COLORING OF MEET-SEMILATTICES

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ABSTRACT. Beck's coloring is studied for meet-semilattices with 0. It is shown that for such semilattices, the chromatic number equals the clique number.

KEY WORDS AND PHRASES: Coloring of a semilattice, clique number, chromatic number, annihilator, semi-ideal.

1. PRELIMINARIES

Beck [3] introduced the notion of coloring in commutative rings as follows. Associate to a commutative ring R a simple graph G whose vertices are labelled by the elements of R and with two vertices x, y adjacent (connected by an edge) in case $xy = 0$. The chromatic number $\chi(R)$ of R is the chromatic number of G, namely the minimal number of colors which can be assigned to the vertices of G in such a way that adjacent vertices of G have different colors. If this number is not finite write $\chi(R) = \infty$. A subset $C = \{x_1, x_2, \cdots\}$ of R is called a clique in case $x_i x_j = 0$ whenever $i \neq j$. If R contains a clique with n elements and every clique has at most n elements, then the clique number of R, is $Clique(R) = n$. If the sizes of the cliques are not bounded, then $Clique(R) = \infty$. The clique number of R is then the least upper bound of the number of vertices in a complete subgraph of G. We always have $\chi(R) \geq Clique(R)$ and Beck [3] conjectured that $\chi(R) = Clique(R)$, but Anderson and Naseer [1] gave an example of a commutative local ring R with 32 elements for which $\chi(R) > Clique(R)$.

Notice that the definitions above depend only on the multiplicative structure of R, and thus can be made just as well for a commutative semigroup S with 0. Assuming this is done, we can consider the coloring problem for semigroups and in particular for meet-semilattices with 0.

A commutative semigroup S in which $x^2 = x$ for all $x \in S$ is a partially ordered set with $x \leq y$ if and only if $x = xy$. It becomes a meet-semilattice with $a \wedge b = ab$. Each meet-semilattice determines and is determined by a commutative semigroup S in which every element is idempotent see; page 24 of Clifford and Preston [4].

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In this article we show that the chromatic number and clique number of a meet-semilattice S with 0 (the least element) are the same. Thus Beck's conjecture holds for such semilattices. Further we show that if these numbers are finite, then these are determined by the number of minimal prime semi-ideals of S. In section 2 we show that an infinite distributive lattice L with 0 (the least element) and 1 (the greatest element) and with at least one nonzero complemented element has infinitely many zerodivisors, see; Ganesan [6] for the case of rings. We also show that if L has infinitely many elements of finite order then the graph of L contains an infinite clique [3]. The undefined terms are from Harary [8] and Grätzer [7].

2. The Chromatic Number of a Meet-semilattice with 0

We begin with an example of a graph for which the clique number and the chromatic number are distinct. However, it cannot be a graph of a meet-semilattice with 0. The clique number of the graph shown in figure 1 is 3 and its chromatic number is 4.

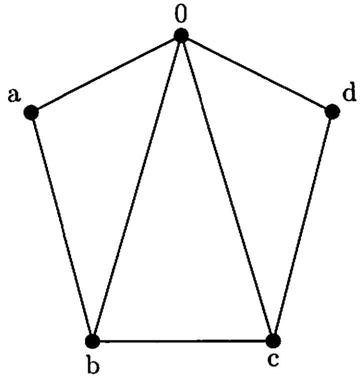

Figure 1

We consider the set $S = \{0, a, b, c, d\}$. The graph determines which meets are zero and which are nonzero. Thus $a \wedge b = b \wedge c = c \wedge d = 0$ and $a \wedge c \neq 0$, $a \wedge d \neq 0$, $b \wedge d \neq 0$. In order that S be a meet-semilattice, all nonzero meets and in particular, $a \wedge d \in S$. We have the following possibilities.

(1) $a \wedge d = a \Longrightarrow a \wedge c = a \wedge d \wedge c = 0$.
(2) $a \wedge d = b \Longrightarrow a \wedge d = a \wedge a \wedge d = a \wedge b = 0$.
(3) $a \wedge d = c \Longrightarrow a \wedge d = a \wedge d \wedge d = c \wedge d = 0$.
(4) $a \wedge d = d \Longrightarrow b \wedge d = b \wedge a \wedge d = 0$.

Thus a contradiction in each case. Therefore $a \wedge d \notin S$ and so S cannot be a meet-semilattice.

The purpose of this article is to prove the following.

Theorem 1. *If S is a meet-semilattice with 0, then $\chi(S) = Clique(S)$.*

The proof of this theorem will be accomplished by a series of lemmas, and our methods loosely follow those used by Beck to prove Theorem 3.8 of [3] which gives Theorem 1 for reduced rings. Note that if $Clique(S) = \infty$ then $\chi(S) \geq Clique(S) = \infty$ and the result follows. Thus it suffices to check the result when $Clique(S)$ is finite.

Raney [9] introduced the concept of a semi-ideal in a partially ordered set. A nonempty subset I of a partially ordered set P is called a *semi-ideal*, if for $x \in I$, $y \in P$, $y \leq x$ implies $y \in I$. It can be similarly defined for a meet-semilattice. A proper semi-ideal I of a meet-semilattice S is called a *prime semi-ideal*, if $a \wedge b \in I$ implies $a \in I$ or $b \in I$. For $x \in S$, the annihilator of x is the set $ann(x) = \{y \in S \mid x \wedge y = 0\}$. It is clear that $ann(x)$ is a semi-ideal for each $x \in S$.

Lemma 1. *Let S be a meet-semilattice with 0. If $x \in S$ and $ann(x)$ is maximal in the set $\{ann(a) \mid a \in S, ann(a) \neq S\}$, then $ann(x)$ is a prime semi-ideal in S.*

Proof. Assume $a \wedge b \in ann(x)$, $a \notin ann(x)$. Then $a \wedge b \wedge x = 0$ and $a \wedge x \neq 0$ so $b \in ann(a \wedge x)$. Let $t \in ann(x)$, then $t \wedge x = 0$ so $t \wedge a \wedge x = 0$ which implies $t \in ann(a \wedge x)$. Therefore, $ann(x) \subseteq ann(a \wedge x)$. By maximality, this implies either $ann(x) = ann(a \wedge x)$ or $ann(a \wedge x) = S$. Since $a \wedge x \neq 0$, the second alternative is not the case. Therefore $b \in ann(x)$ and $ann(x)$ is prime. □

Lemma 2. *Let S be a meet-semilattice with 0. If I is a prime semi-ideal in S and $x \in S$ but $x \notin I$ then $ann(x) \subseteq I$.*

Proof. If $y \in ann(x)$ then $x \wedge y = 0 \in I$ which implies $y \in I$ since $x \notin I$ and I is a prime semi-ideal. □

Lemma 3. *Let S be a meet-semilattice with 0. If for some $x, y \in S$, $ann(x)$ and $ann(y)$ are distinct prime semi-ideals, then $x \wedge y = 0$.*

Proof. Suppose $x \wedge y \neq 0$. Note that for $t \in S$, $t \wedge y \in ann(x)$ if and only if $t \in ann(x \wedge y)$. Since $ann(x)$ is prime and $y \notin ann(x)$, $t \wedge y \in ann(x)$ implies $t \in ann(x)$. Therefore $ann(x) = ann(x \wedge y)$. Similarly $ann(x \wedge y) = ann(y)$. Thus, $ann(x) \neq ann(y)$ implies $x \wedge y = 0$. □

Lemma 4. *If S is a meet-semilattice with 0 and $Clique(S) < \infty$ then every nonempty set of semi-ideals in S of the form $ann(x)$, $x \neq 0$, contains a maximal element.*

Proof. Suppose $ann(x_1) \subset ann(x_2) \subset \cdots$ is a proper ascending chain. Let $a_j \in ann(x_j) - ann(x_{j-1})$ for $j > 1$. Then $y_j = a_j \wedge x_{j-1} \neq 0$ and $y_j \wedge x_j = a_j \wedge x_j \wedge x_{j-1} = 0$. Then $y_j \in ann(x_j) - ann(x_{j-1})$. Thus the y_j

are all distinct. Moreover, for $j < k$, $y_j \wedge y_k = a_j \wedge x_{j-1} \wedge a_k \wedge x_{k-1} = 0$ since $a_j \in ann(x_j) \subset ann(x_{k-1})$. Thus the $y_j's$ form an infinite clique and the result follows. □

Lemma 5. *If S is a meet-semilattice with 0 and $Clique(S) < \infty$, then every minimal prime semi-ideal P in S has the form $ann(x)$ for some $x \in S$.*

Proof. Let P be a minimal prime semi-ideal in S. If $x \notin P$ then $ann(x) \subseteq P$ by Lemma 2. By lemma 4 we can find $ann(y)$ maximal in $\{ann(x) \mid ann(x) \subseteq P,\ x \notin P\}$. If $ann(y)$ is prime then by minimal choice of P, $ann(y) = P$. Thus it suffices to check $ann(y)$ is prime.
Case 1. Assume $a \wedge b \in ann(y)$ and $a \notin P$. Then $a \wedge b \wedge y = 0$ and $a \wedge y \neq 0$ so $b \in ann(a \wedge y) \subsetneq S$. Containment is proper since $a \wedge y \neq 0$. Since $a \notin P$ and $y \notin P$, $a \wedge y \notin P$. By Lemma 2, $ann(a \wedge y) \subseteq P$. Since $ann(y) \subseteq ann(a \wedge y)$, by maximal choice of $ann(y)$, $ann(a \wedge y) = ann(y)$. Hence $b \in ann(y)$ and $ann(y)$ is prime.
Case 2. Let $a \wedge b \in ann(y)$ for $a \in P - ann(y)$. Then $a \wedge b \wedge y = 0$ and $a \wedge y \neq 0$ so $b \in ann(a \wedge y) \subsetneq S$. If $ann(a \wedge y) \subset P$ then $b \in ann(y)$ as in case 1. If $0 \neq c \in ann(a \wedge y) - P$ then $c \wedge a \wedge y = 0$, so $c \wedge a \in ann(y)$ with $c \notin P$. By case 1 with a replaced by c and b replaced by a, we have $a \in ann(y)$ which contradicts the choice of a. Hence $ann(a \wedge y) \subseteq P$ and as above $ann(y)$ is prime so $P = ann(y)$. □

A semi-ideal I of a meet-semilattice S with 0 is called a maximal annihilator semi-ideal, if I is a maximal element of $\{ann(x) \mid x \neq 0, x \in S\}$.

Lemma 6. *If S is a meet-semilattice with 0 and $Clique(S) < \infty$, then the set of distinct proper maximal annihilator semi-ideals of S is finite.*

Proof. Let $A = \{x_i \mid ann(x_i)$ are distinct and maximal, $x_i \neq 0\}$. By Lemma 1, the semi-ideals $ann(x_i)$ are prime. If $x_i, x_j \in A$ with $i \neq j$ then $x_i \wedge x_j = 0$ by Lemma 3. Since $Clique(S) > |A|$ and $Clique(S) < \infty$, $|A| < \infty$. □

Lemma 7. *Let S be a meet-semilattice with 0 and assume $Clique(S) < \infty$. Then the semi-ideal $\{0\}$ in S is a finite intersection of minimal prime semi-ideals. The semi-ideals in the intersection are all the minimal prime semi-ideals of S.*

Proof. By Lemma 6, we let $\{ann(x_i) \mid 1 \leq i \leq n,\ x_i \neq 0\}$ be the finite set of distinct maximal annihilator semi-ideals in S. By Lemma 1, this is a set of prime semi-ideals. By Lemma 3, if $i \neq j$ then $x_i \wedge x_j = 0$. If $a \in \bigcap_{i=1}^{n} ann(x_i)$ and $a \neq 0$ then $a \wedge x_i = 0$ for all i. Therefore $x_i \in ann(a)$ for every i. But $ann(a) \subseteq ann(x_i)$ for some i since every annihilator semi-ideal belongs to a maximal annihilator semi-ideal by Lemma 4. Therefore

$x_i \in ann(x_i)$ which is impossible since $x_i \neq 0$. We denote $ann(x_i)$ by P_i. Thus $\{0\} = \bigcap_{i=1}^n P_i$ with P_i prime semi-ideals. In this intersection we can assume all the P_i are distinct, no P_i contains P_j whenever $i \neq j$, and no P_i contains $\bigcap_{j \neq i} P_j$. To show that P_i are all minimal prime semi-ideals it suffices to check that for any prime semi-ideal P, $P_i \subseteq P$ for some i. Otherwise there exists $y_i \in P_i - P$ for every i. Let $y = \bigwedge_{i=1}^n y_i$. Since $y \leq y_i$, for each i implies $y \in P_i$ for all i, hence $y = 0$. Then $y \in P$ and P is prime implies, $y_i \in P$ for some i, a contradiction. □

Theorem 2. *Let S be a meet-semilattice with 0, and assume $Clique(S) < \infty$. Then the number n of minimal prime semi-ideals in S is finite and $\chi(S) = Clique(S) = n + 1$.*

Proof. By Lemma 7 we can write $\{0\} = P_1 \cap \cdots \cap P_n$ for all the minimal prime semi-ideals P_i of S. By Lemma 5, $P_i = ann(x_i)$ for each i so $\{0\} = \bigcap_{i=1}^n ann(x_i)$. By Lemma 3, $\{x_i \mid 1 \leq i \leq n\} \cup \{0\}$ is a clique in S. Therefore $Clique(S) \geq n + 1$. Define a coloring f on S by $f(0) = 0$ and if $x \neq 0$ let $f(x) = min\ \{i \mid x \notin P_i\}$. If x, y are connected by an edge then $x \wedge y = 0$. If $f(x) = k + 1$, then $x \in P_i$, $1 \leq i \leq k$ and $x \notin P_{k+1}$. It follows that $y \in P_{k+1}$ since $x \wedge y = 0 \in P_{k+1}$ so $f(y) \neq k + 1$ and thus $f(x) \neq f(y)$. Thus f is a coloring of S. Therefore $\chi(S) \leq n + 1$. Thus, $n + 1 \geq \chi(S) \geq Clique(S) \geq n + 1$ so $\chi(S) = Clique(S) = n + 1$. □

Let S be a meet-semilattice with 0. An element $x \in S$ is called a zerodivisor in case $x \neq 0$ and there is a $y \in S$, $y \neq 0$ with $x \wedge y = 0$. In [2] (for commutative rings), and [5] (for commutative semigroups with 0) a simple graph G, called the zerodivisor graph of S, is associated to S. The vertices of G are the nonzero zerodivisors of S with an edge connecting two distinct vertices x, y in case $xy = 0$. When G is non empty, both its chromatic number and clique number are one less than the graph which Beck [3] associated to S.

Corollary 1. *If S is a meet-semilattice with 0 and with at least one zero-divisor, G is the zerodivisor graph associated to S, and the clique number of $G < \infty$ then $\chi(G) = Clique(G) = n$ where n is the number of minimal prime semi-ideals in S.*

3. Complemented Distributive Lattices

In this short section we give two results about the zerodivisors in distributive lattices which have analogs for commutative rings. We say that an element x in a (semi)lattice L is finite in case the principal ideal generated by x, $(x] = \{y \in L \mid y \leq x\}$ is finite. The first result is a variant of a result of Ganesan [6].

Theorem 3. *If L is an infinite distributive lattice with 0 and 1 and which contains at least one nonzero complemented element other than 1, then L contains infinitely many zero divisors.*

Proof. Let $x \neq 0$, $x \neq 1$ be a complemented element in L. There exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' = 1$. If $(x]$ is infinite, then for each $y \in (x]$ we have $y \wedge x' = 0$ and the result follows. Otherwise assume $(x] = \{x_1, \cdots, x_n\}$. For some i, $J_i = \{z \in L \mid z \wedge x = x_i\}$ is infinite. But $x_i \leq x$ implies $z \wedge x_i = x_i$ so $J_i = \{z \mid z \wedge x_i = x_i\}$. We have $z = z \wedge (x \vee x') = (z \wedge x) \vee (z \wedge x') = x_i \vee (z \wedge x')$. Therefore, if $z_1 \neq z_2$ then $z_1 \wedge x' \neq z_2 \wedge x'$. In particular $z \wedge x' \neq 0$ for any $z \notin (x]$. This implies $(z \wedge x') \wedge x = 0$ for infinitely many $z \in L$. Thus, L has infinitely many zerodivisors. □

Example 1

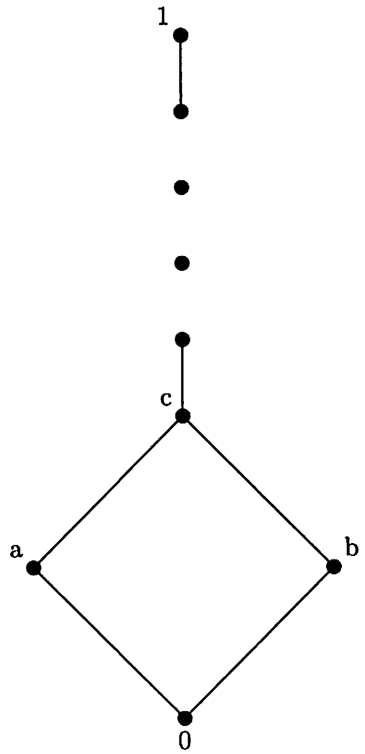

Figure 2

The infinite distributive lattice in Figure 2 shows that the existence of a nonzero complemented element is necessary in Theorem 3. This lattice does not have a nonzero complemented element other than 1 and the only nonzero zerodivisors are a and b.

The following (with a similar proof) is an analog of Lemma 3.1 of [3].

Theorem 4. *If a complemented distributive lattice has an infinite number of finite elements, then L contains an infinite clique.*

Proof. Let $x_1, x_2, \cdots$ be finite elements in L. Then $x_1 \wedge x_2, x_1 \wedge x_3, \cdots$ belong to the finite set $(x_1]$. Hence for some a_j, $x_1 \wedge x_{a_1} = x_1 \wedge x_{a_2} = \cdots$. Consider the elements $x_{a_1}, x_{a_2}, \cdots$ and repeat the procedure. Continuing in this way, construct a set $\{y_1, y_2, \cdots\} \subset \{x_1, x_2, \cdots\}$ such that $y_i \wedge y_j = y_i \wedge y_k$ for $j, k > i$. In this subset $y_1 = x_1, y_2 = x_{a_1}$, etc. Put $z_{ij} = y_i \wedge y_j'$, then $z_{ij} \wedge z_{kr} = 0$ if $1 < j < k < r$. Consider $z_{12} \wedge z_{34} = z_{12} \wedge z_{35} = 0$. Since $z_{34} \neq z_{35}$, at least one of z_{34} and z_{35} is different from z_{12}, say $z_{35} \neq z_{12}$. Then $\{z_{12}, z_{35}\}$ is a clique with two elements. Note z_{67}, z_{68}, z_{69} are different and if say $z_{69} \notin \{z_{12}, z_{35}\}$ then $\{z_{12}, z_{35}, z_{69}\}$ is a clique with three elements. Continuing in this way, we get an infinite clique. □

Example 2 This example shows that the assumption of distributivity is necessary in Theorem 4. An integer a is divisible by an integer b, if $a = bc$ for some integer c. In this case write $b|a$. Thus, $a|0$ for all integers a including $a = 0$. Let $A = \{a \in \mathbb{Z} |\ a > 0,\ 2|a,\ 3 \nmid a\}$ and $B = A \cup \{0, 1, 3\}$. Then B is a complemented lattice under divisibility order with smallest element 1 and largest element 0. Since any complemented distributive lattice is uniquely complemented, this lattice is not distributive as the element 3 has infinitely many complements. Every nonzero element is a finite element, but B does not have an infinite clique.

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