$ for some j. The vertices of jth column can only be dominated by vertices of the (j-1)st and (j+1)st columns (addition of subscripts is modulo n). Thus we have $s_{j-1} + s_{j+1} \ge m$. In general we have $s_{j-1} + 3s_j + s_{j+1} \ge m$. we shall repeatedly make use of these facts. We shall also make use of the following useful lemma.

***Lemma* 1.** There is a minimum dominating set D for $C_m\times C_n$ with dominating sequence $(s_1, s_2, \dots, s_n)$ such that, for all $j = 1, 2, \dots, n$,

$$s_j \le \lfloor 3m/5 \rfloor, \quad m \equiv 0, 1, 3, 4 \pmod 5),\ m \ge 5,$$

$$s_j \le \lfloor 3m/5 \rfloor - 1, \quad m \equiv 2 \pmod 5).$$

Proof. Let D be a minimum dominating set for $C_m\times C_n$ with dominating sequence $(s_1, s_2, \dots, s_n)$. The idea of the proof is as follows. If for some j, s_j is large then we show how to modify D by moving two vertices from column j, one to column j-1 and the other one to column j+1 such that the resulting set is still a dominating one for $C_m\times C_n$. Repeating this process if necessary eventually leads to a dominating set with the required properties.

Consider that $1 \le j \le n$ and $1 \le i \le m$. Put $W_j = D \cap \{(i, j), (i+1, j), \ldots, (i+4, j)\}$. If $|W_j| \ge 4$ then we define $D' = (D - W_j) \cup \{(i, j), (i+2, j-1), (i+2, j+1), (i+4, j)\}$. Then D' is also a dominating set for $C_m \times C_n$. Thus we can assume that every five consecutive vertices of the jth column contain at most three vertices of D. This shows that $s_j \le \lfloor 3m/5 \rfloor$.
Let us now the case $m = 5k + 2$. We assume that $s_j = \lfloor 3m/5 \rfloor = 3k + 1$. We also assume that every five consecutive vertices of the jth column contain at most three vertices of D. The set W_j must contain two adjacent vertices, say $(1, j), (2, j) \in W_j$. If $W_j \cap \{(3, j), (4, j)\} \ne \varnothing$ and $W_j \cap \{(m-1, j), (m, j)\} \ne \varnothing$, then we put $D' = (D - \{(1, j), (2, j)\}) \cup \{(1, j-1), (2, j+1)\}$. Again D' is dominating set for $C_m \times C_n$. To finish the proof, assume that one of $W_j \cap \{(3, j), (4, j)\}$ and $W_j \cap \{(m-1, j), (m, j)\}$ is empty, say $W_j \cap \{(m-1, j), (m, j)\} = \varnothing$. This forces the 5k vertices $(1, j), \ldots, (m-2, j)$ to contain $3k + 1$ vertices of D which implies that some five consecutive vertices of them contain at least four vertices of D. This contradiction completes the proof of the lemma.□

3. The Domination number $\gamma(C_5 \times C_n)$.

***Theorem* 1.**

$\gamma(C_5 \times C_n) = n$ if $n \equiv 0 \pmod 5$ (6),

$\gamma(C_5 \times C_n) = n + 1$ if $n = 3$ or $n \equiv 1, 2, 4 \pmod 5$ (7),

$\gamma(C_5 \times C_n) = n + 2$ if $n \equiv 3 \pmod 5$, $n \ne 3$ (8),

Proof. **(1)** For $n \equiv 0, 1, 2, 4 \pmod 5$, see [1].

(2) $n \equiv 3 \pmod 5$. From (5) we have $\gamma(C_5 \times C_n) \le n + 2$. Aiming to get a contradiction, we assume D is a dominating set for $C_5 \times C_n$ with $|D| = n + 1$ and having a dominating sequence $(s_1, s_2, \ldots, s_n)$. We finish the proof through a series of Facts.

***Fact* 1.** If some $s_j = 0$ then $|D| = \sum_{j=1}^{n} s_j \ge n + 2$.

Proof. We have $n = X_0 + X_1 + X_2 + X_3$ and $|D| = X_1 + 2X_2 + 3X_3$. Suppose that $X_0 > 0$. Observe that if $s_j = 0$ then $s_{j-1} + s_{j+1} \ge 5$ and thus, by Lemma 1, one of s_{j-1}, s_{j+1} is at least 3 and the other is at least 2. We now consider two cases:

***Case* (a).** $X_0 \le 2$.
In this case we have $X_2 + X_3 \ge X_0 + 1$ and $X_3 \ge 1$. This implies that $|D| \ge n + 2$.

***Case* (b).** $X_0 \ge 3$.
In this case we have $X_2 + X_3 \ge X_0$ and $X_3 \ge X_0/2 \ge 2$. Again we can deduce that $|D| \ge n + 2$. This completes the proof of Fact 1.

***Fact* 2.** $\gamma(C_5 \times C_n) = n + 2$.

Proof. By Fact 1, we can assume that all s_j are ≥ 1. Since $\sum_{j=1}^{n} s_j = n+1$, then all s_j are equal to 1 except one of them which equals 2. Suppose that $s_1 = s_2 = \ldots = s_{n-1} = 1$, and that $s_n = 2$. Without loss of generality, we can assume that D contains the vertices (1, 1), (3, 2) and then deduce that D contains also the vertices (5, 3), (2, 4), (4, 5), (1, 6), ..., (3, n-1). In order to cover the vertices (4, 1), (5, n-1) we must have (5, n), (4, n)$\in$D and that must be all of D. However, this is a contradiction, since the vertex (2, n) is not covered by any vertex of D. This contradiction completes the proof of Fact 2 and thereby, finishes the proof of the theorem.□

4. The Domination number $\gamma(C_6 \times C_n)$.

***Theorem* 2.**

$\gamma(C_6 \times C_n) \leq \lceil 4n/3 \rceil$ if $n \equiv 0, 1, 4 \pmod 6$ (9),

$\gamma(C_6 \times C_n) \leq \lceil 4n/3 \rceil + 1$ if $n \equiv 2, 3, 5 \pmod 6$ and $n \not\equiv 5 \pmod{18}$ (10),

$\gamma(C_6 \times C_n) \leq \lceil 4n/3 \rceil$ if $n \equiv 5 \pmod{18}$ (11),

Proof. Let

$$D_1 = \{(1, 6k+1) : 0 \leq k \leq \lfloor (n-1)/6 \rfloor\} \cup \{(4, 6k+2) : 0 \leq k \leq \lfloor (n-2)/6 \rfloor\} \cup \{(2, 6k+3) : 0 \leq k \leq \lfloor (n-3)/6 \rfloor\} \cup \{(6, 6k+3) : 0 \leq k \leq \lfloor (n-3)/6 \rfloor\} \cup \{(4, 6k+4) : 0 \leq k \leq \lfloor (n-4)/6 \rfloor\} \cup \{(1, 6k+5) : 0 \leq k \leq \lfloor (n-5)/6 \rfloor\} \cup \{(3, 6k) : 1 \leq k \leq \lfloor n/6 \rfloor\} \cup \{(5, 6k) : 1 \leq k \leq \lfloor n/6 \rfloor\},$$

$$D_2 = \{(4, 1)\}, \quad D_3 = \{(3, n), (5, n)\},$$

$$D'_1 = \{(j, 3k+1) : 0 \leq k \leq \lfloor (n-1)/3 \rfloor, 1 \leq j \leq 6, j \equiv k+1 \pmod 6\} \cup \{(j, 3k+2) : 0 \leq k \leq \lfloor (n-2)/3 \rfloor, 1 \leq j \leq 6, j \equiv k+3 \pmod 6\} \cup \{(j, 3k) : 1 \leq k \leq \lfloor n/3 \rfloor, 1 \leq j \leq 6, j \equiv k+4 \pmod 6\} \cup \{(j, 3k) : 1 \leq k \leq \lfloor n/3 \rfloor, 1 \leq j \leq 6, j \equiv k+5 \pmod 6\},$$

$$D'_2 = \{(5, n)\}.$$

The sets D_1, D'_2 are illustrated in Figures 1, 2 respectively for n = 9, 18.
We can check that $|D_1| = |D'_1| = \lfloor 4n/3 \rfloor$. Furthermore, each of D_1, D'_1 covers all the vertices belonging to columns 2, 3, ..., n-1. This implies that the following sets are dominating sets for $C_6 \times C_n$ as indicated:

D_1	when $n \equiv 0 \pmod 6$,
$D_1 \cup D_2$	when $n \equiv 1, 4 \pmod 6$,
$D_1 \cup D_3$	when $n \equiv 2, 3, 5 \pmod 6$,
$D'_1 \cup D'_2$	when $n \equiv 5 \pmod{18}$.

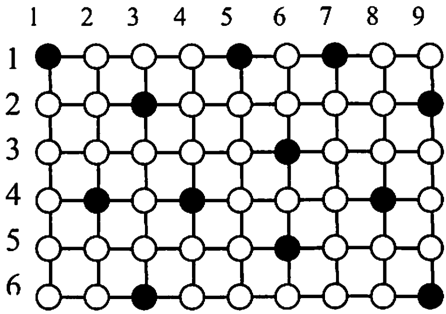

Figure 1. The set D_1.

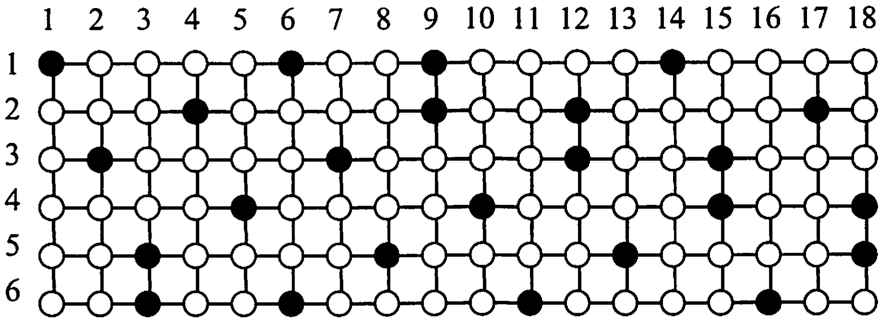

Figure 2. The set D'_2.

We note that these sets have the appropriate cardinalities. This completes the proof of the theorem. □

***Lemma* 2.** $\gamma(C_6 \times C_n) \geq \lceil 4n/3 \rceil$.

Proof. Let D be a minimum dominating set for $C_6 \times C_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$. By Lemma 1, we can assume that each $s_j \leq 3$. Then we have

$$n = X_0 + X_1 + X_2 + X_3 \qquad (12),$$

$$|D| = X_1 + 2X_2 + 3X_3 \qquad (13).$$

Observe that if $s_j = 0$, then $s_{j-1} + s_{j+1} \geq 6$ which implies that $s_{j-1} = s_{j+1} = 3$. Also, if $s_j = 1$, then $s_{j-1} + s_{j+1} \geq 3$ and this implies that at least one of s_{j-1}, s_{j+1} is ≥ 2.

***Case* 1.** $X_0 = n/2$ (where n is even). This implies that $X_3 = n/2$. Then we have $\gamma(C_6 \times C_n) = 3X_3 = 3n/2 \geq \lceil 4n/3 \rceil$.

Case **2.** $0 < X_0 < n/2$. The observations we made above implies that

$$X_0 + 1 \le X_3 \quad (14),$$

$$X_0 + X_1/2 \le X_2 + X_3 \quad (15).$$

We claim that

$$X_2 + X_3 \ge n/3 \quad (16).$$

Suppose that $X_2 + X_3 < n/3$. Then using (12), (14) and (15), we get

$$n < 2X_0 + X_1 + X_2 + X_3 \le 2(X_2 + X_3) + X_2 + X_3 < 3(n/3) = n,$$

which is a contradiction. Now we have

$$\gamma(C_6 \times C_n) = X_1 + 2X_2 + 3X_3 = n - X_0 + X_2 + 2X_3$$
$$\ge n + X_2 + X_3 + 1 \ge n + n/3 + 1 > \lceil 4n/3 \rceil.$$

Case **3.** $X_0 = 0$. Here we have

$$X_2 + X_3 \ge X_1/2.$$

Then we get

$$n = X_1 + X_2 + X_3 \ge 3X_1/2$$

This implies that

$$\gamma(C_6 \times C_n) = X_1 + 2X_2 + 3X_3 = n + X_2 + 2X_3$$
$$\ge n + X_2 + X_3 = 2n - X_1 \ge 4n/3.$$

This completes the proof of the lemma. □

***Theorem* 3.**

$$\gamma(C_6 \times C_n) = \lceil 4n/3 \rceil \qquad n \equiv 0, 1, 4 \pmod 6 \quad (17),$$
$$\gamma(C_6 \times C_n) = \lceil 4n/3 \rceil + 1 \qquad n \equiv 2, 3, 5 \pmod 6 \text{ but } n \not\equiv 5 \pmod{18} \quad (18),$$
$$\gamma(C_6 \times C_n) = \lceil 4n/3 \rceil \qquad n \equiv 5 \pmod{18} \quad (19).$$

Proof. Theorem 2 together with Lemma 2 implies (17) and (19). Assume that $n \equiv 2, 3$ or $5 \pmod 6$, where $n \not\equiv 5 \pmod{18}$. Further assume that $n \ge 8$. Aiming to get a contradiction, we suppose that there is a dominating set D for $C_6 \times C_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$ where $|D| = \lceil 4n/3 \rceil$. By Lemma 1, we can assume that each $s_j \le 3$. The proof of Lemma 2 shows that in cases 1 and 2 we have $\gamma(C_6 \times C_n) > \lceil 4n/3 \rceil$. Thus we can assume that $X_0 = 0$, that is, we are in case 3 of Lemma 2. Since $\gamma(C_6 \times C_n) = \lceil 4n/3 \rceil$ then we must have $X_1 = \lfloor 2n/3 \rfloor$, $X_2 = \lceil n/3 \rceil$ and $X_3 = 0$. Thus the sequence $(s_1, s_2, \ldots, s_n)$ consists of 1's and 2's. Furthermore if $s_j = 1$ then at least one of s_{j-1} and s_{j+1} is equal to 2. Hence we can assume that

$s_j = 1$ if $j \equiv 1$ or $2 \pmod 3) : j \neq n$,
$s_j = 2$ if $j \equiv 0 \pmod 3$ or $j = n$.

Assume, without loss of generality, that $(1, 1) \in D$. If one of the vertices (1, 2), (2, 2), (6, 2) belongs to D, then this vertex together with the vertex (1, 1) and the two vertices of D in column 3 will not suffice to dominate all the vertices of the second column. Therefore we must have that D contains exactly one of (3, 2), (4, 2), (5, 2). The cases $(3, 2) \in D$ and $(5, 2) \in D$ are similar by symmetry. Thus we are left with two cases.

Case **1.** $(3, 2) \in D$.
Here, in order that the two vertices (5, 2), (6, 2) be covered, we must have that $(5, 3), (6, 3) \in D$. Similarly, in order that the vertex (2, 3) be covered, we must have that $(2, 4) \in D$. Continuing in this argument we see that D coincides, up to the (n- 1)st column, with the set D'_1 defined in the proof of Theorem 2. However, then there will be no possible choice for the remaining two vertices of D in the nth column in order that D will be a dominating set for $C_6 \times C_n$.

Case **2.** $(4, 2) \in D$.
Here, we use an argument similar to the Case $(3, 2) \in D$ to show that D contains the vertices (4, 2), (2, 3), (6, 3), (4, 4), ... Now D will coincide, up to the (n-1)st column, with the set D_1 defined in the proof of Theorem 2, and again, there will be no possible choice for the remaining two vertices of D in the nth column. From the above two cases, we see that the assumption that $|D| = \lceil 4n/3 \rceil$ leads to a contradiction. Thus we have $\gamma(C_7 \times C_n) \geq \lceil 4n/3 \rceil + 1$. This inequality together with (10) implies (18) which completes the proof of the theorem. □

5. The Domination number $\gamma(C_7 \times C_n)$.

Theorem **4.** For $n \geq 7$

$$\gamma(C_7 \times C_n) \leq \lceil 3n/2 \rceil \qquad n \equiv 0, 5, 9 \pmod{14} \qquad (20),$$
$$\gamma(C_7 \times C_n) \leq \lceil 3n/2 \rceil + 1 \qquad n \equiv 1, 3, 4, 6, 7, 10, 11, 13 \pmod{14} \qquad (21),$$
$$\gamma(C_7 \times C_n) \leq 3n/2 + 2 \qquad n \equiv 2, 8, 12 \pmod{14} \qquad (22).$$

Proof. Let

$$\begin{aligned} D_1 = \{(k, 2j + 1) &: 0 \leq j \leq (n-1)/2,\ 1 \leq k \leq 7,\ k \equiv j + 1 \pmod 7\} \cup \\ \{(k, 2j) &: 1 \leq j \leq n/2,\ 1 \leq k \leq 7,\ k \equiv j + 3 \pmod 7\} \cup \\ \{(k, 2j) &: 1 \leq j \leq n/2,\ 1 \leq k \leq 7,\ k \equiv j + 5 \pmod 7\} \end{aligned} \qquad (23).$$

It is clear that $|D_1| = \lfloor 3n/2 \rfloor$. We can check that the following sets are dominating sets for $C_7 \times C_n$ as indicated :

D_1	when $n \equiv 0 \pmod{14}$,
$D_1 \cup \{(3, 1)\}$	when $n \equiv 4, 9 \pmod{14}$,
$D_1 \cup \{(4, 1)\}$	when $n \equiv 6 \pmod{14}$,
$D_1 \cup \{(6, 1)\}$	when $n \equiv 5, 10 \pmod{14}$,
$D_1 \cup \{(4, 1)\} \cup \{(5, n)\}$	when $n \equiv 1, 8 \pmod{14}$,
$D_1 \cup \{(4, 1)\} \cup \{(1, n)\}$	when $n \equiv 2, 7 \pmod{14}$,
$D_1 \cup \{(4, 1)\} \cup \{(6, n)\}$	when $n \equiv 3 \pmod{14}$,
$D_1 \cup \{(4, 1)\} \cup \{(3, n)\}$	when $n \equiv 11 \pmod{14}$,
$D_1 \cup \{(4, 1)\} \cup \{(7, n)\}$	when $n \equiv 12 \pmod{14}$,
$D_1 \cup \{(4, 1)\} \cup \{(4, n)\}$	when $n \equiv 13 \pmod{14}$,

(We illustrate the cases n = 14 and n = 12 in Figures 3 and 4 respectively). This completes the proof of the theorem.□

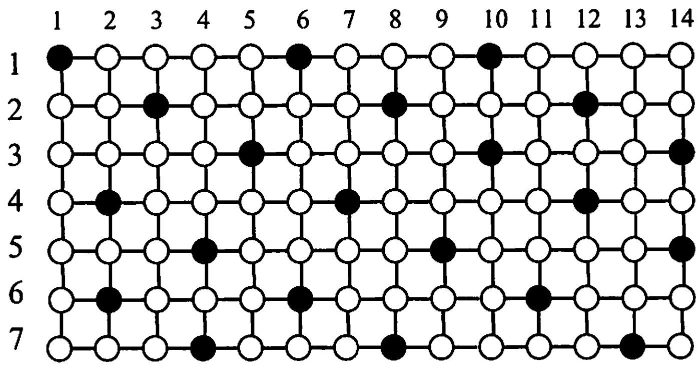

Figure 3. A dominating set for $C_7 \times C_{14}$.

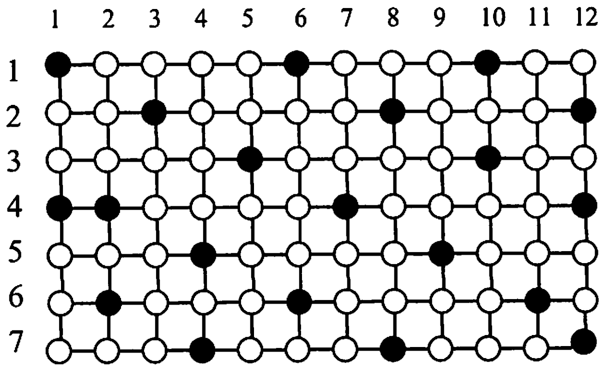

Figure 4. A dominating set for $C_7 \times C_{12}$.

***Lemma* 3.** $\gamma(C_7 \times C_n) \geq \lceil 3n/2 \rceil$ (24).

Proof. Suppose that D is a dominating set for $C_7 \times C_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$. From Lemma 1, we can assume that $1 \leq s_j \leq 3$. we have

$$X_1 + X_2 + X_3 = n \quad (25).$$

The graph $C_7 \times C_n$ is 4-regular so that each vertex in D covers 4 vertices other than itself. Thus we have $5X_1 + 10X_2 + 15X_3 \geq 7n$. However this inequality can be strengthened due to fact that some vertices of $C_7 \times C_n$ are doubly dominated, i.e., dominated by more than one vertex of D. We write this inequality as $5X_1 + 10X_2 + 15X_3 - N \geq 7n$ where N is a lower bound on the number of doubly dominated vertices. We wish to estimate N. Since no $s_j = 0$, then in any column, say the jth one, at least one vertex is dominated by a vertex from W_{j-1} and at least another one is dominated by a vertex from W_{j+1}. Consider now a column j for which $s_j = 2$. Each one of the two vertices of W_j dominates two vertices in the jth column other than itself. Thus the jth column must contain at least one vertex which is doubly dominated. A similar argument shows that if $s_j = 3$ then the jth column contributes 4 to the value of N. Hence we take $N = X_2 + 4X_3$. Then we have

$$5X_1 + 9X_2 + 11X_3 \geq 7n \quad (26).$$

From (25) and (26) we get that $4X_2 + 6X_3 \geq 2n$, that is

$$2X_2 + 3X_3 \geq n \quad (27).$$

Now, using (25) we have $|D| = X_1 + 2X_2 + 3X_3 = n + X_2 + 2X_3$. From (27), we get

$$|D| = n + X_2 + 2X_3 \geq n + n/2 + X_3/2$$

That is

$$|D| \geq 3n/2 + X_3/2 \quad (28).$$

This implies that $|D| \geq \lceil 3n/2 \rceil$ as required.□

***Lemma* 4.** $\gamma(C_7 \times C_n) = \lceil 3n/2 \rceil$ for $n \equiv 0, 5, 9 \pmod{14}$.

Proof. This follows from (20) and (24).□

***Lemma* 5.** If $\gamma(C_7 \times C_n) = \lceil 3n/2 \rceil$ then $X_3 = 0$.

Proof. Assume that D is a dominating set for $C_7 \times C_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$. Where $|D| = \lceil 3n/2 \rceil$. If n is even then the required result follows from (28). Let n be odd. Then (28) implies that $X_3 \leq 1$. Suppose that $X_3 = 1$. We have

$$\lceil 3n/2 \rceil = |D| = n + X_2 + 2X_3 = n + X_2 + 2$$

Then $X_2 = \lceil n/2 \rceil - 2$ and therefore, $X_1 = \lceil n/2 \rceil$. This implies that for some j, we have $s_j = s_{j+1} = 1$. We then have $s_{j-1} = s_{j+2} = 3$ which is a contradiction. This completes the proof of the lemma.□

Let us introduce a definition. Suppose that D is a dominating set for $C_7 \times C_n$ and assume that $1 \leq j, k \leq n$. We say that the kth column of D is an m-shift of its jth column if $(t, j) \in D \Leftrightarrow (t + m, k) \in D$ where the indices t, t + m are reduced modulo 7.

***Lemma* 6.** Let D be a dominating set for $C_7 \times C_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$. Assume that for some j we have $s_{j-2} = s_j = s_{j+2} = 1$ and $s_{j-1} = s_{j+1} = 2$. Then the two vertices of D in each of columns j-1 and j+1 are at distance 2 apart. Moreover, for k = j-2, j-1, j, column k+2 is a 1-shift (res. a 6-shift) of column k.

Proof. Without loss of generality we assume that $W_j = \{(1, j)\}$. Then for each $3 \leq k \leq 6$, we have either $(k, j+1) \in W_{j+1}$ or $(k, j-1) \in W_{j-1}$. If $W_{j+1} = \{(3, j+1), (4, j+1)\}$ then in order to cover the vertices (6, j+1), (7, j+1) we must have $(6, j+2), (7, j+2) \in W_{j+2}$. This contradicts the fact that $s_{j+2} = 1$. Similarly, if $W_{j+1} = \{(4, j+1), (5, j+1)\}$ or $W_{j+1} = \{(5, j+1), (6, j+1)\}$ then $s_{j+2} \geq 2$ which is a contradiction. Finally if $W_{j+1} = \{(3, j+1), (6, j+1)\}$ then $W_{j-1} = \{(4, j-1), (5, j-1)\}$ which implies that $s_{j-2} \geq 2$. Thus we are left with only two possibilities:

> $W_{j+1} = \{(4, j+1), (6, j+1)\}$ and $W_{j-1} = \{(3, j-1), (5, j-1)\}$. This implies that $W_{j-2} = \{(7, j-2)\}$ and $W_{j+2} = \{(2, j+2)\}$ or
>
> $W_{j+1} = \{(3, j+1), (5, j+1)\}$ and $W_{j-1} = \{(4, j-1), (6, j-1)\}$. This implies that $W_{j-2} = \{(2, j-2)\}$ and $W_{j+2} = \{(7, j+2)\}$.

The required result now follows.□

***Lemma* 7.** If n is even and $n \not\equiv 0 \pmod{14}$ then $\gamma(C_7 \times C_n) > 3n/2$. In particular, $\gamma(C_7 \times C_n) = 3n/2 + 1$ for $n \equiv 4, 6, 10 \pmod{14}$.

Proof. Assume that n is even but $n \not\equiv 0 \pmod{14}$. By Lemma 3, we have $\gamma(C_7 \times C_n) \geq 3n/2$. Suppose that D is a dominating set for $C_7 \times C_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$. Where $|D| = 3n/2$. From Lemma 5, we get that $X_3 = 0$. Then $X_1 = X_2 = n/2$. It follows that the sequence $s_1, s_2, \ldots, s_n$ consists of alternating 1's and 2's. From Lemma 6 every jth column is a 1-shift (res. a 6-shift) of the (j-2)nd column. This implies that n is divisible by 14 which is a contradiction. Thus $\gamma(C_7 \times C_n) \geq 3n/2 + 1$. Now if $n \equiv 4, 6, 10 \pmod{14}$ then (21) imply the required result.□

Lemma **8.** $\gamma(C_7 \times C_n) = \lceil 3n/2 \rceil + 1$ for $n \equiv 1, 3, 7, 11, 13 \pmod{14}$.

Proof. Suppose $n \equiv 1, 3, 7, 11, 13 \pmod{14}$. From Lemma 3, we get that $\gamma(C_7 \times C_n) \geq \lceil 3n/2 \rceil$. Suppose that D is a dominating set for $C_7 \times C_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$. Where $|D| = \lceil 3n/2 \rceil$. From Lemma 5, we have $X_3 = 0$. Hence $X_1 = \lfloor n/2 \rfloor$, $X_2 = \lceil n/2 \rceil$. We can assume that

$s_j = 1$ for $1 \leq j \leq n-2$, j odd,
$s_j = 2$ for $2 \leq j \leq n-1$, j even,
$s_n = 2$.

We can also assume that $W_1 = \{(1, 1)\}$, $W_2 = \{(4, 2), (6, 2)\}$ and $W_n = \{(3, n), (5, n)\}$. Now we deduce that column 2 is a 1-shift of column n, column 4 is a 1-shift of column 2, ... column (n - 1) is a 1-shift of column (n – 3). Thus the (n – 1)st column is a k-shift of the nth column where $1 \leq k \leq 7$ and $(n - 1)/2 \equiv k \pmod 7$. We have k = 0, 1, 3, 5 or 6 respectively when $n \equiv 1, 3, 7, 11, 13 \pmod{14}$. Hence $W_{n-1} = \{(3+k, n-1), (5+k, n-1)\}$ where the first index is reduced modulo 7. Now the vertex (7, n) is not dominated since $(7,1) \notin W_1$ and $(7, n-1) \notin W_{n-1}$. This contradiction shows that $\gamma(C_7 \times C_n) \geq \lceil 3n/2 \rceil + 1$. From (21), we get the result.□

Lemma **9.** Let $n \equiv 2, 8$ or $12 \pmod{14}$. Suppose there is a dominating set D for $C_7 \times C_n$ with $|D| = 3n/2 + 1$. Then $X_3 = 0$.

Proof. From (28), we get that $X_3 \leq 2$.

Case (a). $X_3 = 2$.
We have $3n/2 + 1 = |D| = n + X_2 + 2X_3$. Then $X_2 = n/2 - 3$ and $X_1 = n/2 + 1$. This implies that there are two distinct values of j for which $s_j = s_{j+1} = 1$. There cannot be three consecutive terms in the dominating sequence which equals 1. Also if $s_j = s_{j+1} = 1$ then $s_{j-1} = s_{j+2} = 3$. This implies that $X_3 \geq 3$, which is a contradiction.

Case (b). $X_3 = 1$.
Here we have $X_2 = n/2 - 1$ and $X_1 = n/2$. We can assume that the dominating sequence has the form :

$s_j = 1$ for j odd,
$s_j = 2$ for $2 \leq j \leq n - 2$, j even,
$s_n = 3$.

We can also assume that $W_1 = \{(1, 1)\}$ and $W_2 = \{(4, 2), (6, 2)\}$. This implies that the (n -1)st' column is a (n/2 -1)-shift of the first column. Assume first that $n \equiv 2 \pmod{14}$. Then the (n -1)st column coincides with the first column. We conclude that $W_{n-1} = \{(1, n -1)\}$ and $W_{n-2} = \{(3, n-2), (5, n-2)\}$. In order to cover all the vertices of the (n -1)st column we must have $(4, n), (6, n) \in W_n$ and in order to cover the vertices of the first column we must have $(3, n), (5, n) \in W_n$. This is a contradiction since $s_n = 3$. Similar calculations show that $\{(2, n), (7, n), (3, n), (5, n)\} \subseteq W_n$ when $n \equiv 8 \pmod{14}$ and that $\{(2, n), (4, n), (3, n), (5, n)\} \subseteq W_n$, again a contradiction. This completes the proof of the lemma.□

The remaining cases for $\gamma(C_7 \times C_n)$ are when $n \equiv 2, 8, 12 \pmod{14}$. We shall prove only the case $n \equiv 8 \pmod{14}$, the proofs for the other values of n are similar and will be omitted. For this end we further need a lemma.

***Lemma* 10.** $\gamma(C_7 \times C_8) = 14$.

Proof. From (22) and Lemma 7 we have $13 \leq \gamma(C_7 \times C_8) \leq 14$. Suppose that $\gamma(C_7 \times C_8) = 13$ and let D be a dominating set for $C_7 \times C_8$ with dominating sequence $(s_1, s_2, \ldots, s_8)$, where $|D| = 13$. By Lemmas 1 and 9 we can assume that $1 \leq s_j \leq 2$. Thus $X_1 = 3$ and $X_2 = 5$. We have two cases:

***Case* (a).** The dominating sequence has a subsequence ..., 1, 2, 2, 2, 1, ... We can assume that

$s_j = 1$ for $j = 1, 3, 5$,
$s_j = 2$ otherwise.

We further assume that $W_1 = \{(1, 1)\}$ and $W_2 = \{(4, 2), (6, 2)\}$. This implies that $W_3 = \{(2, 3)\}$, $W_4 = \{(5, 4), (7, 4)\}$ and $W_5 = \{(3, 5)\}$. In order to cover the vertices (3, 1), (5, 1) we must have $(3, 8), (5, 8) \in W_8$, i.e., $W_8 = \{(3, 8), (5, 8)\}$. Also to cover the vertices (6, 5), (1, 5) we need $W_6 = \{(6, 6), (1, 6)\}$. Finally, in order to cover the vertices (4, 6) and (7, 8), we deduce that $W_7 = \{(4, 7), (7, 7)\}$. However the vertex (2, 7) is not covered which is a contradiction.

***Case* (b).** No three consecutive s_j are equal to 2. We assume that

$s_j = 1$ for $j = 1, 3, 6$,
$s_j = 2$ otherwise.

Let $W_1 = \{(1, 1)\}$. Since $s_1 = s_3 = 1$, then the two vertices of W_2 must cover at least 5 vertices in the second column, thus these two vertices of W_2 cannot be adjacent. Therefore W_2 is equal to one of the sets $\{(3, 2), (5, 2)\}$, $\{(4, 2), 6, 2)\}$ and $\{(3, 2), (6, 2)\}$. The cases $W_2 = \{(3, 2), (5, 2)\}$ and $W_2 = \{(4, 2), (6, 2)\}$ are similar by symmetry.

***Subcase* (b.1).** $W_2 = \{(4, 2), (6, 2)\}$.
We deduce that $W_8 = \{(3, 8), (5, 8)\}$, $W_3 = \{(2, 3)\}$ and $W_4 = \{(5, 4), (7, 4)\}$. Furthermore, the vertices (3, 4) and (7, 8) require that $(3, 5) \in W_5$ and $(7, 7) \in W_7$ respectively. The set D contains further three other vertices, one on each of the 5th, 6th and 7th columns, and these vertices have to cover those vertices of these columns which are not yet covered. However, as can be seen from Figure 5, there is no possible choice for these extra vertices of D.

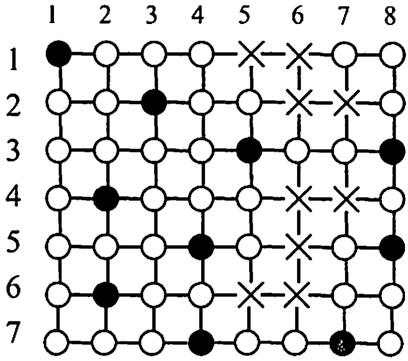

Figure 5. The vertices marked X have to be covered.

***Subcase* (b.2).** $W_2 = \{(3, 2), (6, 2)\}$.
In order to cover the vertices (4, 1), (5, 1) we must have that $W_8 = \{(4, 8), (5, 8)\}$. Also in order that the vertices (2, 8), (7, 8) be covered, we get that $W_7 = \{(2, 7), (7, 7)\}$. Since $s_2 = s_4 = 2$ and $s_3 = 1$, then column 3 contains no doubly dominated vertices and this implies that we have no choice but $W_3 = \{(1, 3)\}$. This implies that $W_4 = \{(4, 4), (5, 4)\}$. Now the vertices (2, 4) and (7, 4) require that $W_5 = \{(2, 5), (7, 5)\}$. However this is a contradiction since no single vertex of column 6 would cover those vertices of this column which are not yet covered. This shows that $\gamma(C_7 \times C_8) = 14$.□

***Theorem* 5.** $\gamma(C_7 \times C_n) = 3n/2 + 2$ for $n \equiv 8 \pmod{14}$.

Proof. We shall use induction on $n \equiv 8 \pmod{14}$. The Theorem is true for n = 8 by the previous Lemma. Assume that $n \equiv 8 \pmod{14}$ where $n \geq 22$. From (22) and Lemma 7 we have $3n/2 + 1 \leq \gamma(C_7 \times C_n) \leq 3n/2 + 2$. Aiming to get a contradiction we assume that $\gamma(C_7 \times C_n) = 3n/2 + 1$ and let D be a dominating set $C_7 \times C_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$ where $|D| = 3n/2 + 1$. By Lemmas 1 and 9 we can assume that $1 \leq s_j \leq 2$. Thus $X_1 = n/2 - 1$ and $X_2 = n/2 + 1$. Suppose that there is an index $1 \leq k \leq n$ such that

$$s_k = s_{k+2} = s_{k+4} = \ldots = s_{k+14} = 1,$$
$$s_{k+1} = s_{k+3} = s_{k+5} = \ldots = s_{k+13} = 2.$$

This implies that column k + 14 is a 0-shift of the kth one. If we delete columns k +1, k +2, ..., k +13 and identify the (k +14)th column with the kth one then we get a dominating set for $C_7 \times C_{n-14}$ having cardinality $3(n-14)/2 + 1$. This contradicts the hypothesis of the induction. Therefore we assume that there is no alternating subsequence of the form 1, 2, 1, 2, 1, ..., 2, 1 whose length is greater than or equal to 15. This can only happen for n = 22 where dominating sequence has the form 2, 2, S_1, 2, 2, S_2 where S_1 and S_2 denote two subsequences of the form 1, 2, 1, ..., 2, 1 whose lengths is less than 15. Thus the length of each of S_1, S_2 is between 5 and 13 inclusive.

We assume that $s_1 = s_2 = 2$, $s_3 = 1$, $s_4 = 2$, $s_5 = 1$, ..., $s_{2m} = s_{2m+1} = 2$, $s_{2m+2} = 1$, $s_{2m+3} = 2$, ..., $s_{21} = 2$ and $s_{22} = 1$ where $4 \le m \le 8$. The subsequence $s_2, s_3, \ldots, s_{2m}$ imply that column 2m is an (m-1)-shift of column 2. Similarly, the subsequence $s_{2m+1}, s_{2m+2}, \ldots, s_n, s_1$ imply that the first column is an (11-m)-shift of column 2m+1. We would like to calculate the possible shift from column 1 to column 2 and from column 2m to column 2m + 1. With no loss of generality, we assume that $W_2 = \{(4, 2), (6, 2)\}$ and $W_3 = \{(2, 3)\}$. Then $(1, 1) \in W_1$. Thus we have $W_1 = \{(1, 1), (3, 1)\}$ or $W_1 = \{(6, 1), (1, 1)\}$. This implies that the second column is either a 3-shift or a 5-shift of the first column. In a similar way column 2m + 1 is a 3-shift or a 5-shift of the first column . Summing up the shift from column 1 to column 2 then to column 2m, then to column 2m +1 and back to column 1 we get the value (3 or 5) + (m - 1) + (3 or 5) + (11 – m). This value is, therefore, equal to 16, 18, or 20. On the other hand this value of the shift has to be a multiple of 7. This contradiction completes the proof of the theorem.□

We summarize the results of this section in the following

***Theorem* 6.** For $n \ge 7$

$\gamma(C_7 \times C_n) = \lceil 3n/2 \rceil$ $\quad n \equiv 0, 5, 9 \pmod{14}$,

$\gamma(C_7 \times C_n) = \lceil 3n/2 \rceil + 1$ $\quad n \equiv 1, 3, 4, 6, 7, 10, 11, 13 \pmod{14}$,

$\gamma(C_7 \times C_n) = \lceil 3n/2 \rceil + 2$ $\quad n \equiv 2, 8, 12 \pmod{14}$.□

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