An Upper Bound on the Number of Independent Sets in a Tree

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Abstract

The main result of this paper is an upper bound on the number of independent sets in a tree in terms of the order and diameter of the tree. This new upper bound is a refinement of the bound given by Prodinger and Tichy [Fibonacci Q., 20 (1982), no. 1, 16-21]. Finally, we give a sufficient condition for the new upper bound to be better than the upper bound given by Brigham, Chandrasekharan and Dutton [Fibonacci Q., 31 (1993), no. 2, 98–104].

1 Introduction

Given a graph G, a subset $S \subseteq V(G)$ is said to be independent, if no two vertices of S are adjacent in G. We follow the notation given by Jou and Chang (2000), that is, the set of all independent sets of a graph G is denoted by $\mathrm{I}(G)$ while the cardinality of $\mathrm{I}(G)$ is denoted by $i(G)$. For undefined concepts the reader may refer to Diestel (1997).

Erdös and Moser were the first to study the problem of determining the number of maximal independent sets in a graph and it is now well-studied. For a survey on this research area see Jou and Chang (1995) and Jou and Chang (2000). Along the same line, Prodinger and Tichy (1982) considered the problem of determining $i(G)$. They proved the following result.

Theorem 1.1 (Prodinger and Tichy, 1982)
For any tree T on n vertices, $\mathrm{fib}(n+2) \leq i(T) \leq 2^{n-1}+1$. *Moreover,* $i(T) = \mathrm{fib}(n+2)$ *if and only if* $T \simeq P_n$, *and* $i(T) = 2^{n-1}+1$ *if and only if* $T \simeq K_{1,n-1}$.

Here $\text{fib}(n)$ denotes the *nth Fibonacci number*, which is defined inductively by $\text{fib}(0) := 0, \text{fib}(1) := 1$ and $\text{fib}(n) := \text{fib}(n-1) + \text{fib}(n-2)$ for $n \geq 2$.

Lin and Lin (1995) considered the problem of determining the trees T with large or small value of the graph parameter $i(T)$. That is, Lin and Lin characterized all trees T of order $n \geq 8$ with $2^{n-2} + 7 \leq i(T) \leq 2^{n-1} + 1$ and they showed that $i(T) \geq 2\text{fib}(n) + 3\text{fib}(n-3)$ for any tree $T \not\simeq P_n$.

For any graph G on n vertices, the power set of $V(G)$ has cardinality 2^n and therefore $i(G) \leq 2^n$. Obviously, equality is obtained only if G consists of n isolated vertices.

Observation 1.2
Let G denote a graph and let H denote any spanning subgraph of G. Then $i(G) \leq i(H)$.

Using this observation together with Theorem 1.1, we find that any connected graph G on n vertices has at most $2^{n-1} + 1$ independent sets, that is, at most half the nonempty subsets of $V(G)$ are independent sets.

Observation 1.3
If G is a graph with components $G_1, \ldots, G_k$, then $i(G) = \prod_{i=1}^{k} i(G_i)$.

This observation gives the following result.

Proposition 1.4
Let G denote a graph. If $i(G)$ is a prime number, then G is connected.

Proposition 1.5
Let G denote a connected graph and let x denote any vertex of G. Then $i(G) < 2i(G - x)$

Proof. Let x denote any vertex of G and let y denote a neighbour of x. We may write $\text{I}(G) = \mathcal{A} \cup \mathcal{B}$, where $\mathcal{A}$ consists of the independent sets of G, which contain x, and $\mathcal{B}$ consists of the independent sets of G, which do not contain x. Observe that $\mathcal{B}$ is equal to the set of independent sets of $G - x$.

Every set $A - \{x\} \in \mathcal{A}$ is also a member of $\mathcal{B}$ and so $|\mathcal{A}| \leq |\mathcal{B}|$. But $\{y\} \in \mathcal{B}$ corresponds to no set $A - \{x\} \in \mathcal{A}$. Thus, $|\mathcal{A}| < |\mathcal{B}|$ and $i(T) = |\mathcal{A}| + |\mathcal{B}| < 2|\mathcal{B}|$. ∎

The main theorem of this paper states that $i(T) \leq \text{fib}(d) + 2^{n-d}\text{fib}(d+1)$ for any tree T of order $n \geq 2$ and diameter d. Moreover, we determine the trees for which equality occurs. In order to prove this theorem we need some preliminary results about a certain type of trees, which we call brooms.

2 Brooms

For any triple of integers (n, d, k) where $d \geq 3, n \geq d+1$ and $1 \leq k \leq n-d$, let $B_{n,d,k}$ denote the graph constructed from $P_{d-1} : x_1 \dots x_{d-1}$ by attaching k pendant edges at x_1 and $n+1-k-d$ pendant edges at x_{d-1}. The graphs $B_{n,d,k}$ are called *brooms* and, in particular, $B_{n,d,1}$ and $B_{n,d,n-d}$ are called *simple brooms.* Thus, $B_{n,d,k}$ is a tree of order n and diameter d, and it contains precisely two stems x_1 and x_{d-1} with k and $n-k-d+1 =: k'$ leaves, respectively. Note that $n = k + k' + d - 1$ and $B_{n,d,k} \simeq B_{n,d,k'}$. As an example, the broom $B_{12,5,5}$ is shown in Figure 1.

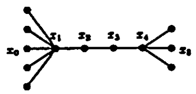

Figure 1: The broom $B_{12,5,5}$.

Lemma 2.1
For any pair of integers (n, d) where $d \geq 3, n \geq d+1$,

$$i(B_{n,d,1}) = i(B_{n,d,n-d}) = \text{fib}(d) + 2^{n-d}\text{fib}(d+1).$$

Proof. Since $B_{n,d,1}$ and $B_{n,d,n-d}$ are isomorphic, we need only consider $B_{n,d,n-d}$. Let $P_{d+1} = x_0 x_1 \dots x_d$ denote a diametrical path of $B_{n,d,n-d}$. Any independent set of $B_{n,d,n-d}$, which does not contain x_1, can be constructed by choosing some of the $n-d$ leaves at x_1 (possibly none) and some independent set of the P_{d-1}-component of $B_{n,d,1} - x_1$. Thus, there are $2^{n-d} i(P_{d-1}) = 2^{n-d}\text{fib}(d+1)$ independent sets of $B_{n,d,n-d}$, which does not contain x_1. The number of independent sets of $B_{n,d,n-d}$, which contain x_1, is equal to the number of independent sets of $B_{n,d,n-d} - N[x_1] \simeq P_{d-2}$. It follows that $i(B_{n,d,n-d}) = 2^{n-d}\text{fib}(d+1) + \text{fib}(d)$. ■

Theorem 2.2
For any triple of integers (n, d, k) where $d \geq 3, n \geq d+1$ and $1 \leq k \leq n-d$,

$$i(B_{n,d,k}) = \text{fib}(d-3) + \left(2^k + 2^{k'}\right)\text{fib}(d-2) + 2^{n-d+1}\text{fib}(d-1), \quad (1)$$

where $k' = n - k - d + 1$. Moreover,

$$i(B_{n,d,k}) \leq \text{fib}(d) + 2^{n-d}\text{fib}(d+1), \quad (2)$$

and equality holds if and only if $k \in \{1, n-d\}$.

Proof. First, we count the number of independent sets in $B_{n,d,k}$. Let P_{d+1} : $x_0x_1\dots x_d$ denote the underlying path such that x_1 denotes the stem with k leaves and x_{d-1} denotes the stem with $k' = n-d+1-k$ leaves.

Any independent set in $B_{n,d,k}$, which do not contain x_1, can be constructed by choosing some of the leaves at x_1 and choosing some independent set of the $B_{n-k-1,d-2,1}$-component of $B_{n,d,1} - x_1$. Thus, there are $2^k i(B_{n-k-1,d-2,1})$ distinct independent subsets of $B_{n,d,k}$ which do not contain x_1.

Clearly, the number of independent sets in $B_{n,d,k}$, which contains x_1, is equal to the number of independent sets in $B_{n,d,k} - N[x_1] \simeq B_{n-k-2,d-3,1}$.

Thus, $i(B_{n,d,k}) = 2^k i(B_{n-k-1,d-2,1}) + i(B_{n-k-2,d-3,1})$ and so, by Lemma 2.1,

$$\begin{aligned} i(B_{n,d,k}) &= 2^k\left(\text{fib}(d-2) + 2^{n-k-1-(d-2)}\text{fib}(d-1)\right) + \\ &\quad \text{fib}(d-3) + 2^{n-k-2-(d-3)}\text{fib}(d-2) \\ &= 2^k\left(\text{fib}(d-2) + 2^{k'}\text{fib}(d-1)\right) + \text{fib}(d-3) + 2^{k'}\text{fib}(d-2) \\ &= \text{fib}(d-3) + \left(2^k + 2^{k'}\right)\text{fib}(d-2) + 2^{k+k'}\text{fib}(d-1). \end{aligned}$$

Hence, (1) is established. Next we establish inequality (2). By (1),

$$i(B_{n,d,1}) = \text{fib}(d-3) + \left(2 + 2^{n-d}\right)\text{fib}(d-2) + 2^{n-d+1}\text{fib}(d-1),$$

and so in order to establish (2), we need only that $2^k + 2^{n-d+1-k} < 2 + 2^{n-d}$ for every integer k, where $1 < k < n-d$. Let $a := n-d$. The required inequality follows by a bit of arithmetic;

$$\begin{aligned} k &< a \implies \\ 2^k &< 2^a \implies \\ (2^{k-1}-1)2^{k+1} &< (2^{k-1}-1)2^{a+1} \implies \\ 2^{2k} + 2^{a+1} &< 2^{a+k} + 2^{k+1} \implies \\ 2^k + 2^{a+1-k} &< 2^a + 2 \implies \\ 2^k + 2^{n-d+1-k} &< 2^{n-d} + 2. \end{aligned}$$

Thus, inequality (2) holds and equality occurs if and only if $k \in \{1, n-d\}$. This completes the proof. ∎

Corollary 2.3
For any tree of order n and diameter d, where $1 \le d \le 3$,

$$i(T) \le \text{fib}(d) + 2^{n-d}\text{fib}(d+1). \tag{3}$$

Furthermore,

(i) *if* $d = 1$, *then* $T \simeq K_2$ *and equality holds in* (3).

(ii) *If* $d = 2$, *then* $T \simeq K_{1,n-1}$ *and equality holds in* (3)

(iii) *If* $d = 3$, *then* $T \simeq B_{n,3,k}$ *for some pair of positive integers* (n, k), *where* $1 \le k \le n - 3$, *and equality holds in* (3) *if and only if* $k \in \{1, n-3\}$.

Proof. Statements (i) and (ii) are easily verified and statement (iii) follows from Theorem 2.2. ■

The following result shows that if n is kept fixed, then $i(B_{n,d,1})$ is a strictly decreasing function of d.

Proposition 2.4
For any $d \ge 3$ *and* $n \ge d + 1$,

$$i(B_{n,d,1}) < i(B_{n,d-1,1})$$

Proof. The inequality is proved by the following calculation.

$$\begin{aligned} 1 &< 2^{n-d} \implies \\ \text{fib}(d-2) &< 2^{n-d}\text{fib}(d-2) \implies \\ \text{fib}(d) - \text{fib}(d-1) &< 2^{n-d}(2\text{fib}(d) - \text{fib}(d+1)) \implies \\ \text{fib}(d) + 2^{n-d}\text{fib}(d+1) &< \text{fib}(d-1) + 2^{n-(d-1)}\text{fib}(d) \implies \\ i(B_{n,d,1}) &< i(B_{n,d-1,1}). \end{aligned}$$

■

3 An Upper Bound on the Number of Independent Sets in a Tree

In this section we give an upper bound on the number of independent sets in a tree. The bound is a function of the order and the diameter of the tree, and it is optimal in the sense that, given any pair of integers (n, d), where $1 \le d \le n - 1$, there exist a tree T of order n and diameter d such that $i(T)$ equals the bound.

Theorem 3.1
Let T *denote a tree of order* $n \ge 2$ *and diameter* d. *Then*

$$i(T) \le \text{fib}(d) + 2^{n-d}\text{fib}(d+1) = i(B_{n,d,1}) \tag{4}$$

and equality occurs if and only $T \simeq B_{n,d,1}$.

Proof. We apply induction on the order of the tree. Let $T_{n,d}$ denote a tree on $n \geq 2$ vertices and with diameter d. If $n \leq 4$, then the diameter of $T_{n,d}$ is at most three and so by Corollary 2.3 the statement is true. Hence we may assume that $n \geq 5$ and that the statement is true for any tree with less than n vertices. By Corollary 2.3, we may also assume that $d \geq 4$.

Let $P : y_1, x_1x_2x_3 \ldots x_d$ denote a longest path in $T_{n,d}$. Let Y denote the set of leaves at x_1 and let $k = |Y| \geq 1$. Note that $k \leq n - d$ and $k = n - d$ if and only if $T_{n,d}$ is a simple broom.

Let $H_1 = T_{n,d} - \{y_1\}$ and $H_2 = T_{n,d} - (Y \cup \{x_1\})$. We observe that $i(T_{n,d}) = i(H_1) + 2^{k-1}i(H_2)$. Since $d \geq 4$ both H_1 and H_2 contain at least two vertices and so the induction hypothesis may be applied to these graphs.

(i) For $k = 1$ we find that H_1 has diameter $d_1 \geq d - 1$ and order $n - 1$ while H_2 has diameter $d_2 \geq d - 2$ and order $n - 2$. The induction hypothesis, along with Proposition 2.4 and Lemma 2.1, implies

$$\begin{aligned} i(H_1) &\leq i(B_{n-1,d_1,1}) \leq i(B_{n-1,d-1,1}) = \text{fib}(d-1) + 2^{(n-1)-(d-1)}\text{fib}(d) \text{ and} \\ i(H_2) &\leq i(B_{n-2,d_2,1}) \leq i(B_{n-2,d-2,1}) = \text{fib}(d-2) + 2^{(n-2)-(d-2)}\text{fib}(d-1). \end{aligned}$$

By using the above inequalities along with the inductive definition of the Fibonacci numbers, we obtain $i(T_{n,d}) \leq \text{fib}(d) + 2^{n-d}\text{fib}(d+1)$. Moreover, equality can only occur if both H_1 and H_2 are simple brooms with diameters $d-1$ and $d-2$, respectively. Consequently, both x_1 and x_2 have degree two in $T_{n,d}$, implying that $T_{n,d}$ is also a simple broom.

(ii) For $k \geq 2$ we find that H_1 has order $n - 1$ and diameter d while H_2 has order $n_2 := n - k - 1$ and diameter $d_2 \geq d - 2$. The induction hypothesis, along with Proposition 2.4, gives us the following inequalities.

$$\begin{aligned} i(H_1) &\leq i(B_{n-1,d,1}) = \text{fib}(d) + 2^{n-1-d}\text{fib}(d+1) \text{ and} \\ i(H_2) &\leq i(B_{n_2,d_2,1}) \leq i(B_{n_2,d-2,1}) = \text{fib}(d-2) + 2^{(n-k-1)-(d-2)}\text{fib}(d-1). \end{aligned}$$

We use the above inequalities to derive an upper bound for $i(T_{n,d})$.

$$\begin{aligned} i(T_{n,d}) &\leq \text{fib}(d) + 2^{n-d-1}\text{fib}(d+1) + 2^{k-1}\text{fib}(d-2) + 2^{k-1}2^{(n-k-1)-(d-2)}\text{fib}(d-1) \\ &= \text{fib}(d) + 2^{n-d-1}\text{fib}(d+1) + 2^{k-1}\text{fib}(d-2) + 2^{n-d}\text{fib}(d-1). \end{aligned} \quad (5)$$

In Lemma 2.1 we have an expression for the number of independent sets in a simple broom. Using this, along with the inductive definition of the Fibonacci numbers, the following expression is obtained through simple calculations.

$$i(B_{n,d,1}) = \text{fib}(d) + 2^{n-d-1}\text{fib}(d+1) + 2^{n-d-1}\text{fib}(d-2) + 2^{n-d}\text{fib}(d-1). \quad (6)$$

Now the inequality $k \le n - d$ together with (6) and (5) implies $i(T_{n,d}) \le i(B_{n,d,1})$. Moreover, if equality occurs then we must have $k = n - d$, that is, $T_{n,d} \simeq B_{n,d,1}$.

In each case we have proved that $i(T_{n,d}) \le i(B_{n,d,1})$ and that equality occurs if and only if $T_{n,d}$ is isomorphic to $B_{n,d,1}$. Hence the proof is complete. ∎

4 A Comparative Study of Two Upper Bounds for $i(T)$

It is easy to show that the upper bound in Theorem 3.1 is better than the bound in Theorem 1.1. In the following we compare the upper bound in Theorem 3.1 with an upper bound given by Dutton et al. (1993).

Theorem 4.1 (Dutton et al., 1993)
Let T denote a nontrivial tree on n vertices. Let β_1 denote the matching number of T. Then

$$i(T) \le \frac{2}{3}2^n\left(\frac{3}{4}\right)^{\beta_1} + 2^{\beta_1 - 1} =: h(n, \beta_1).$$

Now the question is which of the upper bounds $g(n,d) := \mathrm{fib}(d) + 2^{n-d}\mathrm{fib}(d+1)$ and $h(n, \beta_1)$ is better. The main result of this section gives a sufficient condition for the bound g in Theorem 3.1 to be better than the bound h in Theorem 4.1.

Theorem 4.2
Let T denote a tree of order n and diameter $d \ge 3$. Let β_1 denote the matching number of T. If $d > 0.68n + 3$, then $g(n,d) < h(n, \beta_1)$.

The proof of Theorem 4.2 is established through a few lemmas. To simplify notation we write β instead of β_1.

Lemma 4.3
For pairs of integers $n \ge 2$ and $\beta \in \{1, \ldots, \lfloor n/2 \rfloor\}$,

$$h(n, \beta) < h(n, \beta - 1).$$

Proof. A bit of arithmetic establishes the desired inequality.

$$\begin{aligned}
\beta &\le \frac{n}{2} \Longrightarrow \\
\beta &< \frac{7n}{10} - \frac{2}{10} \Longrightarrow \\
\beta &< \frac{n\ln(2)}{\ln(8/3)} + \frac{\ln(8/9)}{\ln(8/3)} \Longrightarrow \\
\ln\left((8/3)^{\beta}\right) &< n\ln(2) + \ln(8/9) \Longrightarrow \\
\left(\frac{8}{3}\right)^{\beta} &< 2^n\frac{8}{9} \Longrightarrow \\
2^{\beta}\left(\frac{4}{3}\right)^{\beta} &< \frac{2^{n+1}}{3}\frac{4}{3} \Longrightarrow \\
\frac{2^{\beta}}{4} &< \frac{2^{n+1}}{3}\left(\frac{3}{4}\right)^{\beta}\frac{1}{3} \Longrightarrow \\
2^{\beta}\left(\frac{1}{2} - \frac{1}{4}\right) &< \frac{2^{n+1}}{3}\left(\frac{3}{4}\right)^{\beta}\left(\frac{4}{3} - 1\right) \Longrightarrow \\
\frac{2^{n+1}}{3}\left(\frac{3}{4}\right)^{\beta} + 2^{\beta-1} &< \frac{2^{n+1}}{3}\left(\frac{3}{4}\right)^{\beta}\frac{4}{3} + 2^{\beta-2} \Longrightarrow \\
h(n,\beta) &< h(n,\beta-1).
\end{aligned}$$

■

Corollary 4.4
For any integer $n \ge 2$ and $\beta \in \{1, \ldots, \lfloor n/2 \rfloor\}$,

$$h(n,\beta) \ge h(n, n/2).$$

It is well-known that the nth Fibonacci number may be written as

$$\mathrm{fib}(n) = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}. \tag{7}$$

See for instance Redmond (1996). Using (7) and the Triangle Inequality we obtain the following result.

Lemma 4.5
For any positive integer n,

$$\frac{(1+\sqrt{5})^n}{2^n\sqrt{5}}\left(1 - \frac{1}{2^n}\right) \le \mathrm{fib}(n) \le \frac{(1+\sqrt{5})^n}{2^n\sqrt{5}}\left(1 + \frac{1}{2^n}\right).$$

Finally, we are able to give a proof of Theorem 4.2.

Proof of Theorem 4.2. Observe that

$$g(n,d) < 2^{n-d}\mathrm{fib}(d) + 2^{n-d}\mathrm{fib}(d+1) = 2^{n-d}\mathrm{fib}(d+2).$$

Since $d \geq 3$, we have $(1 + \frac{1}{2^{d+2}}) < \frac{17}{16}$ and so, according to Lemma 4.5,

$$\mathrm{fib}(d+2) < \left(\frac{1+\sqrt{5}}{2}\right)^{d+2} \frac{17}{16\sqrt{5}}.$$

By Corollary 4.4,

$$h(n,\beta) \geq h(n,n/2) = \frac{2}{3}2^n\left(\frac{3}{4}\right)^{n/2} + 2^{n/2-1} > \frac{2}{3}(\sqrt{3})^n.$$

Thus, to prove $g(n,d) < h(n,\beta)$ it suffices to prove

$$2^{n-d}\left(\frac{1+\sqrt{5}}{2}\right)^{d+2}\frac{17}{16\sqrt{5}} < \frac{2}{3}(\sqrt{3})^n. \tag{8}$$

Define

$$x := \ln\left(\frac{51(1+\sqrt{5})^2}{128\sqrt{5}}\right), \quad y := \ln\left(\frac{1+\sqrt{5}}{4}\right) \quad \text{and} \quad z := \ln\left(\frac{\sqrt{3}}{2}\right).$$

We note that $y \approx -0.212$, $(-x/y) \approx 2.9433 < 3$ and $z/y \approx 0.6787 < 0.68$. By the hypothesis we have $d > 0.68n + 3$, therefore $d > nz/y - x/y$. Using this, we derive inequality (8).

$$\begin{aligned}
d &> nz/y - x/y \implies \\
x + dy &< nz \implies \\
\exp(x)\exp(y)^d &< \exp(z)^n \implies \\
\frac{51(1+\sqrt{5})^2}{128\sqrt{5}}\left(\frac{1+\sqrt{5}}{4}\right)^d &< \left(\frac{\sqrt{3}}{2}\right)^n \implies \\
\frac{2^{n+1}}{3}\frac{3\cdot 17}{4\cdot 2\cdot 16}\frac{(1+\sqrt{5})^2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{4}\right)^d &< \frac{2^{n+1}}{3}\left(\frac{\sqrt{3}}{2}\right)^n \implies \\
\frac{2^n 17}{2^d 16\sqrt{5}}\left(\frac{1+\sqrt{5}}{4}\right)^{d+2} &< \frac{2}{3}(\sqrt{3})^n,
\end{aligned}$$

which is the desired inequality (8). ■

5 A Table of Trees with Less Than Nine Vertices

When studying the behavior of the graph parameter i on the class of trees, it is very helpful to have a list of all non-isomorphic trees of "small" order. Such lists may be found in Harary (1969) and Read and Wilson (1998). All the trees of order ≤ 8 are listed below along with the value of the graph parameter i. The numeration of the trees follows that of Read and Wilson (1998).

It follows from Figure 6 that two non-isomorphic trees T_1 and T_2 may satisfy $i(T_1) = i(T_2)$.

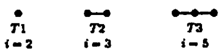

Figure 2: The trees with 1, 2 or 3 vertices.

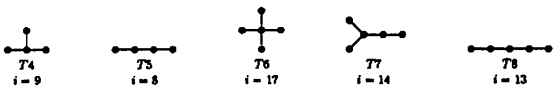

Figure 3: The trees with 4 or 5 vertices.

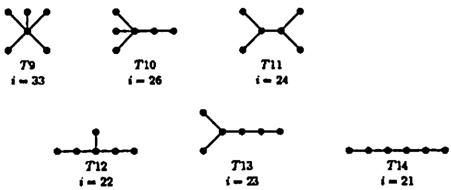

Figure 4: The trees with 6 vertices.

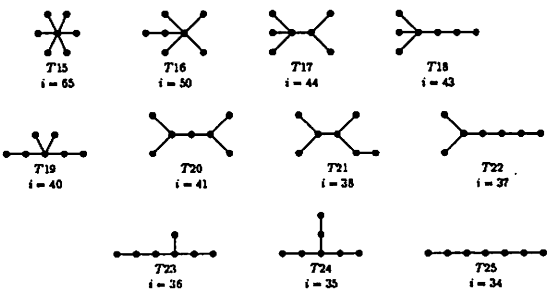

Figure 5: The trees with 7 vertices.

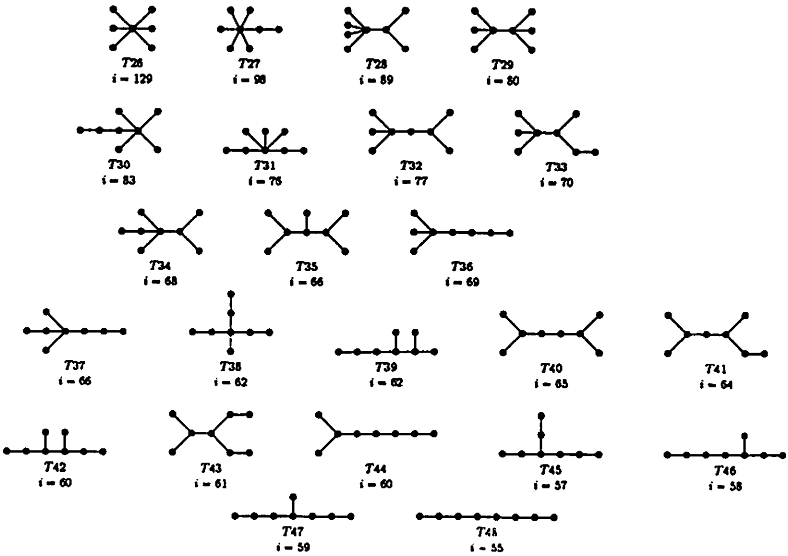

Figure 6: The trees with 8 vertices.

6 Concluding Remarks

In this paper we have obtained an optimal upper bound of $i(T)$ in terms of the order and diameter of the tree T. The analogous problem of obtaining an optimal lower bound of $i(T)$ in terms of the order and diameter is still open.

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