Towards a Characterisation of Lottery Set Overlapping Structures

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Abstract

Consider a lottery scheme consisting of randomly selecting a winning t–set from a universal m–set, while a player participates in the scheme by purchasing a playing set of any number of n–sets from the universal set prior to the draw, and is awarded a prize if k or more elements of the winning t–set occur in at least one of the player's n–sets ($1 \leq k \leq \{n,t\} \leq m$). This is called a k–prize. The player may wish to construct a playing set, called a lottery set, which guarantees the player a k–prize, no matter which winning t–set is chosen from the universal set. The cardinality of a *smallest* lottery set is called the lottery number, denoted by $L(m,n,t;k)$, and the number of such non–isomorphic sets is called the lottery characterisation number, denoted by $\eta(m,n,t;k)$. In this paper an exhaustive search technique is employed to characterise minimal lottery sets of cardinality not exceeding six, within the ranges $2 \leq k \leq 4$, $k \leq t \leq 11$, $k \leq n \leq 12$ and $\max\{n,t\} \leq m \leq 20$. In the process 32 new lottery numbers are found, and bounds on a further 31 lottery numbers are improved. We also provide a theorem that characterises when a minimal lottery set has cardinality two or three. Values for the lottery characterisation number are also derived theoretically for minimal lottery sets of cardinality not exceeding three, as well as a number of growth and decomposition properties for larger lotteries.

Keywords: Lottery, Lottery problem, Lottery design.

AMS Subject Classifacation: 05B05, 05B40, 05C69, 62K05, 62K10.

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1 Introduction

Suppose the lottery scheme $\langle m, n, t; k\rangle$ consists of randomly selecting a winning t–set w from the universal set $\mathcal{U}_m = \{1, 2, \ldots, m\}$, while a player participates in the scheme by selecting a playing set $\mathcal{P}$ of any number of n–sets from $\mathcal{U}_m$ prior to the draw, and is awarded a prize, called a *k–prize*, if at least k elements of w match those of at least one of the player's n–sets in $\mathcal{P}$. Here we assume that $1 \leq k \leq \{n, t\} \leq m$.

Let $\Phi(\mathcal{A}, s)$ denote the set of all (unordered) s–sets from a set $\mathcal{A}$, so that $|\Phi(\mathcal{A}, s)| = \binom{|\mathcal{A}|}{s}$. We consider the following problem.

Definition 1 (The lottery problem) *Define a* lottery set *for* $\langle m, n, t; k\rangle$ *as a subset* $\mathcal{L}(\mathcal{U}_m, n, t; k) \subseteq \Phi(\mathcal{U}_m, n)$ *with the property that, for any element* $\phi_t \in \Phi(\mathcal{U}_m, t)$*, there exists an element* $l \in \mathcal{L}(\mathcal{U}_m, n, t; k)$ *such that* $\Phi(\phi_t, k) \cap \Phi(l, k) \neq \emptyset$ *(i.e., there is a* k–intersection *between* ϕ_t *and* l*). Then the* lottery problem *is: what is the smallest possible cardinality of a lottery set* $\mathcal{L}(\mathcal{U}_m, n, t; k)$*? Denote the answer to this question by the* lottery number $L(m, n, t; k)$*. We refer to a lottery set of cardinality* $L(m, n, t; k)$ *as an* $L(m, n, t; k)$*-set for* $\langle m, n, t; k\rangle$*.* ■

The above–mentioned combinatorial optimisation problem has been studied extensively (see, for example, the references listed in the bibliography of [8]). However, with the exception of a few basic classes, lottery numbers are generally only known for small values of the parameters m, n, t and k. See [1, 7, 11, 12] for listings of known, small lottery numbers and [6, 7] for some new lottery numbers and improvements on the bounds of yet undetermined lottery numbers.

Given an $L(m, n, t; k)$–set $\mathcal{L} = \{T_1, T_2, \ldots, T_{L(m,n,t;k)}\}$ for $\langle m, n, t; k\rangle$, it is possible to interchange the roles of elements in $\mathcal{U}_m$ in order to induce a different $L(m, n, t; k)$–set for $\langle m, n, t; k\rangle$. Although these $L(m, n, t; k)$–sets are different, they still have isomorphic structures in terms of n–set overlappings. We denote the number of non–isomorphic n–set overlapping structures of an $L(m, n, t; k)$–set by the characterisation number $\eta(m, n, t; k)$.

In this paper we seek to characterise all overlapping structures of cardinality not exceeding 6 within the parameter ranges $2 \leq k \leq 4$, $k \leq t \leq 11$, $k \leq n \leq 12$ and $\max\{n, t\} \leq m \leq 20$. In §2 we give a characterisation, as well as theoretically determined values for $\eta(m, n, t; k)$ for all lotteries whose minimal lottery set cardinalities do not exceed 3. In §3 some growth properties of $\eta(m, n, t; k)$ are established, and in §4 and §5 we develop an exhaustive search procedure for characterising overlapping structures of small $L(m, n, t; k)$–sets. This technique is employed to establish 32 new lottery numbers and to improve upon best known bounds for a further 31 lottery numbers, as listed in §7. In [6] the same procedure was extended to the ranges of the parameters of designs listed in [1], where 192 of the listed designs were shown to be optimal, 204 new lottery numbers were determined, and a further 304 upper bounds were improved. Also, the same procedure is employed in [4] to characterise solution sets to a novel, incomplete version of the lottery problem.

2 Notation and characterisation theorems

We require an efficient coding scheme whereby the structure of a lottery set, $\mathcal{L} = \{T_1, T_2, \ldots, T_{L(m,n,t;k)}\}$ may be captured. This may be achieved by defining the function

$$x^{(L)}_{(t_L t_{L-1} \cdots t_2 t_1)_2} = \left| \bigcap_{i=1}^{L} \left\{ \begin{array}{ll} T_i & \text{if } t_i = 1 \\ T_i' & \text{if } t_i = 0 \end{array} \right. \right|,$$

where $(t_L t_{L-1} \cdots t_2 t_1)_2$ denotes the binary representation of an integer in the range $\{0, \ldots, 2^L - 1\}$ and where T_i' denotes the complement $\mathcal{U}_m \setminus T_i$. This function induces the 2^L–integer vector

$$\vec{X}^{(L)} = \left(x^{(L)}_{(000\cdots00)_2}, x^{(L)}_{(000\cdots01)_2}, \ldots, x^{(L)}_{(111\cdots11)_2} \right),$$

which represents all the information needed to describe the n–set overlapping structure of any playing set (and hence any lottery set of minimum cardinality) for $\langle m, n, t; k \rangle$. The entries of the vector $\vec{X}^{(L)}$ add up to m and may be interpreted as follows:

- there are $x^{(L)}_{(000\cdots00)_2}$ elements of $\mathcal{U}_m$ in no n–set of $\mathcal{L}$;
- there are $x^{(L)}_{(000\cdots01)_2}$ elements of $\mathcal{U}_m$ in only T_1;
- there are $x^{(L)}_{(000\cdots10)_2}$ elements of $\mathcal{U}_m$ in only T_2;
- there are $x^{(L)}_{(000\cdots11)_2}$ elements of $\mathcal{U}_m$ in both T_1 and T_2, *etc.*

Sometimes it is more convenient to write the subscripts of the $x^{(L)}$ entries in decimal form. In this paper we shall mostly use an abbreviated notation for the vector $\vec{X}^{(L)}$, namely a vector in which only non–zero entries are specified (*i.e.*, if vector entry $x^{(L)}_d = a$ (subscript in decimal form) is non–zero it will be denoted by d^a in the abbreviated vector notation). [Square brackets] will be used to distinguish this abbreviated vector notation from the full vector notation (in round brackets). We illustrate the

$\mathcal{U}_{26}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
T_1		×	×	×	×	×	×	×																		
T_2									×	×	×	×					×	×	×							
T_3													×	×	×	×	×	×	×							
T_4																				×	×	×	×	×	×	×
	1	7							4				4				3			7						

Figure 2.1: Tabular representation of the $L(26,7,14;4)$–set in Example 1.

above method of lottery set structure encoding and notation by means of a simple example.

Example 1 *The set* $\vec{X}^{(4)} = (1,7,4,0,4,0,3,0,7,0,0,0,0,0,0,0) = [0^1, 1^7, 2^4, 4^4, 6^3, 8^7]$ *is a lottery set with minimum cardinality* 4 *for* $\langle 26,7,14;4\rangle$. *An instance adhering to this lottery set structure may be found by focusing on the non-zero entries in the vector* $\vec{X}^{(4)}$: $x_0^{(4)} = 1$, $x_1^{(4)} = 7$, $x_2^{(4)} = 4$, $x_4^{(4)} = 4$, $x_6^{(4)} = 3$ *and* $x_8^{(4)} = 7$. *In binary form these are* $x_{(0000)_2}^{(4)} = 1$, $x_{(0001)_2}^{(4)} = 7$, $x_{(0010)_2}^{(4)} = 4$, $x_{(0100)_2}^{(4)} = 4$, $x_{(0110)_2}^{(4)} = 3$ *and* $x_{(1000)_2}^{(4)} = 7$, *which yield the structure of all corresponding* $L(26,7,14;4)$–*sets, in terms of the number of elements from the universal set* $\mathcal{U}_{26}$ *in each term of the inclusion–exclusion counting principle. The set* $\mathcal{L} = \{\{2,3,4,5,6,7,8\}, \{9,10,11,12,17,18,19\}, \{13,14,15,16,17,18,19\}, \{20,21,22,23,24,25,26\}\}$ *emerges as an example of an* $L(26,7,14;4)$*–set of this structure, which is represented in tabular form in Figure 2.1.* ■

We state, without proof, the following well–known lottery isomorphism result that appears in several papers (see for example [14]).

Theorem 1 (Lottery Isomorphism)
$\langle m,n,t;k\rangle \simeq \langle m, m-n, m-t; m+k-n-t\rangle$, *so that*

$$\begin{aligned} L(m,n,t;k) &= L(m,m-n,m-t;m+k-n-t) \\ \text{and } \eta(m,n,t;k) &= \eta(m,m-n,m-t;m+k-n-t) \end{aligned}$$

for all $1 \leq k \leq \{n,t\} < m$ *satisfying* $m+k > n+t$. ■

It is possible to characterise small lottery numbers, as we do in the following theorem.

Theorem 2 **(Characterisation of Small Lottery Numbers)**
For all $1 \leq k \leq \{n, t\} \leq m$,
(a) $L(m, n, t; k) = 1$ *if and only if* $n + t \geq m + k$.
(b) $L(m, n, t; k) = 2$ *if and only if* $2k - 1 + \max\{m - 2n, 0\} \leq t \leq m + k - n - 1$.
(c) $L(m, n, t; k) = 3$ *if and only if*

$$t \leq \min\{2k - 2 + \max\{m - 2n, 0\}, m - n + k - 1\} \tag{2.1}$$

and

$$t \geq \begin{cases} 3k - 2 + \max\{m - 3n, 0\} & \text{if } m \geq 2n \\ \frac{3}{2}k - 1 + \max\{m - \frac{3}{2}n, 0\} & \text{if } m < 2n. \end{cases} \tag{2.2}$$

The result of Theorem 2(a) is known and a proof of this result may be found in [14]. The results of Theorem 2(b) and 2(c) are novel. However, the proof of Theorem 2(b) is similar to, but much simpler than that of Theorem 2(c). Hence we omit the proof of Theorem 2(b).

Proof: (c) It follows, by (a) and (b) of this theorem, that $L(m, n, t; k) \neq 1, 2$ if and only if $t \leq m - n + k - 1$ and $t \leq 2k - 2 + \max\{m - 2n, 0\}$. Therefore

$$L(m, n, t; k) > 2 \text{ iff } t \leq \min\{2k - 2 + \max\{m - 2n, 0\}, m - n + k - 1\}. \tag{2.3}$$

We first prove the theorem for the case $m \geq 2n$. Suppose that (2.1) and that the first inequality in (2.2) holds. Then it follows, by (2.3), that $L(m, n, t; k) > 2$. We now show that $L(m, n, t; k) \leq 3$ by constructing a playing set $\mathcal{L}^{(2)} = \{T_1^{(2)}, T_2^{(2)}, T_3^{(2)}\}$ of cardinality 3 for $\langle m, n, t; k \rangle$, for which $x^{(3)}_{(000)_2}$ is a minimum. In such a case $x^{(3)}_{(000)_2} = \max\{m - 3n, 0\}$ and hence $\Phi(T_i^{(2)}, k) \cap \Phi(w, k) \neq \emptyset$ for at least one $i \in \{1, 2, 3\}$, where w is an arbitrary winning t–set for $\langle m, n, t; k \rangle$, since $t > \max\{m - 3n, 0\} + 3(k - 1)$. Therefore $\mathcal{L}^{(2)}$ is a lottery set for $\langle m, n, t; k \rangle$, and we conclude that $L(m, n, t; k) = 3$.

Conversely, suppose $L(m, n, t; k) = 3$. Then (2.1) follows from (2.3). We show that $t \geq 3k - 2 + \max\{m - 3n, 0\}$ by proving that, for any $L(m, n, t; k)$–set,

$$\max\{m - 3n, 0\} \leq x^{(3)}_{(000)_2} \leq t - 3k + 2. \tag{2.4}$$

The first inequality in (2.4) is obvious. To prove the second inequality in (2.4), suppose, to the contrary, that $x^{(3)}_{(000)_2} > t - 3k + 2$ for some $L(m, n, t; k)$–set $\mathcal{L}^{(3)} = \{T_1^{(3)}, T_2^{(3)}, T_3^{(3)}\}$ of $\langle m, n, t; k \rangle$. We first consider the case $x^{(3)}_{(000)_2} = t - 3k + 3$. Note that, from (2.1), we have $m \geq 2n + t - 2k + 2 = 2n + (k-1) + t - 3k + 3$, so that $|T_1^{(3)} \cup T_2^{(3)} \cup T_3^{(3)}| \geq 2n + (k-1)$. This means each n–set in $\mathcal{L}^{(3)}$ has at least $k-1$ elements that are utilised only once, *i.e.*, $x^{(3)}_{(100)_2}, x^{(3)}_{(010)_2}, x^{(3)}_{(001)_2} \geq k-1$. In this case, if the winning t–set, w, consists of $t - 3k + 3$ elements from $\mathcal{U}_m \setminus (T_1^{(3)} \cup T_2^{(3)} \cup T_3^{(3)})$, and $k-1$ elements from each of $T_1^{(3)} \setminus (T_2^{(3)} \cup T_3^{(3)})$, $T_2^{(3)} \setminus (T_1^{(3)} \cup T_3^{(3)})$ and $T_3^{(3)} \setminus (T_1^{(3)} \cup T_2^{(3)})$, it follows that $\Phi(T_i^{(3)}, k) \cap \Phi(w, k) = \emptyset$ for all $i \in \{1, 2, 3\}$, contradicting the fact that $\mathcal{L}^{(3)}$ is a lottery set for $\langle m, n, t; k \rangle$. For the case $x^{(3)}_{(000)_2} > t - 3k + 3$, say $x^{(3)}_{(000)_2} = t - 3k + 3 + y$, it is easy to see that the "worst case" is when one n–set is disjoint and the other two (say T_1 and T_2) each has $k - 1 - y$ elements that are utilised once. In this case, if the winning t–set, w, consists of $t - 3k + 3 + y$ elements from $\mathcal{U}_m \setminus (T_1^{(3)} \cup T_2^{(3)} \cup T_3^{(3)})$, and $k-1-y$, $k-1-y$, $k-1$ elements from respectively $T_1^{(3)} \setminus (T_2^{(3)} \cup T_3^{(3)})$, $T_2^{(3)} \setminus (T_1^{(3)} \cup T_3^{(3)})$ and $T_3^{(3)} \setminus (T_1^{(3)} \cup T_2^{(3)})$, and y elements from $T_1^{(3)} \cap T_2^{(3)}$ it follows that $\Phi(T_i^{(3)}, k) \cap \Phi(w, k) = \emptyset$ for all $i \in \{1, 2, 3\}$, contradicting the fact that $\mathcal{L}^{(3)}$ is a lottery set for $\langle m, n, t; k \rangle$. This completes the proof for the case $m \geq 2n$.

For the case $m < 2n$ we consider the complementary lottery problem $\langle m', n', t'; k' \rangle \equiv \langle m, m - n, m - t; m + k - n - t \rangle$ by virtue of Theorem 1. We then have $m' > 2n'$, which is the first case, proved above. Therefore $L(m', n', t'; k') = 3$ if and only if

$$t' \leq \min\{2k' - 2 + \max\{m' - 2n', 0\}, m' - n' + k' - 1\} \tag{2.5}$$

and

$$t' \geq 3k' - 2 + \max\{m' - 3n', 0\}. \tag{2.6}$$

We only have to show that (2.6) is equivalent to the second inequality in (2.2). From (2.6) it follows that

$$\begin{aligned} m - t &\geq 3(m + k - n - t) - 2 + \max\{m - 3(m - n), 0\} \\ \Leftrightarrow \quad 2t &\geq 3k - 2 + 2m - 3n + \max\{3n - 2m, 0\} \\ \Leftrightarrow \quad 2t &\geq 3k - 2 + \max\{2m - 3n, 0\}, \end{aligned}$$

which is equivalent to the second inequality in (2.2). ■

The characterisation number $\eta(m, n, t; k)$ may be determined analytically for $L(m, n, t; k) = 1, 2, 3$. Note that if $L(m, n, t; k) = 1$, then $\eta(m, n, t; k) = 1$. However, additional notation needs to be introduced before $\eta(m, n, t; k)$ may be determined if $L(m, n, t; k) = 2, 3$. Let $\zeta_\ell(m, n)$ denote the number of distinct ways in which a set of m elements may be covered by ℓ distinct sets, each of cardinality n (excluding permutations/symmetries). It is clear that $\zeta_2(m,n) = 1$ if $n < m \leq 2n$ and $\zeta_2(m,n) = 0$ otherwise. It seems a hard problem to find a closed form formula for $\zeta_\ell(m,n)$ if $\ell \geq 3$. Hence we tabulate values for $\zeta_3(m,n)$ in Table 2.1 within the ranges for m and n necessary for our results.

Theorem 3 (Basic values for $\eta(m,n,t;k)$)
(a) When $L(m,n,t;k) = 2$,

$$\eta(m,n,t;k) = \sum_{i=0}^{t-2k+1} \zeta_2(m-i,n) = \begin{cases} t-2k+2-m+2n, & \text{if } m \geq 2n \\ t-2k+2, & \text{if } m < 2n. \end{cases} \tag{2.7}$$

(b) When $L(m,n,t;k) = 3$,

$$\eta(m,n,t;k) = \begin{cases} \displaystyle\sum_{i=0}^{t-3k+2} \zeta_3(m-i,n), & \text{if } m \geq 2n \\ \displaystyle\sum_{i=0}^{2t-3k-2m+3n+2} \zeta_3(m-i,m-n), & \text{if } m < 2n. \end{cases} \tag{2.8}$$

Proof: (a) Each feasible value for $x_0^{(2)}$ in the $\vec{X}^{(2)}$–vector of an $L(m,n,t;k)$–set corresponds to a different overlapping structure. Because $\max\{m-2n,0\} \leq x_0^{(2)} \leq t-2k+1$, the result follows immediately.

(b) Each feasible value for $x_0^{(3)}$ in the construction of $\mathcal{L}^{(3)}$ in the proof of Theorem 2(c) corresponds to $\zeta_3(m-x_0^{(3)},n)$ different overlapping n–set structures for $\mathcal{L}^{(3)}$. Note that $\zeta_3(m,n) = 0$ if $m > 3n$. We have $\max\{m-3n,0\} \leq x_0^{(3)} \leq t-3k+2$ if $m \geq 2n$. Therefore the first equality in (2.8) follows. For $m < 2n$, Theorem 1 may be used to obtain the second equality in (2.8). ∎

An interesting result is that the characterisations of $L(m,m-n,m-t;m+k-n-t)$–sets are given by the mirror images of the $\vec{X}$–vectors characterising $L(m,n,t;k)$–sets for $\langle m,n,t;k\rangle$, as dictated by the following theorem.

Theorem 4 *If an $L(m,n,t;k)$-set for $\langle m,n,t;k\rangle$ conforms to the overlapping structure*

$$\vec{X}^{(L)} = \left(x_0^{(L)}, x_1^{(L)}, \ldots, x_{2L-2}^{(L)}, x_{2L-1}^{(L)}\right),$$

$m\backslash n$	3	4	5	6	7	8	9	10	11	12
4	1	0	0	0	0	0	0	0	0	0
5	3	1	0	0	0	0	0	0	0	0
6	3	4	1	0	0	0	0	0	0	0
7	3	5	4	1	0	0	0	0	0	0
8	1	6	6	4	1	0	0	0	0	0
9	1	4	9	7	4	1	0	0	0	0
10	0	3	8	11	7	4	1	0	0	0
11	0	1	7	12	12	7	4	1	0	0
12	0	1	4	13	15	13	7	4	1	0
13	0	0	3	9	18	17	13	7	4	1
14	0	0	1	7	16	22	18	13	7	4
15	0	0	1	4	14	23	25	19	13	7
16	0	0	0	3	9	23	28	27	19	13
17	0	0	0	1	7	17	31	32	28	19
18	0	0	0	1	4	14	28	38	35	29
19	0	0	0	0	3	9	24	38	43	37
20	0	0	0	0	1	7	17	37	46	47

Table 2.1: Values for $\zeta_3(m,n)$ for $4 \leq m \leq 20$ and $3 \leq n \leq 12$.

for some $1 \leq k \leq \{n,t\} < m$ *satisfying* $m+k > n+t$, *then the set corresponding to the overlapping structure*

$$\vec{X}^{(L)} = \left(x_{2^L-1}^{(L)}, x_{2^L-2}^{(L)}, \ldots, x_1^{(L)}, x_0^{(L)}\right)$$

is an $L(m,m-n,m-t;m+k-n-t)$*-set for* $\langle m,m-n,m-t;m+k-n-t\rangle$.

Proof: Consider a two–dimensional tabular representation similar to that in Figure 2.1, but for an $L(m,n,t;k)$–set, $\mathcal{L}$, for $\langle m,n,t;k\rangle$, consisting of $L(m,n,t;k)$ rows denoting the n–sets in $\mathcal{L}$ and m columns denoting the elements of $\mathcal{U}_m$, in which the (i,j)–th cell contains a cross if $j \in \mathcal{U}_m$ is an element of the i–th n–set of $\mathcal{L}$. Then the complement of the tabular representation (where crosses are replaced by empty spaces and *vice versa*) represents the corresponding $L(m,m-n,m-t;m+k-n-t)$–set for $\langle m,m-n,m-t;m+k-n-t\rangle$. For any specific element of $\mathcal{U}_m$, a cross in its column indicates that the element is present in some n–set of $\mathcal{L}$. These crosses correspond to 1–bits in the binary index of the $\vec{X}^{(L)}$–vector capturing the overlapping n–set structure of $\mathcal{L}$. Thus the corresponding element in the vector $\vec{X}^{(L)}$ for the $L(m,m-n,m-t;m+k-n-t)$–set may be obtained by taking the complement of each of the bits in the binary indices of that element. Therefore the $\vec{X}^{(L)}$–vector for the $L(m, m-n, m-t; m+k-n-t)$–set is the $\vec{X}^{(L)}$–vector for $\mathcal{L}$ in reverse order. ■

3 Properties of $\eta(m, n, t; k)$

An interesting saw–tooth growth pattern of the lottery characterisation number $\eta(m, n, t; k)$ emerges when any three of its parameters are held constant, while the other parameter is allowed to increase. The saw–tooth jumps occur when the corresponding lottery number value changes as the variable parameter increases. For example, when n, t and k are fixed and m is allowed to increase, the saw–tooth widths are always asymptotically non–increasing, whilst there seems to be no rule as to whether the saw–tooth heights increase or decrease, as may be seen in Figure 3.1 for the special case $n = t = 6$ and $k = 1, 2$.

To prove these observations in a more general setting, we require the notion of a *jump sequence*. Informally, such a sequence represents values of one of the parameters m, n, t or k at which saw–teeth separations occur in the sequence $\{\eta(m, n, t; k)\}$ (*i.e.*, parameter values at which the corresponding lottery number value changes). Formally, define the increasing sequence $\max\{n, t\} = m_1^{(n,t,k)}, m_2^{(n,t,k)}, m_3^{(n,t,k)}, \ldots$, to which we refer as the *m–jump sequence*, as those integers $m_{i+1}^{(n,t,k)}$ satisfying

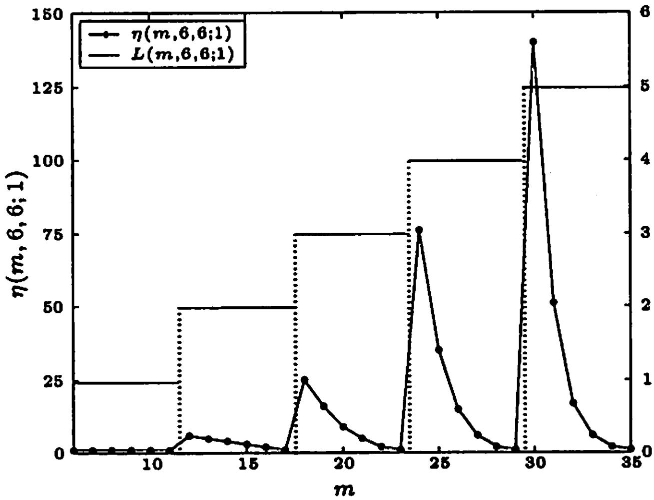
$$L(m_{i+1}^{(n,t,k)}, n, t; k) > L(m_i^{(n,t,k)}, n, t; k), \qquad i = 1, 2, 3, \ldots$$

but for which

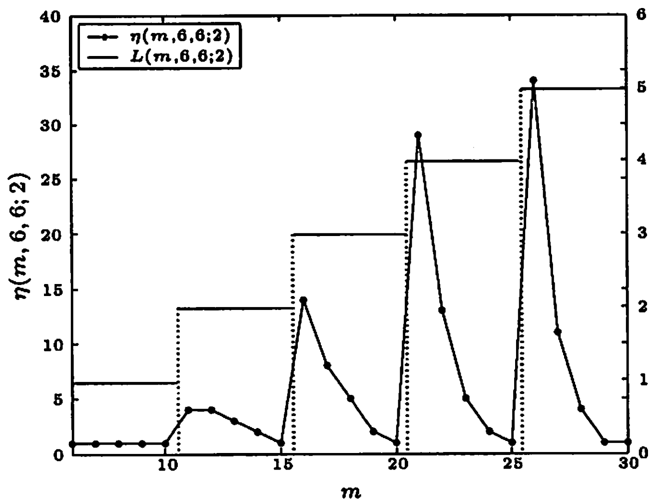
$$L(m_{i+1}^{(n,t,k)} - 1, n, t; k) = L(m_i^{(n,t,k)}, n, t; k)$$

in cases where $m_i^{(n,t,k)}$ and $m_{i+1}^{(n,t,k)}$ are non–adjacent integers. An n–jump sequence, a t–jump sequence and a k–jump sequence may be defined similarly. Then it is possible to prove the following properties of these jump sequences.

Theorem 5 (Properties of the jump sequences)
(a) The m–jump sequence is infinite, for all $1 \leq k \leq \{n, t\}$.
(b) The density of the m–jump sequence is asymptotically increasing for all $2 \leq k \leq \{n, t\}$, *in the sense that there exists an* $i^{(n,t,k)} \in \mathbb{N}$ *such that* $m_{i+1}^{(n,t,k)} = m_i^{(n,t,k)} + 1$ *for all* $i \geq i^{(n,t,k)}$. *After this point the m–jump sequence is said to have* maximum density.
(c) The n–jump sequence is finite, for all $1 \leq k \leq t \leq m$.
(d) The t–jump sequence is finite, for all $1 \leq k \leq n \leq m$.
(e) The k–jump sequence is finite, for all $1 \leq \{n, t\} \leq m$.



(a) $\eta(m, 6, 6; 1)$ and $L(m, 6, 6; 1)$



(b) $\eta(m, 6, 6; 2)$ and $L(m, 6, 6; 2)$

Figure 3.1: The number of non–isomorphic $L(m, 6, 6; k)$–set structures, $\eta(m, 6, 6; k)$, for $\langle m, 6, 6; k\rangle$ where (a) $6 \leq m \leq 35$ and $k = 1$, and where (b) $6 \leq m \leq 30$ and $k = 2$.

Note that, for $k = 1$ and $n = t$, the m–jump sequence is simply the sequence of all positive multiples of n, since $L(m,n,n;1) = \lfloor \frac{m}{n} \rfloor$, (see [8]). On the other hand, for $n = t = k$, the m–jump sequence is the sequence of all integers exceeding $n-1$, since $L(m,n,n;n) = \binom{m}{n}$, (again see [8]). These two cases represent two extreme growth patterns of the characterisation sequence in $\{\eta(m,n,t;k)\}_{m=\max\{n,t\}}^{\infty}$ with respect to increasing m: In the former case the saw–teeth all have maximum width (namely n), the saw–teeth are as blunt as possible (as will be shown), the m–jump sequence never reaches maximum density, and the growth in saw–tooth height is positive and constant. In the latter case the saw–teeth all have minimum width (namely 1, in other words the maximum density is achieved by the m–jump sequence right from the start), the saw–teeth have become so sharp (infinitely sharp) that the individual teeth themselves have become indistinguishable, and the growth in saw–tooth height is positive and binomial.

Proof of Theorem 5: (a) By contradiction. Suppose the m–jump sequence is finite for some values of $1 \leq k \leq \{n,t\}$. Then there exists an $m_f^{(n,t,k)} \in \mathbb{N}$ such that

$$L(m,n,t;k) = L(m_f^{(n,t,k)},n,t;k) \in \mathbb{N} \quad \text{for all} \quad m \geq m_f^{(n,t,k)}.$$

But this contradicts the (easily established) lower bound

$$L(m,n,t;k) \geq \left\lfloor \frac{m-t}{n} \right\rfloor + 1$$

for large values of m.

(b) In 1964 Hanani, *et al.* [9] proved that

$$L(m,n,n;2) \geq \frac{m(m-n+1)}{n(n-1)^2}$$

for all $m \geq n \geq 2$, and additionally showed that this bound is asymptotically best possible, in the sense that

$$\lim_{m\to\infty} L(m,n,n;2)\frac{n(n-1)^2}{m(m-n+1)} = 1.$$

For all $2 \leq k \leq \{n,t\} \leq m$ it therefore follows that

$$L(m,n,t;k) \geq L(m,s,s;k) \geq L(m,s,s;2) \geq \frac{m(m-s+1)}{s(s-1)^2},$$

where $s = \max\{n, t\}$ by Theorem 1(b), (e) and (f) in [6], and hence the growth of the lottery sequence $\{L(m, n, t; k)\}_{m=\max\{n,t\}}^{\infty}$ is super–quadratic, from which it follows that the density of the m–jump sequence is asymptotically increasing, and that maximum density is achieved at some finite value of m.

(c)–(e) Since $1 \leq k \leq \{n, t\} \leq m$ and since m is fixed for any n–jump sequence, any t–jump sequence and any k–jump sequence, these (increasing) sequences must be finite. ■

The following growth properties of the characterisation number $\eta(m, n, t; k)$ may now be proved for inter–jump sequence values of the parameters m, n, t and k.

Theorem 6 (Growth properties of $\eta(m, n, t; k)$)
(a) $\eta(m, n, t; k) \geq \eta(m', n, t; k)$ *for all* $m_i^{(n,t,k)} \leq m \leq m' < m_{i+1}^{(n,t,k)}$ *and all* $1 \leq k \leq \{n, t\} \leq m$, *when* $m_i^{(n,t,k)}$ *and* $m_{i+1}^{(n,t,k)}$ *are non–adjacent integers.*
(b) $\eta(m, n, t; k) \leq \eta(m, n', t; k)$ *for all* $n_i^{(m,t,k)} \leq n \leq n' < n_{i+1}^{(m,t,k)}$ *and all* $1 \leq k \leq \{n \leq n', t\} \leq m$, *when* $n_i^{(m,t,k)}$ *and* $n_{i+1}^{(m,t,k)}$ *are non–adjacent integers.*
(c) $\eta(m, n, t; k) \leq \eta(m, n, t'; k)$ *for all* $t_i^{(m,n,k)} \leq t \leq t' < t_{i+1}^{(m,n,k)}$ *and all* $1 \leq k \leq \{n, t \leq t'\} \leq m$, *when* $t_i^{(m,n,k)}$ *and* $t_{i+1}^{(m,n,k)}$ *are non–adjacent integers.*
(d) $\eta(m, n, t; k) \geq \eta(m, n, t; k')$ *for all* $k_i^{(m,n,t)} \leq k \leq k' < k_{i+1}^{(m,n,t)}$ *and all* $1 \leq k \leq k' \leq \{n, t\} \leq m$, *when* $k_i^{(m,n,t)}$ *and* $k_{i+1}^{(m,n,t)}$ *are non–adjacent integers.*

Proof: (a) Suppose two adjacent elements $m_i^{(n,t,k)}$ and $m_{i+1}^{(n,t,k)}$ of the m–jump sequence are non–adjacent integers for some $i \in \mathbb{N}$ and consider two intermediate values m and m' satisfying $m_i^{(n,t,k)} \leq m \leq m' < m_{i+1}^{(n,t,k)}$. Then clearly $L(m, n, t; k) = L(m', n, t; k)$. Now consider the following construction technique to obtain an $L(m, n, t; k)$–set for $\langle m, n, t; k\rangle$ from an $L(m', n, t; k)$–set for $\langle m', n, t; k\rangle$.

> **Construction Method 1.** Consider a two–dimensional tabular representation similar to that in Figure 2.1, but for an $L(M, n, t; k)$–set $\mathcal{L}$ for $\langle M, n, t; k\rangle$, consisting of $L(M, n, t; k)$ rows denoting the n–sets in $\mathcal{L}$ and M columns denoting the elements of $\mathcal{U}_M$, in which the (i, j)–th cell contains a cross if

$j \in \mathcal{U}_M$ is an element of the i–th n–set of $\mathcal{L}$, and is empty otherwise. Remove any column from this representation, and place a cross in any empty cell of each row that now contains only $n-1$ crosses as a result of the column deletion. The result is a tabular representation of a lottery set of cardinality $L(M,n,t;k)$ for $\langle M-1,n,t;k\rangle$.

By (possibly repeated) application of Construction Method 1, to any $L(m',n,t;k)$–set for $\langle m',n,t;k\rangle$, an $L(m,n,t;k)$–set for $\langle m,n,t;k\rangle$ is obtained. We conclude that $\eta(m,n,t;k) \geq \eta(m',n,t;k)$.

(b) The proof is similar to that in (a), except that the above construction method should be replaced by the following alternative method to obtain an $L(m,n',t;k)$–set for $\langle m,n',t;k\rangle$ from an $L(m,n,t;k)$–set for $\langle m,n,t;k\rangle$:

Construction Method 2. Consider a tabular representation of an $L(m,N,t;k)$–set for $\langle m,N,t;k\rangle$, as described in Construction Method 1. Place a cross in any empty cell of each row. The result is a tabular representation of a lottery set for $\langle m,N+1,t;k\rangle$, using a similar argument as in Construction Method 1.

(c) Suppose two adjacent elements $t_i^{(m,n,k)}$ and $t_{i+1}^{(m,n,k)}$ of the t–jump sequence are non–adjacent integers for some $i \in \mathbb{N}$ and consider two intermediate values t and t', satisfying $t_i^{(m,n,k)} \leq t \leq t' < t_{i+1}^{(m,n,k)}$. Then $L(m,n,t;k) = L(m,n,t';k)$. Let $\mathcal{L} = \{\ell_1,\ldots,\ell_{L(m,n,t;k)}\}$ be an $L(m,n,t;k)$–set for $\langle m,n,t;k\rangle$ and let $T' \in \Phi(\mathcal{U}_m,t')$ be an arbitrary winning t'–set for $\langle m,n,t';k\rangle$. There exists (by the definition of $\mathcal{L}$), for any $T \in \Phi(T',t)$, an $i \in \{1,\ldots,L(m,n,t;k)\}$ such that $\Phi(T,k) \cap \Phi(\ell_i,k) = T''$ (say) $\neq \emptyset$. But since $T \subseteq T'$, it follows that $T'' \in \Phi(T',k) \cap \Phi(\ell_i,k)$, and hence $\mathcal{L}$ is also an $L(m,n,t';k)$–set for $\langle m,n,t';k\rangle$. We conclude that $\eta(m,n,t';k) \geq \eta(m,n,t;k)$.

(d) The proof of this result is similar to that in (c). ∎

Note that $\eta(m,n,t;k)$ exhibits exactly opposite growth properties to those of $L(m,n,t;k)$ with respect to the arguments m, n, t and k, as may be seen by comparing the growth properties in [10] and the above theorem.

4 Characterisation procedure

One method of enumerating all $L(m,n,t;k)$–set structures for $\langle m,n,t;k\rangle$, consists of constructing a rooted tree (referred to as the *lottery tree*) of evolving overlapping set structures, the nodes of which resemble overlap specifications similar to that in Figure 4.1. Level i of the lottery tree contains all possible (non–isomorphic) overlapping n–set structures of cardinality i and is constructed from the nodes on level $i-1$ of the lottery tree by adding an i–th n–set, T_i, in all possible ways to each of the nodes. The corresponding $\vec{X}^{(i)}$–vector for each new node is then calculated and a so–called *isomorphism test* is performed to determine whether the overlapping structure has been generated before, in which case the node is removed from the tree. The isomorphism test for each node is performed by permuting the n–sets in all possible ways, and by testing whether any of the permuted sets has an $\vec{X}^{(i)}$–vector that is lexicographically smaller than the concerned $\vec{X}^{(i)}$–vector.

Suppose the lottery tree has $\ell+1$ levels in total. The first level of the tree consists of the node $\vec{X}^{(1)} = (m-n, n)$ only (the root), while the nodes $\vec{X}^{(\ell)}$ on level ℓ of the tree represent potential lottery set structures of cardinality ℓ for $\langle m,n,t;k\rangle$. An $(\ell+1)$–st level of nodes is added to the tree (in such a manner that $|T_{\ell+1} \cap T_j| \leq t$ for all $j \leq \ell$) in order to carry out a so–called *domination test* (*i.e.*, to test which of the nodes on level ℓ actually represent valid lottery sets). This domination test is achieved by testing whether *all* nodes on level $\ell+1$ overlap in at least k positions with at least one n–set of the existing ℓ n–set overlapping structure (represented by its parent node $\vec{X}^{(\ell)}$) in the tree. If this were the case, then the n–set overlapping structure represented by the parent node $\vec{X}^{(\ell)}$ would constitute a lottery set for $\langle m,n,t;k\rangle$ (and hence $L(m,n,t;k) \leq \ell$). However, if there exists at least one node on level $\ell+1$ of the tree for which the corresponding final t–set overlaps in fewer than k positions with *all* n–sets of the parent node overlapping structure, the parent node does not represent a lottery set. If *no* parent node represents a valid lottery set, the bound $L(m,n,t;k) > \ell$ is established.

We do not give the full implementation details here of the characterisation procedure described above, but rather demonstrate the tree construction by means of the simple schematic representation in Figure 4.1.

The number of nodes on level i of the tree typically grows very rapidly as i increases, even when permutations of node structures are avoided. See [8] for examples of the growth in the number of nodes per level and the resulting execution time required to construct the lottery tree. However, with the use of pruning rules, we are able to save considerable execution

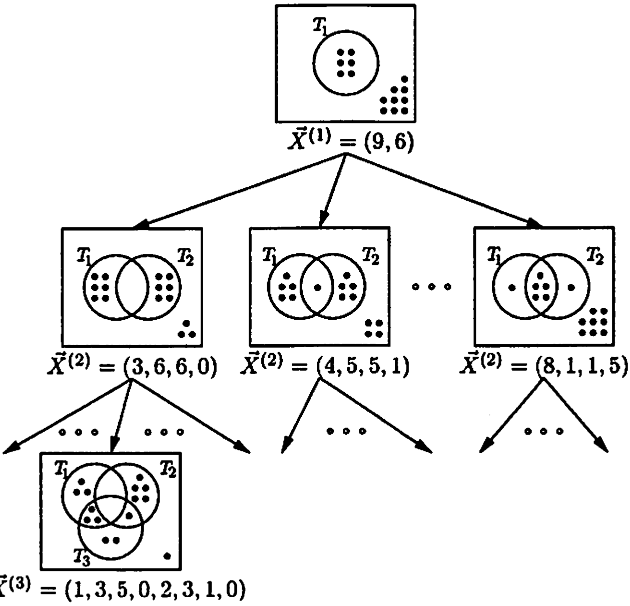


Figure 4.1: Part of the lottery tree construction to determine all non–isomorphic $L(15,6,6;3)$–set overlapping structures for the lottery $\langle 15,6,6;3\rangle$.

time. We describe the pruning rules used in the following section.

5 Pruning the characterisation search tree

Before we give the pruning rules, we prove the following theorem, in which we use the notation $\eta(m,n,t;k)|_{x_0>0}$ to denote the number of non–isomorphic lottery sets for $\langle m,n,t;k\rangle$ of cardinality $L(m,n,t;k)$ with the property that at least one of the elements of $\mathcal{U}_m$ are not utilised. Similarly, the subscript $x_0 = 0$ denotes the corresponding number of non–isomorphic lottery sets for $\langle m,n,t;k\rangle$ with the property that all the elements of $\mathcal{U}_m$ are utilised.

Lemma 1 *For all* $1 \leq k \leq \{n,t\} \leq m$, *if* $L(m-1,n,t-1;k) = L(m,n,t;k)$, *then* $\eta(m,n,t;k)|_{x_0>0} = \eta(m-1,n,t-1;k)$.

Proof: Suppose $L(m-1,n,t-1;k) = L(m,n,t;k)$. We show that any $L(m,n,t;k)$–set for $\langle m,n,t;k\rangle$ with the property that at least one element of $\mathcal{U}_m$ is not utilised is also an $L(m-1,n,t-1;k)$–set for $\langle m-1,n,t-1;k\rangle$, and *vice versa.*

First, suppose that $\mathcal{L}^{(1)}$ is an $L(m,n,t;k)$–set for $\langle m,n,t;k\rangle$ with the property that at least one element of $\mathcal{U}_m$ is not utilised in $\mathcal{L}^{(1)}$. Without loss of generality we may assume m is such an element. Since all $\phi_t \in \Phi(\mathcal{U}_m,t)$ with $m \in \phi_t$ have a k–intersection with some n–set in $\mathcal{L}^{(1)}$, all $\phi_t \in \Phi(\mathcal{U}_{m-1},t-1)$ also have k–intersections with some n–set in $\mathcal{L}^{(1)}$, implying that $\mathcal{L}^{(1)}$ is a lottery set for $\langle m-1,n,t-1;k\rangle$. From our assumption that $L(m-1,n,t-1;k) = L(m,n,t;k)$, this set is of minimum cardinality. Thus $\mathcal{L}^{(1)}$ is an $L(m-1,n,t-1;k)$–set for $\langle m-1,n,t-1;k\rangle$.

Conversely, suppose $\mathcal{L}^{(2)}$ is an $L(m-1,n,t-1;k)$–set for $\langle m-1,n,t-1;k\rangle$. Then it is easy to see that all $\phi_t \in \Phi(\mathcal{U}_m,t)$ have k–intersections with some n–set in $\mathcal{L}^{(2)}$, implying that $\mathcal{L}^{(2)}$ is also a lottery set for $\langle m,n,t;k\rangle$ in which the element m is not utilised. Again, this set is of minimal cardinality, because $L(m-1,n,t-1;k) = L(m,n,t;k)$. ■

It is now possible to prove the following useful theorem.

Theorem 7 (Eta decomposition)
Suppose $L(m,n,t;k) = \ell$ and $L(m-1,n,t-1;k) = \ell'$. Then, for all $1 \leq k \leq \{n,t\} \leq m$,

$$\eta(m,n,t;k) = \begin{cases} \eta(m,n,t;k)|_{x_0=0} & \text{if } \ell' > \ell \\ \eta(m,n,t;k)|_{x_0=0} + \eta(m-1,n,t-1;k) & \text{if } \ell' = \ell. \end{cases}$$

Proof: Note that $\eta(m,n,t;k) = \eta(m,n,t;k)|_{x_0=0} + \eta(m,n,t;k)|_{x_0>0}$. Also, Li proved in [10] that $\ell' = L(m-1,n,t-1;k) \geq L(m,n,t;k) = \ell$, so that we have two cases. Firstly, if $\ell' > \ell$, then all elements of $\mathcal{U}_m$ must be utilised in a lottery set for $\langle m,n,t;k\rangle$, because if not, then we have a lottery set for $\langle m-1,n,t-1;k\rangle$ of cardinality ℓ (as explained in the proof of Lemma 1), contradicting our assumption that $\ell' > \ell$. Secondly, if $\ell' = \ell$, the result follows directly from Lemma 1. ■

Note that Theorem 7 may be applied recursively, so that we only have to search for solutions where all elements are utilised (*i.e.*, $x_0 = 0$). This saves considerable execution time when performing the tree characterisation procedure. We now give the pruning rules for the tree search that were implemented in our approach.

(1) If $x_0^{(\ell)} > 0$, then the structure corresponding to the vector $\vec{X}^{(\ell)}$ may be omitted from the tree.

(2) If $\min\{x^{(\ell)}_{(100\cdots0)_2}, k-1\} + \cdots + \min\{x^{(\ell)}_{(000\cdots1)_2}, k-1\} + x_0^{(\ell)} \geq t$, then the structure corresponding to the vector $\vec{X}^{(\ell)}$ is *not* a lottery set, and may hence be omitted from the tree.

(3) If $x_0^{(\ell-1)} \geq n+1$, then all possible structures corresponding to the vector $\vec{X}^{(\ell-1)}$ may be omitted from the tree.

(4) If $\min\{x^{(\ell-1)}_{(100\cdots0)_2}, k-1\} + \cdots + \min\{x^{(\ell-1)}_{(000\cdots1)_2}, k-1\} + x_0^{(\ell-1)} \geq n+t$, then all possible structures corresponding to the vector $\vec{X}^{(\ell-1)}$ are *not* lottery sets, and may hence be omitted from the tree.

Rules (1) and (2) are implemented just after level ℓ of the tree, before the domination test. Rule (1) follows from Theorem 7. Note that in the case where $L(m-1, n, t-1; k) = \ell$, the lottery sets for $\langle m-1, n, t-1; k\rangle$ need to be added to the solution set. In rule (2) we add up the number of elements (not exceeding $k-1$ per set) that are in at most one n–set of the structure corresponding to $\vec{X}^{(\ell)}$. If there are t or more such elements, there exists a t–set having no k–intersection with any of the n–sets in the structure $\vec{X}^{(\ell)}$, and hence the structure does not represent a lottery set. Rules (3) and (4) are are implemented just after level $\ell - 1$ and are based on the same idea as in rules (1) and (2) respectively, in an obvious manner. The test for isomorphism was found computationally much more intensive than the domination test. It was therefore better to perform the domination test before the isomorphism test on level $\ell + 1$ of the tree.

6 The $L(m, n, t; k)$–set characterisations

Tables listing bounds for $L(m, n, t; k)$, values for $\eta(m, n, t; k)$, as well as their corresponding set characterisations may be found online in [3, 7]. These tables are rather bulky and are therefore omitted here.

7 New lottery numbers and bounds

According to [12] the lottery number for $\langle 18, 6, 6; 3\rangle$ falls in the range $6 \leq L(18, 6, 6; 3) \leq 7$. The tree for $\langle 18, 6, 6; 3\rangle$ is too deep (large) to traverse in a realistic time–span, so that it is not feasible to determine the value of $L(18, 6, 6; 3)$ using the tree characterisation procedure described in §4–5. However, it is possible to prove that $L(18, 6, 6; 3) \neq 6$ via a construction method described below, from which the following result may then be deduced.

Theorem 8 $L(18, 6, 6; 3) = 7$.

Consider the following method for constructing lottery sets of the same cardinality for $\langle m-1, n, n; k\rangle$ from any lottery set for $\langle m, n, n; k\rangle$.

> **Construction.** Consider a tabular representation (such as in Figure 2.1) of a lottery set for $\langle m, n, n; k\rangle$. Remove any column from this representation, and add an arbitrary element to the original n–sets that now have only $n-1$ elements as a result of the deletion. The result is a tabular representation of a lottery set for $\langle m-1, n, n; k\rangle$.

In order to prove Theorem 8, we require the following intermediate result.

Lemma 2 *If there exist lottery sets for* $\langle 18, 6, 6; 3\rangle$ *of cardinality* 6*, all such sets must contain exactly one disjoint* 6*–set.*

Proof: Suppose there exist lottery sets of cardinality 6 for $\langle 18, 6, 6; 3\rangle$. Then such lottery sets may have at most one disjoint 6–set, otherwise some of their 6–sets will be forced to coincide exactly. Now suppose one such lottery set contains no disjoint 6–set. Then, keeping in mind that all $L(17, 6, 6; 3)$–sets contain one disjoint 6–set (see [3]), it is not difficult to see that it is only possible to construct lottery sets for $\langle 17, 6, 6; 3\rangle$ via the above construction method if at least one element of $\mathcal{U}_{18}$ is not utilised in the $L(18, 6, 6; 3)$–set (*i.e.*, if the tabular representation of the $L(18, 6, 6; 3)$–set contains at least one empty column). However, if the tabular representation of the $L(18, 6, 6; 3)$–set contains at least one empty column, then it must be true that $L(17, 6, 5; 3) \leq 6$. But $7 \leq L(17, 6, 5; 3) \leq 11$ [11], which is a contradiction. ■

We are now in a position to prove Theorem 8.

Proof of Theorem 8: By contradiction. Suppose that $L(18, 6, 6; 3) = 6$. Then it follows, by Lemma 2, that any $L(18, 6, 6; 3)$–set must contain exactly one disjoint 6–set from $\mathcal{U}_{18}$. If, in the construction technique outlined

$L(m,n,t;k)$	Previous $L(m,n,t;k)$	New $L(m,n,t;k)$	$\eta(m,n,t;k)$	Time (sec)
$L(14,6,7;4)$	$5:6^d$	6^a	1	45 665
$L(14,7,6;4)$	$6:7$	7^a	≥ 1	101 044
$L(14,8,5;4)$	$5:7$	7^a	≥ 1	32 988
$L(15,7,7;4)$	$4:5$	5^a	26	281
$L(15,9,5;4)$	$6:7$	7^a	≥ 1	117 195
$L(16,5,10;4)$	$5:6$	6^a	4	7 058
$L(16,6,8;4)$	$4:7$	7^a	≥ 1	182 798
$L(16,7,7;4)$	$4:6^d$	6^a	4	1 326 165
$L(16,10,5;4)$	$4:5$	5^a	1	136
$L(17,4,9;3)$	$6:7$	7^a	≥ 1	9
$L(17,5,7;3)$	$6:7$	7^a	≥ 1	1 551
$L(17,6,9;4)$	$4:6$	6^a	102	379 521
$L(17,8,7;4)$	$4:5$	5^a	67	1 994
$L(17,10,5;4)$	$6:7^d$	7^a	≥ 1	1 737 999
$L(17,11,5;4)$	$4:5$	5^a	11	171
$L(18,4,10;3)$	$5:6$	6^a	4	8
$L(18,5,11;4)$	$5:7$	7^a	≥ 1	3 237
$L(18,6,6;3)$	$6:7$	7^e	≥ 1	-
$L(18,6,9;4)$	$5:6$	6^c	1	257 473
$L(18,7,8;4)$	$4:6$	6^a	≥ 1	824
$L(18,8,7;4)$	$4:6$	$5^{a,b}$	1	2 926
$L(19,4,10;3)$	$6:7$	7^a	≥ 1	6
$L(19,5,8;3)$	$5:7$	7^a	≥ 1	459
$L(19,6,10;4)$	$5:6$	6^a	21	348 970
$L(19,7,6;3)$	$4:5$	5^a	2	778
$L(19,7,9;4)$	$4:5$	5^a	20	1 067
$L(19,8,5;3)$	$5:6$	6^a	≥ 1	2 985
$L(19,9,7;4)$	$4:6$	$5^{a,f}$	154	13 409
$L(19,12,5;4)$	$4:5$	5^a	2	1 191
$L(20,6,10;4)$	$5:7^d$	7^a	-	170 928
$L(20,7,9;4)$	$4:6$	6^a	≥ 1	944
$L(20,9,7;4)$	$4:6$	$5^{a,g}$	3	18 479

Table 7.1: 32 New lottery numbers found via the characterisation technique described in §4–5. The second column contains previously best known bounds on lottery numbers, taken from [1], [11] and [12] using the notation lower bound : upper bound (both inclusive). The corresponding n–set overlapping structures are available at [3] or [7]. Bounds and new lottery numbers in columns 2 and 3 are motivated as follows: ^{a}No lottery sets of smaller cardinality found by the characterisation procedure. ^{b}One lottery set of cardinality 5 found by the characterisation procedure, namely $[6^6 9^4 17^4 26^2 28^2]$. ^{c}Since $L(m+1,n,t;k) \geq L(m,n,t;k)$ [10]. ^{d}Upper bound due to Belic [2]. ^{e}By Theorem 8. ^{f}Lottery sets of cardinality 5 found by the characterisation procedure, for example $[6^3 9^2 14^5 17^7 26^1 28^1]$. ^{g}Lottery sets of cardinality 5 found by the characterisation procedure, for example $[6^6 9^4 14^1 17^5 26^2 28^2]$.

$L(m,n,t;k)$	Previous Best	Current Best	Time (sec)
$L(12,5,7;4)$	$6:8$	$7^a:8$	856
$L(12,7,5;4)$	$6:8$	$7^d:8$	–
$L(13,7,5;4)$	$6:10$	$7^c:10$	–
$L(14,4,7;3)$	$6:8$	$7^a:8$	32
$L(14,5,8;4)$	$6:10$	$7^a:10$	6 200
$L(15,5,9;4)$	$5:8$	$7^a:8$	4 759
$L(15,6,7;4)$	$6:10$	$7^a:10$	77 517
$L(15,8,5;4)$	$6:11$	$7^c:11$	–
$L(16,5,9;4)$	$6:10$	$7^a:10$	2 924
$L(16,9,5;4)$	$6:9$	$7^c:9$	–
$L(17,5,10;4)$	$6:9$	$7^a:9$	3 381
$L(17,6,8;4)$	$6:10$	$7^c:10$	–
$L(17,7,5;3)$	$5:7$	$6^a:7$	222
$L(17,7,7;4)$	$5:9^f$	$6^c:9$	–
$L(18,5,7;3)$	$6:9$	$7^a:9$	358
$L(18,6,8;4)$	$6:14$	$7^b:14$	–
$L(18,7,5;3)$	$5:8$	$7^a:8$	1 115 644
$L(18,10,5;4)$	$6:9$	$7^c:9$	–
$L(18,11,5;4)$	$4:7$	$6^a:7$	905
$L(19,6,6;3)$	$6:9$	$7^c:9$	–
$L(19,6,9;4)$	$5:10$	$7^e:10$	–
$L(19,7,5;3)$	$6:10$	$7^c:10$	–
$L(19,7,8;4)$	$4:8^f$	$6^c:8$	–
$L(19,8,7;4)$	$4:8^f$	$6^a:8$	3 603
$L(19,10,5;4)$	$6:14$	$7^c:14$	–
$L(19,11,5;4)$	$5:8^f$	$6^c:8$	–
$L(20,6,9;4)$	$6:13$	$7^b:13$	–
$L(20,7,6;3)$	$4:7$	$6^a:7$	825
$L(20,8,5;3)$	$5:7$	$6^c:7$	–
$L(20,8,7;4)$	$4:10$	$6^c:10$	–
$L(20,12,5;4)$	$5:7$	$6^c:7$	5 357

Table 7.2: 31 Improved bounds found via the characterisation technique described in §4–5. The second column contains previously best known bounds on lottery numbers, taken directly from [12] using the notation `lower bound` : `upper bound` (both inclusive). The bound improvements or new lottery numbers obtained are listed in the third column, while the last column shows the execution time (in seconds) required to implement the tree characterisation. Bounds and new lottery numbers in columns 2 and 3 are motivated as follows: [a]No lottery sets of smaller cardinality found by the characterisation procedure. [b]Since $L(m+1,n+1,t+1;k+1) \geq L(m,n,t;k)$ [10]. [c]Since $L(m+1,n,t;k) \geq L(m,n,t;k)$ [10]. [d]By Theorem 1. [e]By Theorem 9. [f]Upper bound due to Belic [2].

above, a column involving the disjoint 6–set is deleted, the resulting lottery set for $\langle 17, 6, 6; 3\rangle$ will have no disjoint 6–set, which contradicts the characterisation of minimal lottery sets for $\langle 17, 6, 6; 3\rangle$. We conclude that $6 < L(18, 6, 6; 3) \leq 7$, which yields the desired result. ■

Apart from the new lottery number in Theorem 8, a further 31 new lottery numbers and improvements on previously best known bounds for a further 31 lottery numbers, as listed in [1] and [11], are given in Tables 7.1 and 7.2. These results were all established using the tree characterisation method described in §4–5, and were implemented on an AMD 1.8GHz processor with 256Mb of memory.

Finally, the following theorem illustrates a technique that may be used to improve lower bounds on lottery numbers, utilising the value of η for smaller lotteries.

Theorem 9 $L(19, 6, 9; 4) > 6$.

Proof: By contradiction. Suppose there is at least one lottery set $\mathcal{L}$ of cardinality 6 for $\langle 19, 6, 9; 4\rangle$. As described in [5], it is possible to construct a lottery set of the same cardinality for $\langle 18, 6, 9; 4\rangle$ by deleting any column of the tabular representation of $\mathcal{L}$, and by adding crosses in any empty cell of each row that contains only $n-1$ crosses as a result of the column deletion. First note that $\mathcal{L}$ cannot have an empty column, because then we have a lottery set for $\langle 18, 6, 8; 4\rangle$ of cardinality 6, contradicting $L(18, 6, 8; 4) \geq 7$ (see Table 7.2).

We consider two cases, namely when there exists a column with two crosses and when there does not. If there is a column with 2 crosses, we delete such a column. Then there are at least 8 ($= 19 - 6 - 6 + 1$) columns that have no other crosses in the rows corresponding to the deletion. Of these, choose two columns where the frequencies of the elements do not differ by 1. Since there are at least 8 columns to choose from, it is always possible to make such a choice. For one construction put one cross in each of the two chosen columns and for another construction put both the crosses in any one column (of the two). The two constructions are then non–isomorphic: suppose the frequencies of the elements in each column after the deletion are $\{f_1, f_2, f_3, \ldots, f_{m-1}\}$. If one adds two crosses in the two different ways described above, only two frequencies are affected, say f_1 and f_2. For two constructions to be isomorphic, both must have the same set of frequencies for the elements. Thus one would need $\{f_1+2, f_2\} = \{f_1+1, f_2+1\}$, which can only be true if $f_2 - f_1 = 1$. However, we chose f_2 and f_1 not to differ by 1.

Secondly, if there is no column with two crosses, then there must be at least 10 columns with only one cross each. We then may delete such a column and put the deleted cross either in a column with frequency 1 or any other frequency, which again will give two non-isomorphic constructions.

We have constructed two isomorphically different lottery set for $\langle 18,6,9;4\rangle$ of cardinality 6. But this contradicts the facts that $\eta(18,6,9;4) = 1$ and $L(18,6,9;4) = 6$, hence the result follows. ∎

8 Unresolved cases

Lotteries for which $L(m,n,t;k)$–set characterisations could not be achieved (due to computational complexity), are listed in Tables 8.1 and 8.2.

$L(16,8,6;4) = 6$	$L(17,9,4;3) = 6$	$L(17,9,6;4) = 6$	$L(18,7,8;4) = 6$
$L(18,9,4;3) = 6$	$L(19,8,5;3) = 6$	$L(19,10,4;3) = 6$	$L(19,10,6;4) = 6$
$L(20,7,9;4) = 6$	$L(20,10,6;4) = 6$		

Table 8.1: Lotteries $\langle m,n,t;k\rangle$ for which it is known that $L(m,n,t;k) = 6$, but for which $L(m,n,t;k)$–set characterisations could not be achieved.

$6 \le L(17,7,5;3) \le 7$	$6 \le L(17,7,7;4) \le 9$	$6 \le L(18,7,7;4) \le 12$
$6 \le L(18,9,6;4) \le 7$	$6 \le L(18,11,5;4) \le 7$	$6 \le L(19,7,8;4) \le 8$
$6 \le L(19,8,7;4) \le 7^{\dagger}$	$6 \le L(19,11,5;4) \le 8$	$6 \le L(20,7,6;3) \le 7$
$6 \le L(20,7,8;4) \le 11$	$6 \le L(20,8,5;3) \le 7$	$6 \le L(20,8,7;4) \le 10$
$6 \le L(20,10,4;3) \le 8$	$6 \le L(20,11,5;4) \le 12$	$6 \le L(20,12,5;4) \le 7$

Table 8.2: Lotteries $\langle m,n,t;k\rangle$ for which $L(m,n,t;k)$ is unknown, with lower bound 6, and for which $L(m,n,t;k)$–set characterisations could not be achieved, or bounds could not be improved. †Upper bound due to Li and Van Rees [13].

9 Conclusion

In this paper we considered lotteries for which lottery numbers (i) are either known not to exceed 6, or (ii) for which the lower bounds are known not to exceed 6. We characterised the overlapping structures of minimal lottery sets (with cardinality at least 3) for 501 of these lotteries, which are listed in [3]. In the process 32 new lottery numbers were established, and a further 31 lower bounds were improved. We also provided a theorem that characterises when a minimal lottery set has cardinality two or three.

There are 10 lotteries for which lottery numbers within the ranges $2 \leq k \leq 4$, $k \leq t \leq 11$, $k \leq n \leq 12$ and $\max\{n,t\} \leq m \leq 20$ are known to be 6 (see Table 8.1), but for which the tree characterisation procedure could not be implemented within a reasonable amount of time, due to the computational complexity of the procedure. It is, however, anticipated that these characterisations will be possible with the introduction of further pruning rules, whereby the size of the characterisation tree can be reduced further. Finally, there are 7 lotteries within the ranges $2 \leq k \leq 4$, $k \leq t \leq 11$, $k \leq n \leq 12$ and $\max\{n,t\} \leq m \leq 20$ for which lottery numbers are either 6 or 7 (see Table 8.2); these lottery numbers can be established if new pruning rules were to facilitate traversal of the characterisation tree for all lotteries up to a depth of level 6 (excluding the domination test level) — these cases therefore present an attractive opportunity for further work.

Some of the new lottery numbers and bounds listed in Tables 7.1 and 7.2 were also found independently by Li and Van Rees [13] more or less at the same time that the results in this paper were obtained, but though theoretical analyses. The characterisations determined in this paper also provided useful ideas for finding new general upper bound constructions (see [13]). An opportunity for future work might be to combine theoretical results with computer searches.

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