The Doyen-Wilson Theorem for Extended Directed Triple Systems

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Abstract

In this paper, it is shown that every extended directed triple system of order v can be embedded in an extended directed triple system of order n for all $n \geq 2v$. This produces a generalization of the Doyen-Wilson theorem for extended directed triple systems.

Keywords: Doyen-Wilson Theorem; Extended directed triple system; Decomposition; Latin square.

1 Introduction.

A directed triple system of order n, DTS(n), is a pair (V, T), where T is a collection of transitive triples from a n-set V, such that every ordered pair of distinct elements of V is contained in exactly one transitive triple of T (The transitive triple $[a, b, c]$ contains the ordered pairs ab, bc, ac but not ab, bc, ca). This concept was introduced by Huang and Mendelsohn [6], who proved that a DTS(n) exists if and only if $n \not\equiv 2 \pmod 3$. In the same way, Steiner triple systems and Mendelsohn triple systems have been generalized to extended triple systems [3, 7] and extended Mendelsohn triple systems [2], respectively. The concept of such a system, similar to a DTS, is introduced in which a triple may have repeated elements. An extended directed triple system of order n, EDTS(n), is a pair (V, B), where B is a collection of ordered triples from a n-set V (each ordered triple may have repeated elements) such that every ordered pair of elements of V, not necessarily distinct, is contained in exactly one ordered triple of B. The elements of B are called blocks. There are five types of blocks: (1)$[a,b,c]$, (2)$[a,b,a]$, (3)$[a,a,b]$, (4)$[b,a,a]$ and (5)$[a,a,a]$ in which they are the set of ordered pairs $\{ab, bc, ac\}$, $\{ab, ba, aa\}$, $\{aa, ab\}$, $\{ba, aa\}$ and $\{aa\}$, respectively. For convenience, we call the transitive triple for type (1), 2-arc lollipop (2-lollipop for brevity) for type (2), 1-arc lollipop (1-lollipop for brevity) for type (3) or (4), and loop for type (5). In the following paragraphs, b_3, b_2, b_1, and b_0 are used to denote the number of blocks of (V, B) that are of the type (1), (2), (3) or (4), and (5), respectively. A simple counting argument shows that if (V, B) is EDTS(n), then

$$b_3 = \frac{1}{3}(n(n-1) - 2b_2 - b_1) \quad (1)$$

$$b_0 = n - b_2 - b_1 \quad (2)$$

Evidently b_3 and b_0 are determined by b_2 and b_1. Let $\{n; b_2, b_1\}$ denote the class of EDTS(n) with parameters b_2 and b_1. We say that $\{n; b_2, b_1\}$ exists if there is a design with the specified parameters. A necessary condition for the existence of $\{n; b_2, b_1\}$ is

$$n(n-1) - 2b_2 - b_1 \equiv 0 (mod\ 3) \quad (3)$$

and

$$0 \le b_2 + b_1 \le n. \quad (4)$$

In [5], it was shown that the necessary and sufficient conditions for the existence of a $\{n; b_2, b_1\}$, with $b_1 \neq 1$ and $0 \le b_2 + b_1 \le n$, are:

(1) $b_2 \equiv b_1 (mod\ 3)$ for $n \not\equiv 2 (mod\ 3)$;

(2) $b_2 \equiv b_1 + 1 (mod\ 3)$ for $n \equiv 2 (mod\ 3)$.

In graph notation, a DTS(n) is equivalent to the decomposition of digraph D_n into directed triples, where D_n is the complete symmetric digraph of order n. And an EDTS(n) is equivalent to the decomposition of digraph D_n^+ into directed triples, 2-lollipops, 1-lollipops and loops, where D_n^+ is the digraph obtained by attaching a loop to each vertex of D_n. From now onwards, the decomposition of a digraph G is a decomposition of G into directed triples, 2-lollipops, 1-lollipops and loops.

Extended directed triple systems are the brothers of extended Mendelsohn triple systems. We define an extended Mendelsohn triple to be a loop, a loop with symmetric arcs attached (known as a lollipop), or a directed 3-cycle (known as a cyclic triple). An extended Mendelsohn triple system of order n is an ordered pair (V, B), where B is a set of extended Mendelsohn triples defined on the vertex set V which partitions the edges and loops of D_n^+. It has been shown by Castellana and Raines [4] that every extended Mendelsohn triple system of order v can be embedded into an extended Mendelsohn triple system of order n for all $n \geq 2v$. In the same way, we want to produce the embedding theorem for extended directed triple systems.

An extended directed triple system (V, B) is said to be embedded in an extended directed triple system (V', B') if $V \subseteq V'$ and $B \subseteq B'$. Suppose $|V| = v$ and $|V'| = n$. Then such an embedding can be thought of as a decomposition of D_n^+ with the arcs and loops of D_v^+ removed. We use the notation $D_n^+ \backslash D_v^+$ to denote this digraph. The focus of this paper is to prove the following main theorem, thus obtaining a result analogous to the results obtained by Castellana and Raines for extended Mendelsohn triple systems [4].

Main Theorem *Every EDTS(v) can be embedded in an EDTS(n) for all $n \geq 2v$.*

2 Preliminary results

We start with some definitions and preliminary results. The directed 2-factors P_k of D_n^+, for $k = -[n/2], \cdots, [n/2]$ is defined as follows,

$$(i, j) \in P_k \Longleftrightarrow j - i \equiv k (mod\ n)$$

It can easily be verified that P_0 is a set of n loops, P_k is a set of m directed n/m-cycles, where $m = gcd(k,n)$, and $P_{-[n/2]}$ and $P_{[n/2]}$ are the same if n is even. Thus, the arcs of D_n^+ fall into n disjoint classes $\mathcal{P}$ or $\mathcal{P} \setminus \{P_{-[n/2]}\}$ for odd n or even n, respectively, where $\mathcal{P} = \{P_0, P_{\pm 1}, P_{\pm 2}, \cdots, P_{\pm [n/2]}\}$, and we call them the difference partition of D_n^+. The set $\mathcal{A} \cup (-\mathcal{A})$ or $\mathcal{A} \cup (-\mathcal{A}) \setminus \{-[n/2]\}$ is called the differences of D_n^+ for odd n or even n, respectively, where $\mathcal{A} = \{0, 1, 2, \cdots, [n/2]\}$.

Now, we define three operators on those directed 2-factors of D_n^+. Let $C = (a_1, a_2, \cdots, a_m)$ and $x \neq a_i$ for all i, $\mathcal{O}_1(C, x) = \{[a_1, x, a_2], [a_2, x, a_3], \cdots, [a_{m-1}, x, a_m], [a_m, x, a_1]\}$. Let $P_k = \{C_i \mid i = 1, 2, \cdots, m\}$ and $x \notin V(P_k)$, we define $\mathcal{O}_1(P_k, x) = \{\mathcal{O}_1(C_i, x) \mid i = 1, 2, \cdots, m\}$ for $k \neq 0$ and $\mathcal{O}_1(P_0, x) = \{[i, x, i] \mid i = 1, \cdots, n\}$. Let $\mathcal{O}_2(P_k, P_0) = \{[i, i, i+k] \mid i = 1, 2, \cdots, n\}$ for $k \neq 0$ and $\mathcal{O}_3(P_a, P_b, P_c) = \{[i, i+a, i+a+b] \mid i = 1, 2, \cdots, n\}$ for $c = a + b$, where the sum is modulo n.

We need the following Skolem partition and O'Keefe partition to obtain the main theorem.

Definition 2.1 *A Skolem partition of order n is a partition of $\{1, 2, \cdots, 3n\}$ into n triples $\{i, a_i, i + a_i\}$, $1 \leq i \leq n$.*

Lemma 2.2 *[1] A Skolem partition of order n exists if and only if $n \equiv 0$ or 1 (mod 4).*

Definition 2.3 *An O'Keefe partition of order n is a partition of $\{1, \cdots, 3n-1; 3n+1\}$ into n triples $\{i, a_i, i + a_i\}$, $1 \leq i \leq n$.*

Lemma 2.4 *[1] An O'Keefe partition of order n exists if and only if $n \equiv 2$ or 3 (mod 4).*

3 The embedding

Now, we want to show that every EDTS(v) can be embedded in an EDTS(n) for all $n \geq 2v$. So, we write $n = 2v + q$. The main theorem is equivalent to the decomposition of the digraph $D^+_{2v+q}\backslash D^+_v$. The digraph $D^+_{2v+q}\backslash D^+_v$ can be regarded as a union of subgraph $D_{v,v+q}$ and D^+_{v+q}, where the partition set of the directed bipartite graph $D_{v,v+q}$ are $V_1 \cup V_2$, $V_1 = \{x_1, \cdots, x_v\}$ and $V_2 = V(D^+_{v+q}) = \{1, 2, \cdots, v+q\}$. The main theorem is proved by the following three lemmas.

Lemma 3.1 *A decomposition of $D^+_{2v+3k}\backslash D^+_v$ exists.*

Proof. Case 1: Let $v = 2l + 1$, for some positive integer l.

If $k = 2m$, the set of differences of D^+_{v+6m} is $\mathcal{A} \cup (-\mathcal{A}) \cup \{0, \pm(3m+1), \pm(3m+2), \cdots, \pm(3m+l)\}$, where $\mathcal{A} = \{1, 2, \cdots, 3m\}$. When $m \equiv 0$ or 1 $(mod\ 4)$, by Lemma 2.2, there exists a Skolem partition $\{i, a_i, i+a_i\}$ of $\mathcal{A}$. Let $T_1 = \{\mathcal{O}_3(P_{\pm i}, P_{\pm a_i}, P_{\pm(i+a_i)}) \mid i = 1, \cdots, m\}$ and $T_2 = \{\mathcal{O}_1(P_0, x_1)\} \cup \{\mathcal{O}_1(P_{3m+i}, x_{i+1}), \mid i = 1, 2, \cdots, l\} \cup \{\mathcal{O}_1(P_{-(3m+i)}, x_{l+i+1}), \mid i = 1, 2, \cdots, l\}$, then $T_1 \cup T_2$ is a decomposition of $D^+_{2v+6m}\backslash D^+_v$. When $m \equiv 2$ or 3 $(mod\ 4)$, by Lemma 2.4, there exists an O'Keefe partition $\{i, b_i, i+b_i\}$ of $\mathcal{A} \cup \{3m+1\} \setminus \{3m\}$. Let $T_1 = \{\mathcal{O}_3(P_{\pm i}, P_{\pm b_i}, P_{\pm(i+b_i)}) \mid i = 1, \cdots, m\}$ and $T_2 = \{\mathcal{O}_1(P_0, x_1), \mathcal{O}_1(P_{3m}, x_2), \mathcal{O}_1(P_{-3m}, x_3)\} \cup \{\mathcal{O}_1(P_{3m+i}, x_{i+2}) \mid i = 2, 3, \cdots, l\} \cup \{\mathcal{O}_1(P_{-(3m+i)}, x_{l+i+1}) \mid i = 2, 3, \cdots, l\}$, then $T_1 \cup T_2$ is a decomposition of $D^+_{2v+6m}\backslash D^+_v$.

If $k = 2m+1$, the set of differences of D^+_{v+6m+3} is $\mathcal{A} \cup (-\mathcal{A}) \cup \{0, \pm(3m+4), \pm(3m+5), \cdots, \pm(3m+l+1), 3m+l+2\}$, where $\mathcal{A} = \{1, 2, \cdots, 3m+3\}$. If $m+1 \equiv 0$ or 1 $(mod\ 4)$, by Lemma 2.2, there exists a Skolem partition $\{i, a_i, i+a_i\}$ of $\mathcal{A}$. Let $T_1 = \{\mathcal{O}_3(P_{\pm i}, P_{\pm a_i}, P_{\pm(i+a_i)}) \mid i = 1, \cdots, m+1\}\backslash\{\mathcal{O}_3(P_1, P_{a_1}, P_{1+a_1})\}$ and $T_2 = \{\mathcal{O}_1(P_0, x_1),\ \mathcal{O}_1(P_1, x_2),\ \mathcal{O}_1(P_{a_1}, x_3),\ \mathcal{O}_1(P_{1+a_1}, x_4), \mathcal{O}_1(P_{3m+l+2}, x_5)\} \cup \{\mathcal{O}_1(P_{3m+i}, x_{i+2}) \mid i = 4, 5, \cdots, l+1\} \cup \{\mathcal{O}_1(P_{-(3m+i)}, x_{l+i}) \mid i = 4, 5, \cdots, l+1\}$, then $T_1 \cup T_2$ is a decomposition of $D^+_{2v+6m+3}\backslash D^+_v$. If $m+1 \equiv 2$ or 3 $(mod\ 4)$, by Lemma 2.4, there exists an O'Keefe partition $\{i, b_i, i+b_i\}$ of $\mathcal{A} \cup \{3m+4\} \setminus \{3m+3\}$. Let $T_1 = \{\mathcal{O}_3(P_{\pm i},\ P_{\pm b_i},\ P_{\pm(i+b_i)}) \mid i = 1, \cdots, m+1\}\backslash\{\mathcal{O}_3(P_1, P_{b_1}, P_{1+b_1})\}$ and $T_2 = \{\mathcal{O}_1(P_0, x_1),\ \mathcal{O}_1(P_1, x_2),\ \mathcal{O}_1(P_{b_1}, x_3),\ \mathcal{O}_1(P_{1+b_1}, x_4), \mathcal{O}_1(P_{3m+l+2}, x_5),\ \mathcal{O}_1(P_{3m+3}, x_6),\ \mathcal{O}_1(P_{-(3m+3)}, x_7)\} \cup \{\mathcal{O}_1(P_{3m+i}, x_{i+3}) \mid i = 5, 6, \cdots, l+1\} \cup \{\mathcal{O}_1(P_{-(3m+i)}, x_{l+i}) \mid i = 5, 6, \cdots, l+1\}$, then $T_1 \cup T_2$ is a decomposition of $D^+_{2v+6m+3}\backslash D^+_v$.

Case 2: Let $v = 2l$, for some positive integer l.

If $k = 2m$, the set of differences of D^+_{v+6m} is $\mathcal{A} \cup (-\mathcal{A}) \cup \{0, \pm(3m+1), \pm(3m+2), \cdots, \pm(3m+l-1), 3m+l\}$, where $\mathcal{A} = \{1, 2, \cdots, 3m\}$. When $m \equiv 0$ or 1 (mod 4), by Lemma 2.2, there exists a Skolem partition $\{i, a_i, i+a_i\}$ of $\mathcal{A}$. Let $T_1 = \{\mathcal{O}_3(P_{\pm i}, P_{\pm a_i}, P_{\pm(i+a_i)}) \mid i = 1, \cdots, m\}$ and $T_2 = \{\mathcal{O}_1(P_0, x_1), \mathcal{O}_1(P_{3m+l}, x_2)\} \cup \{\mathcal{O}_1(P_{3m+i}, x_{i+2}), \mid i = 1, 2, \cdots, l-1\} \cup \{\mathcal{O}_1(P_{-(3m+i)}, x_{l+i+1}), \mid i = 1, 2, \cdots, l-1\}$, then $T_1 \cup T_2$ is a decomposition of $D^+_{2v+6m} \backslash D^+_v$. When $m \equiv 2$ or 3 (mod 4), by Lemma 2.4, there exists an O'Keefe partition $\{i, b_i, i+b_i\}$ of $\mathcal{A} \cup \{3m+1\} \setminus \{3m\}$. Let $T_1 = \{\mathcal{O}_3(P_{\pm i}, P_{\pm b_i}, P_{\pm(i+b_i)}) \mid i = 1, \cdots, m+1\}$ and $T_2 = \{\mathcal{O}_1(P_0, x_1), \mathcal{O}_1(P_{3m+l}, x_2), \mathcal{O}_1(P_{3m}, x_3), \mathcal{O}_1(P_{-3m}, x_4)\} \cup \{\mathcal{O}_1(P_{3m+i}, x_{i+3}) \mid i = 2, 3, \cdots, l-1\} \cup \{\mathcal{O}_1(P_{-(3m+i)}, x_{l+i+1}) \mid i = 2, 3, \cdots, l-1\}$, then $T_1 \cup T_2$ is a decomposition of $D^+_{2v+6m} \backslash D^+_v$.

If $k = 2m+1$, the set of differences of D^+_{v+6m+3} is $\mathcal{A} \cup (-\mathcal{A}) \cup \{0, \pm(3m+4), \pm(3m+5), \cdots, \pm(3m+l+1)\}$, where $\mathcal{A} = \{1, 2, \cdots, 3m+3\}$. If $m+1 \equiv 0$ or 1 (mod 4), by Lemma 2.2, there exists a Skolem partition $\{i, a_i, i+a_i\}$ of $\mathcal{A}$. Let $T_1 = \{\mathcal{O}_3(P_{\pm i}, P_{\pm a_i}, P_{\pm(i+a_i)}) \mid i = 1, \cdots, m+1\} \backslash \{\mathcal{O}_3(P_1, P_{a_1}, P_{1+a_1})\}$ and $T_2 = \{\mathcal{O}_1(P_0, x_1), \mathcal{O}_1(P_1, x_2), \mathcal{O}_1(P_{a_1}, x_3), \mathcal{O}_1(P_{1+a_1}, x_4)\} \cup \{\mathcal{O}_1(P_{3m+i}, x_{i+1}) \mid i = 4, 5, \cdots, l+1\} \cup \{\mathcal{O}_1(P_{-(3m+i)}, x_{l+i-1}) \mid i = 4, 5, \cdots, l+1\}$, then $T_1 \cup T_2$ is a decomposition of $D^+_{2v+6m+3} \backslash D^+_v$. If $m+1 \equiv 2$ or 3 (mod 4), by Lemma 2.4, there exists an O'Keefe partition $\{i, b_i, i+b_i\}$ of $\mathcal{A} \cup \{3m+4\} \setminus \{3m+3\}$. Let $T_1 = \{\mathcal{O}_3(P_{\pm i}, P_{\pm b_i}, P_{\pm(i+b_i)}) \mid i = 1, \cdots, m+1\} \backslash \{\mathcal{O}_3(P_1, P_{b_1}, P_{1+b_1})\}$ and $T_2 = \{\mathcal{O}_1(P_0, x_1), \mathcal{O}_1(P_1, x_2), \mathcal{O}_1(P_{b_1}, x_3), \mathcal{O}_1(P_{1+b_1}, x_4), \mathcal{O}_1(P_{3m+3}, x_5), \mathcal{O}_1(P_{-(3m+3)}, x_6)\} \cup \{\mathcal{O}_1(P_{3m+i}, x_{i+2}) \mid i = 5, 6, \cdots, l+1\} \cup \{\mathcal{O}_1(P_{-(3m+i)}, x_{l+i-1}) \mid i = 5, 6, \cdots, l+1\}$, then $T_1 \cup T_2$ is a decomposition of $D^+_{6m+3+2v} \backslash D^+_v$. ◇

Let $\mathcal{A} \cup \{d\}$ be the set of differences of D^+_{v+3k+1}, where $\mathcal{A}$ is the set of differences of D^+_{v+3k} and d is $[(3k+v+1)/2]$ if $3k+v+1$ is even; $-[(3k+v+1)/2]$ if $3k+v+1$ is odd. Using the proof of Lemma 3.1, we obtain the decomposition T of $(D^+_{2v+3k+1} \backslash D^+_v) \backslash E$, where E is the edge set of P_d. In the construction of Lemma 3.1, the connection of P_0 by operation $\mathcal{O}_1$ is always at the vertex x_1. So, $(T \backslash \mathcal{O}_1(P_0, x_1)) \cup \{\mathcal{O}_1(P_d, x_1), P_0\}$ form a decomposition of $D^+_{2v+3k+1} \backslash D^+_v$. Therefore, we obtain the following lemma.

Lemma 3.2 *A decomposition of* $D^+_{2v+3k+1} \backslash D^+_v$ *exists.*

Similarly, the set of differences of D^+_{v+3k+2} is $\mathcal{A} \cup \{a, b\}$, where $\mathcal{A}$ is the set of differences of D^+_{v+3k}. Using the proof of Lemma 3.1, we obtain the decomposition T of $(D^+_{2v+3k+2} \backslash D^+_v) \backslash E$, where E is the edge set of P_a and P_b. Therefore, $(T \backslash \mathcal{O}_1(P_0, x_1)) \cup \{\mathcal{O}_1(P_a, x_1), \mathcal{O}_2(P_b, P_0)\}$ form a decomposition of $D^+_{2v+3k+2} \backslash D^+_v$.

Lemma 3.3 *A decomposition of* $D^+_{2v+3k+2} \backslash D^+_v$ *exists.*

Combining Lemmas 3.1-3.3, the following main result is obtained.

Main Theorem *Every EDTS(v) can be embedded in an EDTS(n) for all* $n \geq 2v$.

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