Edge Complete (p,2) Semigraphs

K. Kayathri
Department of Mathematics, Thiagarajar College, Madurai-625 009.
e-mail: kayathrikanagavel@yahoo.com
S. Pethanachi Selvam
Department of Mathematics, The Standard Fireworks Rajaratnam College for Women, Sivakasi-626 123. e-mail: pethanachiselvam@yahoo.com

Abstract. A semigraph G is an ordered pair (V,X) where V is a non-empty set whose elements are called vertices of G and X is a set of n-tuples ($n \geq 2$), called edges of G, of distinct vertices satisfying the following conditions:

i) any edge $(v_1, v_2, ..., v_n)$ of G is the same as its reverse $(v_n, v_{n-1}, ..., v_1)$, and ii) any two edges have atmost one vertex in common.

Two edges are adjacent if they have a common vertex. G is edge complete if any two edges in G are adjacent. In this paper, we enumerate the non-isomorphic edge complete (p,2) semigraphs.

Keywords: Semigraph, edge complete, m-vertex, end vertex.

Introduction

Sampathkumar [1] introduced a new generalization of Graphs called **Semigraphs**. The edges of a Graph G can be interpreted in the following two ways:
A. Each edge uv of G is a 2-element subset of the vertex set V of G.
B. Edges of G are 2-tuples (u, v) of vertices of G satisfying the following: (u, v) and (u', v') are equal iff either $u = u'$ and $v = v'$ or $u = v'$ and $v = u'$.

The hypergraph theory generalizes graphs using the Approach A, whereas the semigraph theory generalizes graphs using the Approach B.

A semigraph G is an ordered pair (V,X) where V is a non-empty set whose elements are called vertices of G and X is a set of n-tuples ($n \geq 2$), called edges of G, of distinct vertices satisfying the following conditions:
i) any edge $(v_1, v_2, ..., v_n)$ of G is the same as its reverse $(v_n, v_{n-1}, ..., v_1)$, and ii) any two edges have at most one vertex in common.

Linear hypergraphs are hypergraphs where each edge has cardinality at least two, and two edges have at most one vertex in common. Semigraphs can be regarded as linear hypergraphs, where **the vertices in each edge are arranged in a given order**.

Let G = (V, X) be a semigraph and let $E = (v_1, v_2, ..., v_n)$ be an edge of G. The **end vertices** of E are v_1 and v_n and the **middle vertices** or **m-vertices** of E are v_i, $2 \leq i \leq n-1$. In diagrammatical representations of semigraphs **thick dots** denote end vertices of an edge and **small circles** denote middle vertices of an edge. If an m-vertex of an edge E_1 is an end vertex of another edge E_2, we draw a small tangent to the circle at the end of the edge E_2. Two edges are **adjacent** if they have a common vertex.

Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two semigraphs. G_1 is **isomorphic** to G_2 if there exists a bijection $f : V_1 \to V_2$ such that $E = (v_1, v_2, ..., v_n)$ is an edge in G_1 iff $(f(v_1), f(v_2), ..., f(v_n))$ is an edge in G_2. In this case, we denote $(f(v_1), f(v_2), ..., f(v_n))$ as $f(E)$.

A semigraph G is **edge complete** if any two edges in G are adjacent. A semigraph with p vertices and q edges is referred to as a (p,q) semigraph.

Sampathkumar posed the problem of enumerating the edge complete semigraphs with $p \geq 6$. In this paper, we consider all the possible edge complete $(p, 2)$ semigraphs.

If G is an edge complete $(p, 2)$ semigraph, then the two edges of G have a common vertex (say) v. Now, these semigraphs can be classified into three types as follows:

Type 1: v is an end vertex of both the edges.

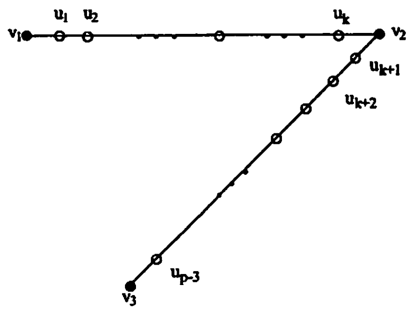
Type 2: v is an end vertex of one edge and is an m-vertex of another edge.

Type 3: v is an m-vertex of both the edges.

Theorem 1: *If G is a semigraph of Type i and H is a semigraph of Type j $(j \neq i)$, i, $j \in \{1, 2, 3\}$, then G is not isomorphic to H.*

Proof: If f is an ismorphism between G and H, then a vertex v is an end vertex (m-vertex, respectively) of E iff $f(v)$ is an end vertex (m-vertex, respectiely) of $f(E)$. Hence the result follows easily.

Type 1: The common vertex is an end vertex of both the edges



G_k

Let G_k denote a $(p, 2)$ semigraph of Type 1, where E_1 has k middle vertices (and E_2 has $p - 3 - k$ middle vertices). Note that for a fixed k, there is only one non-isomorphic semigraph G_k (of Type 1).

Let $G_k = (V, X)$ with

$V = \{v_1,\ v_2,\ v_3,\ u_1,\ u_2,\ \ldots,\ u_{p-3}\}$,

$X = \{E_{1k},\ E_{2k}\}$,

$E_{1k} = (v_1,\ u_1,\ u_2,\ \ldots,\ u_k,\ v_2)$ and

$E_{2k} = (v_2,\ u_{k+1},\ u_{k+2},\ \ldots,\ u_{p-3},\ v_3)$, where $k \in Z$, $0 \leq k \leq p-3$.

Theorem 2: *For* $0 \leq k \leq p-3$, $G_k \cong G_l$, *where* $l = p - k - 3$.

Proof: $G_l = (V',\ X')$ is the edge complete (p,2) semigraph with
$V' = \{v_1',\ v_2',\ v_3', u_1',\ u_2',\ \ldots,\ u_{p-3}'\}$,
$X' = \{E_{1l}',\ E_{2l}'\}$, where $E_{1l}' = (v_1',\ u_1',\ u_2',\ \ldots,\ u_l',\ v_2')$ and
$E_{2l}' = (v_2',\ u_{l+1}',\ u_{l+2}',\ \ldots,\ u_{p-3}',\ v_3')$.

Define $f : V \to V'$ by $f(v_1) = v_3'$, $f(v_2) = v_2'$, $f(v_3) = v_1'$,

$$f(u_r) = \begin{cases} u_{p-2-r}', & 1 \leq r \leq k \\ u_{l-k-r+1}', & k+1 \leq r \leq p-3. \end{cases}$$

In particular, $f(u_k) = u_{p-k-2}' = u_{l+1}'$ and $f(u_{p-3}) = u_1'$. Now f is a bijection.

$$\begin{aligned} f(E_{1k}) &= (f(v_1),\ f(u_1),\ f(u_2),\ \ldots,\ f(u_k),\ f(v_2)) \\ &= (v_3',\ u_{p-3}',\ u_{p-4}',\ \ldots,\ u_{l+1}',\ v_2') = E_{2l}' \\ \text{Also, } f(E_{2k}) &= (f(v_2),\ f(u_{k+1}),\ \ldots,\ f(u_{p-3}),\ f(v_3)) \\ &= (v_2', u_l', u_{l-1}', \ldots, u_1', v_1') = E_{1l}'. \end{aligned}$$

Thus, $G_k \cong G_l$.

Theorem 3: *G_k is not isomorphic to G_m, when $k \neq m$ and* $0 \leq k,\ m \leq \lfloor \frac{p-3}{2} \rfloor$.

Proof: Suppose $G_k \cong G_m$, for some $k \neq m$ and $0 \leq k,\ m \leq \lfloor \frac{p-3}{2} \rfloor$.

Then there exists a bijection $f:\ V \to V'$ such that

$\{f(E_{1k}),\ f(E_{2k})\} = \{E_{1m}',\ E_{2m}'\}$.

E_{1k} has k middle vertices and E_{1m}' has m middle vertices and $k \neq m$.

So, $f(E_{1k}) \neq E_{1m}'$ and $f(E_{2k}) \neq E_{2m}'$. Hence, $f(E_{1k}) = E_{2m}'$.

Then $(f(v_1),\ f(u_1),\ \ldots,\ f(u_k),\ f(v_2)) = (v_2',\ u_{m+1}',\ u_{m+2}',\ \ldots,\ u_{p-3}',\ v_3')$

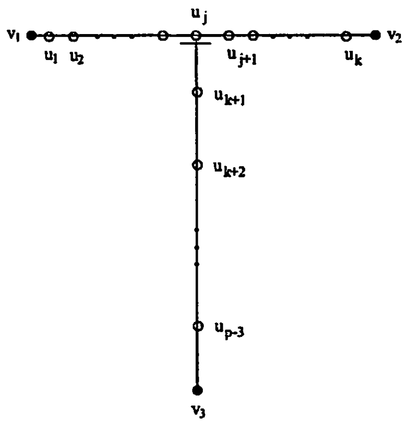
and so $k = p - m - 3 \geq p - 3 - \lfloor \frac{p-3}{2} \rfloor \geq \lceil \frac{p-3}{2} \rceil$. Also, $k \leq \lfloor \frac{p-3}{2} \rfloor$

Then $k = \frac{p-3}{2} = m$, which is a contradiction.

Theorem 4: *The number of non-isomorphic edge complete (p,2) semigraphs of Type 1 is* $\lfloor \frac{p-1}{2} \rfloor$.

Proof: When $k > \lfloor \frac{p-3}{2} \rfloor$ then $l = p-k-3 \leq \lfloor \frac{p-3}{2} \rfloor$ and using Theorem 2, $G_k \cong G_l$. Thus G_0, G_1, G_2, ... $G_{\lfloor \frac{p-3}{2} \rfloor}$ are the only non-isomorphic edge complete (p,2) semigraphs of Type 1. Hence, number of non-isomorphic edge complete (p,2) semigraphs of Type 1 is $\lfloor \frac{p-3}{2} \rfloor + 1 = \lfloor \frac{p-1}{2} \rfloor$.

Type 2: The common vertex is an end vertex of one edge and a m-vertex of another edge



G_{kj}

Let G_{kj} denote a $(p, 2)$ semigraph of Type 2, where E_1 has k middle vertices (and E_2 has $p-3-k$ middle vertices); u_j is the common vertex, where u_j is an m-vertex of E_1 and an end vertex of E_2.

Note that for a fixed k and j, there is only one non-isomorphic semigraph G_{kj} (of Type 2).

Let $G_{kj} = (V, X)$ with $V = \{v_1, v_2, v_3, u_1, u_2, ...u_{p-3}\}$, $X = \{E_{1kj}, E_{2kj}\}$,

$E_{1kj} = (v_1, u_1, u_2, ..., u_j, ..., u_k, v_2)$ and $E_{2kj} = (u_j, u_{k+1}, ..., u_{p-3}, v_3)$,

where $k,\ j \in Z,\ 1 \leq k \leq p-3$ *and* $1 \leq j \leq k$.

Theorem 5: *For a fixed* $k,\ 1 \leq k \leq p-3,\ G_{kj} \cong G_{kl}$, *where* $l = k-j+1,\ \ 1 \leq j \leq k$.

Proof: Proof is similar to Theorem 2.

Theorem 6: *For a fixed* $k,\ 1 \leq k \leq p-3,\ G_{kj}$ *is not isomorphic to* G_{kl}, *when* $j \neq l$ *and* $1 \leq j,\ l \leq \lfloor \frac{k+1}{2} \rfloor$.

Proof: On the contrary, suppose that $G_{kj} \cong G_{kl}$ for some $j \neq l$, $1 \leq j,\ l \leq \lfloor \frac{k+1}{2} \rfloor$. Then there exists a bijection $f : V \rightarrow V'$

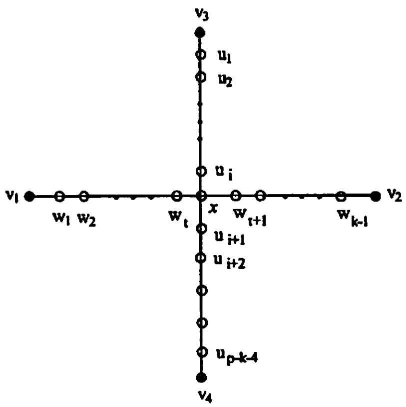
such that $\{f(E_{1kj}), f(E_{2kj})\} = \{E'_{1kl}, E'_{2kl}\}$. Now u_j is the common vertex in G_{kj} and u'_l is the common vertex in G_{kl} and so $f(u_j) = u'_l$.
Since v_1, v_2 are end vertices in E_{1kj}, $f(v_1), f(v_2)$ are end vertices of an edge in G_{kl}. So, $f(E_{1kj}) = E'_{1kl}$ and $f(E_{2kj}) = E'_{2kl}$.
Also, v_3 is an end vertex in G_{kj} and so $f(v_3)$ is also an end vertex in G_{kl}.
Then $f(E_{2kj}) = E'_{2kl}$ implies that
$(f(u_j),\ f(u_{k+1}),\ ...,\ f(u_{p-3}),\ f(v_3)) = (u'_l,\ u'_{k+1},\ ...,\ u'_{p-3},\ v'_3)$
So, $f(u_j) = u'_l$ and $f(u_r) = u'_r$, for $k+1 \leq r \leq p-3$.
$f(E_{1kj}) = E'_{1kl}$ implies that
$(f(v_1), f(u_1), ..., f(u_j), ..., f(u_k), f(v_2)) = (v'_1, u'_1, u'_2, ..., u'_l, ..., u'_k, v'_2)$ ...(1)
or $(f(v_1), f(u_1), ..., f(u_j), ..., f(u_k), f(v_2)) = (v'_2, u'_k, ..., u'_l, ..., u'_1, v_1)$ ...(2)
If (1) is true, then $f(u_r) = u'_r$, for $1 \leq r \leq k$.
In particular, $f(u_j) = u'_j$. This contradicts the fact that $f(u_j) = u'_l, j \neq l$.
If (2) is true, then $f(u_1) = u'_k$, $f(u_2) = u'_{k-1}$, ..., $f(u_k) = u'_1$,
Hence $f(u_r) = u'_{k-r+1}$. In particular, $f(u_j) = u'_{k-j+1}$. So, $u'_l = u'_{k-j+1}$.
Then, $l = k-j+1$ and so $l+j = k+1$. This contradicts the fact that $j \neq l$ and $1 \leq j,\ l \leq \lfloor \frac{k+1}{2} \rfloor$.
Thus, G_{kj} is not isomorphic to G_{kl}, for $j \neq l$ and $1 \leq j,\ l \leq \lfloor \frac{k+1}{2} \rfloor$.
Theorem 7: *For a fixed k, $1 \leq k \leq p-3$, the number of non-isomorphic edge complete (p,2) semigraphs of Type 2 is* $\lfloor \frac{k+1}{2} \rfloor$.
Proof: Proof is similar to Theorem 4.
Theorem 8: *The number of non-isomorphic edge complete (p,2) semigraphs of Type 1 and Type 2 is* $\left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor$.
Proof: **case 1: p is odd**
Using Theorem 4, the number of non-isomorphic edge complete (p,2) semigraphs of Type 1 is $\lfloor \frac{p-1}{2} \rfloor = \frac{p-1}{2}$.
Using Theorem 7,
when k = 1 (or) 2, number of non-isomorphic edge complete (p,2) semigraphs of Type 2 is 1;
when k = 3 (or) 4, number of non-isomorphic edge complete (p,2) semigraphs of Type 2 is 2; etc.,
when k = p-4 (or) p-3, number of non-isomorphic edge complete (p,2) semigraphs of Type 2 is $\frac{p-3}{2}$.
So, number of non-isomorphic edge complete (p,2) semigraphs of Type 2 is $2\,(1+2+3+...+\frac{p-3}{2}) = (\frac{p-1}{2})\ (\frac{p-3}{2})$.
Hence, total number of non-isomorphic edge complete (p,2) semigraphs of Type 1 and Type 2 is $= (\frac{p-1}{2}) + (\frac{p-1}{2})\,(\frac{p-3}{2}) = (\frac{p-1}{2})^2 = \left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor$.
case 2: p is even
Using Theorem 4, number of non-isomorphic edge complete (p,2) semigraphs of Type 1 is $\lfloor \frac{p-1}{2} \rfloor = \frac{p-2}{2}$.
Using Theorem 7,

when k = 1 (or) 2, number of non-isomorphic edge complete (p,2) semigraphs of Type 2 is 1;
when k = 3 (or) 4, number of non-isomorphic edge complete (p,2) semigraphs of Type 2 is 2; etc.,
when k = p-5 (or) p-4, number of non-isomorphic edge complete (p,2) semigraphs of Type 2 is $\frac{p-4}{2}$;
when k = p-3, number of non-isomorphic edge complete (p,2) semigraphs of Type 2 is $\frac{p-2}{2}$.

Therefore, number of non-isomorphic edge complete (p,2) semigraphs of Type 2 is $2\,(1+2+\ldots+\frac{p-4}{2}) + \frac{p-2}{2}$.

Thus, total number of non-isomorphic edge complete (p,2) semigraphs of Type 1 and Type 2 is $\frac{p^2-2p}{4} = \left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor$.

Type 3: The common vertex is an m-vertex of both the edges



G_{it}^{k}

Let G_{it}^{k} denote a $(p,2)$ semigraph of Type 3, where E_1 has k middle vertices (and E_2 has $p-3-k$ middle vertices); x is an m-vertex of both E_1 and E_2.

Note that for a fixed k, t and i, there is only one non-isomrophic semigraph G_{it}^{k} (of Type 3).

Let $G_{it}^{k} = (V, X)$ with

$V = \{v_1, v_2, v_3, v_4, x, w_1, w_2, \ldots, w_{k-1}, u_1, \ldots, u_{p-k-4}\},$

$X = \{E_{1it}^k, E_{2it}^k\}$, where
$E_{1it}^k = (v_1, w_1, w_2, ..., w_t, x, w_{t+1}, ..., w_{k-1}, v_2)$,
$E_{2it}^k = (v_3, u_1, u_2, ..., u_i, x, u_{i+1}, ..., u_{p-k-4}, v_4)$,

where $t, i, k \in Z$ and $0 \leq t \leq k-1$,

$0 \leq i \leq p-k-4$, $1 \leq k \leq p-4$ and x is the common m-vertex.

Theorem 9: *For a fixed k and t, $G_{it}^k \cong G_{lt}^k$ when $l = p-k-4-i$ and $0 \leq i \leq p-k-4$.*

Proof: Proof is similar to Theorem 2.

Theorem 10: *For a fixed k and t, G_{it}^k is not isomorphic to G_{jt}^k, when $i \neq j$ and $0 \leq i, j \leq \left\lfloor \frac{p-k-4}{2} \right\rfloor$.*

Proof: Suppose $G_{it}^k \cong G_{jt}^k$, for some $i \neq j$ and $0 \leq i, j \leq \left\lfloor \frac{p-k-4}{2} \right\rfloor$.

Then there exists a bijection $f : V \to V'$

such that $\{f(E_{1it}^k), f(E_{2it}^k)\} = \{E_{1jt}^k, E_{2jt}^k\}$.

Suppose $f(E_{1it}^k) = E_{2jt}^k$. Then by equating the number of m-vertices in

E_{1it}^k and E_{2jt}^k, we get $k = p-k-3$ ...(1)

$(f(v_1), f(w_1), f(w_2), ..., f(w_t), f(x), f(w_{t+1}), ..., f(w_{k-1}), f(v_2))$
$= (v_3', u_1', u_2', ..., u_j', x', u_{j+1}', ..., u_{p-k-4}', v_4')$ or
$(f(v_1), f(w_1), f(w_2), ..., f(w_t), f(x), f(w_{t+1}), ..., f(w_{k-1}), f(v_2))$
$= (v_4', u_{p-k-4}', ..., u_{j+1}', x', u_j', ..., u_1', v_3')$
$\Rightarrow j = t$ or $j = k-1-t$ ...(2)

Similarly, $f(E_{2it}^k) = E_{1jt}^k$ implies that $i = t$ and $i = k-1-t$.

Since $i \neq j$, either $j = t$ and $i = k-1-t$ or $j = k-1-t$ and $i = t$.

In both the cases, j+i = k-1 = p-k-4 (by 1)

i.e., j+i = p-k-4, which contradicts the fact that $i \neq j$, and
$0 \leq i, j \leq \left\lfloor \frac{p-k-4}{2} \right\rfloor$.

Therefore, $f\left(E_{1it}^k\right) \neq E_{2jt}^k$ and $f\left(E_{2it}^k\right) \neq E_{1jt}^k$.

Hence $f\left(E_{1it}^k\right) = E_{1jt}^k$ and $f\left(E_{2it}^k\right) = E_{2jt}^k$.

$f\left(E_2^k it\right) = E_{2jt}^k$ and $i \neq j$ implies that

$(f(v_3), f(u_1), f(u_2), ..., f(u_i), f(x), f(u_{i+1}), ..., f(u_{p-k-4}), f(v_4))$

$= (v'_4, u'_{p-k-4}, ..., u'_{j+1}, x', u'_j, ..., u'_1, v'_3)$

$\Rightarrow p - k - 4 - i = j$

$\Rightarrow i + j = p - k - 4$, again a contradiction.

Thus, G^k_{it} is not isomorphic to G^k_{jt}, when $i \neq j$ and $0 \leq i,\ j \leq \left\lfloor \frac{p-k-4}{2} \right\rfloor$.

Theorem 11: *For a fixed k and i,* $G^k_{it} \cong G^k_{is}$, *where* $s = k - 1 - t,\ 0 \leq t \leq k - 1$.

Proof: Proof is similar to Theorem 2.

Theorem 12: *For a fixed k and i,* G^k_{it} *is not isomorphic to* G^k_{im}, *when* $t \neq m$ *and* $0 \leq t,\ m \leq \lfloor \frac{k-1}{2} \rfloor$.

Proof: Proof is similar to Theorem 10.

Theorem 13: *For a fixed i and t,* $G^k_{it} \cong G^n_{ti}$, *where* $n = p - k - 3$, $1 \leq k \leq p - 4$.

Proof: Proof is similar to Theorem 2.

Theorem 14: *The semigraphs* G^k_{it}, *where* $0 \leq i \leq \left\lfloor \frac{p-k-4}{2} \right\rfloor$, $0 \leq t \leq \lfloor \frac{k-1}{2} \rfloor$ *and* $1 \leq k < \frac{p-3}{2}$ *are all non-isomorphic.*

Proof: Suppose $G^k_{it} \cong G^h_{jm}$, for some $(i, t, k) \neq (j, m, h)$, $0 \leq j \leq \left\lfloor \frac{p-h-4}{2} \right\rfloor$, $0 \leq m \leq \lfloor \frac{h-1}{2} \rfloor$ and $1 \leq h < \frac{p-3}{2}$.

Then there exists a bijection $f : V(G^h_{jm}) \to V(G^k_{it})$ such that

$\{f(E^h_{1jm}), f(E^h_{2jm})\} = \{E^k_{1it}, E^k_{2it}\}$.

If $f(E^h_{1jm}) = E^k_{1it}$ and $f(E^h_{2jm}) = E^k_{2it}$, then equating the number of m-vertices, we have $h = k$.

Also $f(E^h_{1jm}) = E^k_{1it} \Rightarrow$ either m=t or m=k-1-t.

m=k-1-t $\Rightarrow$ m+t=k-1 $\Rightarrow m = t = \frac{k-1}{2}$. Thus $m = t$.

Similarly, $f(E^h_{2jm}) = E^k_{2it}$ implies that $i = j$.

Thus, (i,t,k)=(j,m,h), which is a contradiction.

If $f(E^h_{1jm}) = E^k_{2it}$ and $f(E^h_{2jm}) = E^k_{1it}$, then equating the number of m-vertices, we have h+k = p-3, a contradiction.

Theorem 15: *When p is odd, let* $0 \leq i,\ j \leq \left\lfloor \frac{p-k-4}{2} \right\rfloor$, $0 \leq t,\ m \leq \lfloor \frac{k-1}{2} \rfloor$ *and* $k = \frac{p-3}{2}$. *Then* $G^k_{it} \cong G^k_{jm}$ *iff* $\{i, t\} = \{j, m\}$.

Proof: Suppose $G^k_{it} \cong G^k_{jm}$, where $0 \le i,j \le \left\lfloor \frac{p-k-4}{2} \right\rfloor$ and $0 \le t,m \le \lfloor \frac{k-1}{2} \rfloor$.

Then there exists a bijection $f : V(G^k_{jm}) \to V(G^k_{it})$

such that $\{f(E^k_{1jm}), f(E^k_{2jm})\} = \{E^k_{1it}, E^k_{2it}\}$.

If $f(E^k_{1jm}) = E^k_{1it}$ and $f(E^k_{2jm}) = E^k_{2it}$, as in Theorem 14, we get **m=t** and **i=j**.

Suppose $f(E^k_{1jm}) = E^k_{2it}$ and $f(E^k_{2jm}) = E^k_{1it}$.

We note that $\left\lfloor \frac{p-k-4}{2} \right\rfloor = \lfloor \frac{k-1}{2} \rfloor$ and so $0 \le i,j \le \lfloor \frac{k-1}{2} \rfloor$.

$f(E^k_{1jm}) = E^k_{2it}$ implies that

$(f(v_1), f(w_1), f(w_2), ..., f(w_m), f(x), f(w_{m+1}), ..., f(w_{k-1}), f(v_2))$

$= (v_3, u_1, u_2, .., u_i, x, u_{i+1}, ..., u_{k-1}, v_4)$ or
$(f(v_1), f(w_1), f(w_2), ..., f(w_m), f(x), f(w_{m+1}), ..., f(w_{k-1}), f(v_2))$

$= (v_4, u_{k-1}, ..., u_{i+1}, x, u_i, ..., u_1, v_3)$. Then either m=i or m=k-1-i

Now, $0 \le i, m \le \lfloor \frac{k-1}{2} \rfloor$ implies that $m = i$.

Similarly, $f(E^k_{2jm}) = E^k_{1it}$ implies that $j = t$. Thus, $\{i,t\} = \{j,m\}$.

Conversely, let $\{i,t\} = \{j,m\}$. If $(j,m) = (i,t), G^k_{jm} = G^k_{it}$.
If $(j,m) = (t,i)$, by Theorem 13, $G^k_{it} \cong G^n_{ti}$, when $n = p-k-3$.

Since $k = \frac{p-3}{2}, n = \frac{p-3}{2} = k$ and so $G^k_{it} \cong G^k_{ti}$. Thus, $G^k_{it} \cong G^k_{jm}$.

Theorem 16: *The number of non-isomorphic edge complete (p,2) semigraphs of Type 3 is given by*

$$|\mathcal{A}_1| = \begin{cases} \frac{1}{48}p(p-2)(p-4), & if\ p\ is\ even \\ \frac{1}{48}(p-1)(p^2-5p+12), & if\ p\ is\ odd\ and\ p \equiv 1\ (mod\ 4) \\ \frac{1}{48}(p-3)(p^2-3p+8), & if\ p\ is\ odd\ and\ p \equiv 3\ (mod\ 4). \end{cases}$$

Proof: Let $\mathcal{A}$ denote the family of all edge complete (p,2) semigraphs of Type 3. Then
$\mathcal{A} = \{G^k_{it} \ /\ i,\ t,\ k \in Z, 0 \le i \le p-k-4,\ 0 \le t \le k-1,\ 1 \le k \le p-4\}$.

Using Theorem 14, the semigraphs G^k_{it}
where $0 \le i \le \left\lfloor \frac{p-k-4}{2} \right\rfloor$, $0 \le t \le \lfloor \frac{k-1}{2} \rfloor$ and $1 \le k < \frac{p-3}{2}$
are all non-isomorphic. ...(1)

When $i > \left\lfloor \frac{p-k-4}{2} \right\rfloor$, using Theorem 9, $G^k_{it} \cong G^k_{lt}$,
where $l = p - k - 4 - i \leq \left\lfloor \frac{p-k-4}{2} \right\rfloor$. ...(2)

When $t > \left\lfloor \frac{k-1}{2} \right\rfloor$, using Theorem 11, $G^k_{it} \cong G^k_{is}$,
where $s = k - 1 - t \leq \left\lfloor \frac{k-1}{2} \right\rfloor$. ...(3)

When $k > \frac{p-3}{2}$, using Theorem 13, $G^k_{it} \cong G^n_{ti}$,
where $n = p - k - 3 < \frac{p-3}{2}$. ...(4)

Let $\mathcal{A}_1$ denote the family of all non-isomorphic edge complete (p,2) semigraphs of Type 3. Using (1), (2), (3) and (4), we have

$$\mathcal{A}_1 = \{G^k_{it}/i,t,k \in Z, 0 \leq i \leq \left\lfloor \tfrac{p-k-4}{2} \right\rfloor,\ 0 \leq t \leq \left\lfloor \tfrac{k-1}{2} \right\rfloor, 1 \leq k \leq \left\lfloor \tfrac{p-3}{2} \right\rfloor\}$$

Case 1: p is even

Now $\frac{p-3}{2}$ is not an integer and so k cannot be equal to $\frac{p-3}{2}$.

Hence $|\mathcal{A}_1| = \sum\limits_{k=1}^{\lfloor \frac{p-3}{2} \rfloor} \left\lfloor \frac{p-k-2}{2} \right\rfloor \quad \left\lfloor \frac{k+1}{2} \right\rfloor$

$$= \left\lfloor \tfrac{p-3}{2} \right\rfloor \left\lfloor \tfrac{2}{2} \right\rfloor + \left\lfloor \tfrac{p-4}{2} \right\rfloor \left\lfloor \tfrac{3}{2} \right\rfloor + \ldots + \left\lfloor \frac{p-2-\lfloor \frac{p-3}{2} \rfloor+1}{2} \right\rfloor \left\lfloor \frac{\lfloor \frac{p-3}{2} \rfloor}{2} \right\rfloor +$$

$$+ \left\lfloor \frac{p-2-\lfloor \frac{p-3}{2} \rfloor}{2} \right\rfloor \left\lfloor \frac{\lfloor \frac{p-3}{2} \rfloor+1}{2} \right\rfloor \quad \text{............(5)}$$

Case 1.1: $p \equiv 0 \pmod 4$

$\left\lfloor \frac{p-3}{2} \right\rfloor = \frac{p-4}{2} = \frac{p}{2} - 2$, $\left\lfloor \frac{p-4}{2} \right\rfloor = \frac{p-4}{2} = \frac{p}{2} - 2$, $\left\lfloor \frac{p-5}{2} \right\rfloor = \frac{p-6}{2} = \frac{p}{2} - 3$ and so on.

Now, $\left\lfloor \frac{p-2-\lfloor \frac{p-3}{2} \rfloor}{2} \right\rfloor = \left\lfloor \frac{p-2-(\frac{p-4}{2})}{2} \right\rfloor = \left\lfloor \frac{p}{4} \right\rfloor = \frac{p}{4}$,

$\left\lfloor \frac{\lfloor \frac{p-3}{2} \rfloor+1}{2} \right\rfloor = \left\lfloor \frac{(\frac{p-4}{2})+1}{2} \right\rfloor = \left\lfloor \frac{p-2}{4} \right\rfloor = \frac{p-4}{4} = \frac{p}{4} - 1$,

$\left\lfloor \frac{p-2-\lfloor \frac{p-3}{2} \rfloor+1}{2} \right\rfloor = \left\lfloor \frac{p-2-(\frac{p-4}{2})+1}{2} \right\rfloor = \left\lfloor \frac{p+2}{4} \right\rfloor = \frac{p}{4}$,

$\left\lfloor \frac{\lfloor \frac{p-3}{2} \rfloor}{2} \right\rfloor = \left\lfloor \frac{p-4}{4} \right\rfloor = \frac{p-4}{4} = \frac{p}{4} - 1$.

$$|\mathcal{A}_1| = \left[\left(\tfrac{p}{2}-2\right)1 + \left(\tfrac{p}{2}-2\right)1\right] + \left[\left(\tfrac{p}{2}-3\right)2 + \left(\tfrac{p}{2}-3\right)2\right] + \ldots$$

$$+ \left[\left(\tfrac{p}{4}-0\right)\left(\tfrac{p}{4}-1\right) + \left(\tfrac{p}{4}-0\right)\left(\tfrac{p}{4}-1\right)\right]$$

$$= 2\left[\left(\tfrac{p}{2}-2\right)1 + \left(\tfrac{p}{2}-3\right)2 + \ldots + \left(\tfrac{p}{2}-\tfrac{p}{4}\right)\left(\tfrac{p}{4}-1\right)\right]$$

$$= 2\left\{\tfrac{p}{2}\left(1+2+3+\ldots+\left(\tfrac{p}{4}-1\right)\right) - \left(1.2 + 2.3 + \ldots + \tfrac{p}{4}\left(\tfrac{p}{4}-1\right)\right)\right\}$$

$$= 2\left\{\frac{\frac{p}{2}\left(\frac{p}{4}-1\right)\frac{p}{4}}{2} - \sum_{n=1}^{\frac{p}{4}-1} n(n+1)\right\}$$

$$= \tfrac{p^2}{8}\left(\tfrac{p}{4}-1\right) - 2\left\{\frac{\left(\frac{p}{4}-1\right)\frac{p}{4}\left(\frac{p-2}{2}\right)}{6} + \frac{\left(\frac{p}{4}-1\right)\frac{p}{4}}{2}\right\}$$

$$= \tfrac{p}{24}\left(\tfrac{p}{4}-1\right)[3p-(p-2)-6]$$

$$|A_1| = \tfrac{1}{48}p(p-2)(p-4).$$

Case 1.2: $p \equiv 2 \pmod 4$

$$\left\lfloor\frac{p-2-\lfloor\frac{p-3}{2}\rfloor}{2}\right\rfloor = \left\lfloor\tfrac{p}{4}\right\rfloor = \tfrac{p-2}{4}, \quad \left\lfloor\frac{\lfloor\frac{p-3}{2}\rfloor+1}{2}\right\rfloor = \left\lfloor\tfrac{p-2}{4}\right\rfloor = \tfrac{p-2}{4},$$

$$\left\lfloor\frac{p-2-\lfloor\frac{p-3}{2}\rfloor+1}{2}\right\rfloor = \left\lfloor\tfrac{p+2}{4}\right\rfloor = \tfrac{p+2}{4} = \tfrac{p}{4}+\tfrac{1}{2}$$

$$\left\lfloor\frac{\lfloor\frac{p-3}{2}\rfloor}{2}\right\rfloor = \left\lfloor\tfrac{p-4}{4}\right\rfloor = \tfrac{p-6}{4} = \tfrac{p}{4}-\tfrac{3}{2}.$$

Now, $|\mathcal{A}_1| = \left[\left(\tfrac{p}{2}-2\right)1 + \left(\tfrac{p}{2}-2\right)1\right] + \left[\left(\tfrac{p}{2}-3\right)2 + \left(\tfrac{p}{2}-3\right)2\right] + \ldots$

$$+ \left[\left(\tfrac{p}{4}+\tfrac{1}{2}\right)\left(\tfrac{p}{4}-\tfrac{3}{2}\right) + \left(\tfrac{p}{4}+\tfrac{1}{2}\right)\left(\tfrac{p}{4}-\tfrac{3}{2}\right)\right] + \tfrac{(p-2)^2}{16}$$

On simplification, we get

$|\mathcal{A}_1| = \frac{1}{48}p(p-2)(p-4).$

Hence $|\mathcal{A}_1| = \frac{1}{48}p(p-2)(p-4)$, if p is even and $p \equiv 0$ or $2 \pmod 4$.

Case 2: p is odd

$\frac{p-3}{2}$ is an integer. Using Theorem 15, when $k = \frac{p-3}{2}$,

$G_{it}^k \cong G_{jm}^k$ iff $\{i,t\} = \{j,m\}$.

We have $\left\lfloor \frac{p-k-4}{2} \right\rfloor = \lfloor \frac{k-1}{2} \rfloor = \lfloor \frac{p-5}{4} \rfloor$ and so $0 \le i,j \le \lfloor \frac{p-5}{4} \rfloor$.

Number of distinct 2-element sets $\{i,t\}$, where $0 \le i,t \le \lfloor \frac{p-5}{4} \rfloor$ is given by $\binom{\lfloor \frac{p-1}{4} \rfloor}{2}$.

For every such $\{i,t\}$, $G_{it}^k \cong G_{ti}^k$.

Hence number of non-isomorphic edge complete (p,2) semigraphs of Type 3 is

$$|\mathcal{A}_1| = \left(\sum_{k=1}^{\frac{p-3}{2}} \left\lfloor \frac{p-k-2}{2} \right\rfloor \lfloor \frac{k+1}{2} \rfloor \right) - \binom{\lfloor \frac{p-1}{4} \rfloor}{2}.$$

$$|\mathcal{A}_1| = \lfloor \frac{p-3}{2} \rfloor \lfloor \frac{2}{2} \rfloor + \lfloor \frac{p-4}{2} \rfloor \lfloor \frac{3}{2} \rfloor + \dots + \left\lfloor \frac{p-2-(\frac{p-5}{2})}{2} \right\rfloor \left\lfloor \frac{(\frac{p-5}{2})+1}{2} \right\rfloor$$

$$+ \left\lfloor \frac{p-2-(\frac{p-3}{2})}{2} \right\rfloor \left\lfloor \frac{\frac{p-3}{2}+1}{2} \right\rfloor - \binom{\lfloor \frac{p-1}{4} \rfloor}{2}. \qquad \dots(6)$$

On simplification, we get,

$$|\mathcal{A}_1| = \begin{cases} \frac{1}{48}(p-1)(p^2-5p+12), & when\ p \equiv 1\ (mod\ 4) \\ \frac{1}{48}(p-3)(p^2-3p+8), & when\ p \equiv 3\ (mod\ 4). \end{cases}$$

Reference

1. Sampathkumar, E.: Semigraphs and their applications, DST Project, DST/MS/022/94.