# On Generalized Schur Numbers for $x_1 + x_2 + c = kx_3$

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## Abstract

For every integer $c$ and every positive integer $k$, let $n = r(c, k)$ be the least integer, provided that it exists, such that for every coloring

$$\Delta: \{1, 2, \ldots, n\} \rightarrow \{0, 1\},$$

there exist three integers, $x_1, x_2, x_3$, (not necessarily distinct) such that

$$\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$$

and

$$x_1 + x_2 + c = kx_3.$$

If such an integer does not exist, then let $r(c, k) = \infty$. The main result of this paper is that

$$r(c, 2) = \begin{cases} |c| + 1 & \text{if } c \text{ is even} \\ \infty & \text{if } c \text{ is odd.} \end{cases}$$

for every integer $c$. In addition, a lower bound is found for $r(c, k)$ for all integers $c$ and positive integers $k$ and linear upper and lower bounds are found for $r(c, 3)$ for all positive integers c.

Note: The major work for this paper occurred when the first author was an undergraduate student at Clarion University of Pennsylvania, under the direction of the second author.

## Introduction

A function $\Delta : \{1, 2, \ldots, n\} \rightarrow \{0, 1, \ldots, t-1\}$ is called a *coloring* of the set $\{1, 2, \ldots, n\}$ with $t$ colors. If $L$ is a system of equations in $m$ variables, then we say that a solution $(x_1, x_2, \ldots, x_m)$ to $L$ is *monochromatic* if and only if

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m).$$

In 1916, I. Schur [16] proved that for every $t \geq 2$, there exists a least integer $n = S(t)$ such that for every coloring of the set $\{1, 2, \ldots, n\}$ with $t$ colors, there exists a monochromatic solution to

$$x_1 + x_2 = x_3.$$

The integers $S(t)$ are called *Schur numbers* and are known for only a few small values of $t$ [17]. In 1933, R. Rado, who was a student of Schur, generalized the work of Schur to arbitrary systems of linear equations. Rado was able to find necessary and sufficient conditions to determine if an arbitrary system of linear equations admits a monochromatic solution for every coloring of the natural numbers with a finite number of colors [4, 10 - 12]. For a given system of linear equations $L$, the least integer $n$, provided that it exists, such that for every coloring of the set $\{1, 2, \ldots, n\}$ with $t$ colors, there exists a monochromatic solution to $L$, is referred to as the $t$-color *generalized Schur number* or the $t$-color *Rado number* for the system $L$.

For a given system of linear equations $L$ and a given natural number $t$, the results of Rado may tell us that the $t$-color generalized Schur number for the system $L$ exists, but they do not tell us what this integer is. The problem of determining the exact generalized Schur numbers for various system has recently received renewed interest [1,2,5 - 9,13 - 15]. In [3], Burr and Loo were able to determine the 2-color generalized Schur numbers for the equations

$$x_1 + x_2 + c = x_3$$

and

$$x_1 + x_2 = kx_3$$

for every integer $c$ and for every positive integer $k$. In this paper, the generalization of the above two equations is considered. We will need the follows definitions.

**Definition 1**: For every integer $c$ and every positive integer $k$, let $L(c, k)$ represent the equation

$$L(c, k): \ x_1 + x_2 + c = kx_3.$$

**Definition 2**: For every integer $c$ and for every positive integer $k$, let $n = r(c, k)$ be the least integer, provided that it exists, such that for every coloring of the set $\{1, 2, \ldots, n\}$ with two colors, there exists a monochromatic solution to the equation $L(c, k)$. If no such integer exists, then let $r(c, k) = \infty$.

Using the notation of Definitions 1 and 2, the following two theorems are the above mentioned results of Burr and Loo.

**Theorem 1**: *For every integer c,*

$$r(c,1) = \begin{cases} 4c+5 & c \geq 0 \\ c - \lceil \frac{c}{5} \rceil + 1 & c < 0. \end{cases}$$

**Theorem 2**: *For every positive integer k,*

$$r(0,k) = \begin{cases} 5 & k=1 \\ 1 & k=2 \\ 9 & k=3 \\ \frac{1}{2}k(k+1) & k \geq 4. \end{cases}$$

In this paper the generalized Schur numbers for the equation

$$L(c,2): \ x_1 + x_2 + c = 2x_3$$

are determined for every integer $c$. Also, linear upper and lower bounds are found for the generalized Schur numbers for the equation

$$L(c,3): \ x_1 + x_2 + c = 3x_3$$

for every integer $c$, and a conjecture is made as to the exact value of these numbers. It is also shown that $r(c,k)$ is infinite if $c$ is odd and $k$ is even, and a lower bound for $r(c,k)$ is given for every integer $c$ and every positive integer $k$.

## Main Results

Before we determine the generalized Schur numbers for the equation $L(c,2)$, we shall first prove the following three lemmas.

**Lemma 1**: *If $c$ is any odd integer and $k$ is any even positive integer, then*

$$r(c,k) = \infty.$$

**Proof of Lemma 1**: Let an odd integer $c$ and an even positive integer $k$ be given. We shall exhibit a coloring of the natural numbers with two colors that avoids a monochromatic solution to $L(c,k)$. Let $\Delta : \mathbb{N} \to \{0,1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \text{ is odd} \\ 1 & \text{if } x \text{ is even.} \end{cases}$$

If $x_1$ and $x_2$ are natural numbers such that $\Delta(x_1) = \Delta(x_2)$, then $x_1 + x_2 + c$ will be an odd integer. Since $kx_3$ is even for every natural number $x_3$, there does not exist a monochromatic solution to $L(c,k)$. Therefore,

$$r(c,k) = \infty$$

and the proof of Lemma 1 is complete. □

We shall now prove two lemmas that give lower bounds for the values of $r(c,k)$ for all integers $c$ and positive integers $k$.

**Lemma 2**: *For all positive integers c and k,*

$$r(c,k) \geq \left\lceil \frac{2\lceil \frac{2+c}{k} \rceil + c}{k} \right\rceil.$$

**Proof of Lemma 2**: Let positive integers $c$ and $k$ be given and let

$$n = \left\lceil \frac{2\lceil \frac{2+c}{k} \rceil + c}{k} \right\rceil.$$

We will exhibit a coloring of the set $\{1,2,\ldots,n-1\}$ with two colors that avoids a monochromatic solution to $L(c,k)$. Let

$$\Delta : \{1,2,\ldots,n-1\} \to \{0,1\}$$

be defined by

$$\Delta(x) = \begin{cases} 0 & 1 \leq x \leq \lceil \frac{2+c}{k} \rceil - 1 \\ 1 & \lceil \frac{2+c}{k} \rceil \leq x \leq n-1. \end{cases}$$

If $x_1$, $x_2$ and $x_3$ are integers such that $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = 0$, then

$$\begin{aligned} x_1 + x_2 + c &\geq 1 + 1 + c \\ &= k\left(\frac{2+c}{k}\right) \\ &> k\left(\left\lceil \frac{2+c}{k} \right\rceil - 1\right) \\ &\geq kx_3. \end{aligned}$$

If $x_1$, $x_2$ and $x_3$ are integers such that $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = 1$, then

$$\begin{aligned} x_1 + x_2 + c &\geq \left\lceil \frac{2+c}{k} \right\rceil + \left\lceil \frac{2+c}{k} \right\rceil + c \\ &= k\left(\frac{2\lceil \frac{2+c}{k} \rceil + c}{k}\right) \\ &> k\left(\left\lceil \frac{2\lceil \frac{2+c}{k} \rceil + c}{k} \right\rceil - 1\right) \\ &= k(n-1) \\ &\geq kx_3. \end{aligned}$$

Hence, there does not exist a monochromatic solution to $L(c,k)$ and the proof of Lemma 2 is complete. □

**Lemma 3**: *For every negative integer $c$ and every positive integer $k$,*

$$r(c,k) \geq \left\lceil \frac{k\left\lceil \frac{k-c}{2} \right\rceil - c}{2} \right\rceil.$$

**Proof of Lemma 3**: Let a negative integer $c$ and a positive integer $k$ be given. Let

$$n = \left\lceil \frac{k\left\lceil \frac{k-c}{2} \right\rceil - c}{2} \right\rceil.$$

We will exhibit a coloring of the set $\{1,2,\ldots,n-1\}$ with two colors that avoids a monochromatic solution to $L(c,k)$. Let

$$\Delta : \{1,2,\ldots,n-1\} \rightarrow \{0,1\}$$

be defined by

$$\Delta(x) = \begin{cases} 0 & 1 \leq x \leq \left\lceil \frac{k-c}{2} \right\rceil - 1 \\ 1 & \left\lceil \frac{k-c}{2} \right\rceil \leq x \leq n-1. \end{cases}$$

If $x_1$, $x_2$ and $x_3$ are integers such that $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = 0$, then

$$\begin{aligned} x_1 + x_2 + c &\leq \left(\left\lceil \frac{k-c}{2} \right\rceil - 1\right) + \left(\left\lceil \frac{k-c}{2} \right\rceil - 1\right) + c \\ &= 2\left(\left\lceil \frac{k-c}{2} \right\rceil - 1\right) + c \\ &< 2\left(\frac{k-c}{2}\right) + c \\ &= k \cdot 1 \\ &\leq kx_3. \end{aligned}$$

If $x_1$, $x_2$ and $x_3$ are integers such that $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = 1$, then

$$\begin{aligned} x_1 + x_2 + c &\leq (n-1) + (n-1) + c \\ &= 2\left(\left\lceil \frac{k\left\lceil \frac{k-c}{2} \right\rceil - c}{2} \right\rceil - 1\right) + c \end{aligned}$$

$$< 2\left(\frac{k\left\lceil \frac{k-c}{2} \right\rceil - c}{2}\right) + c$$

$$= k\left\lceil \frac{k-c}{2} \right\rceil$$

$$\leq kx_3.$$

Hence, there does not exist a monochromatic solution to $L(c,k)$ and the proof of Lemma 3 is complete. □

We are now ready to determine the generalized Schur numbers for the equation

$$L(c,2): \; x_1 + x_2 + c = 2x_3.$$

**Theorem 3:** *For every integer c,*

$$r(c,2) = \begin{cases} |c| + 1 & \text{if } c \text{ is even} \\ \infty & \text{if } c \text{ is odd.} \end{cases}$$

**Proof of Theorem 3:** First we will consider the case where $c$ is an odd integer. From Lemma 1 it follows that

$$r(c,2) = \infty$$

whenever $c$ is an odd integer.

Next we shall consider the case where $c = 0$. Since $(1,1,1)$ is a solution to $L(0,2)$, it follows immediately that

$$r(0,2) = 1 = |c| + 1.$$

Next we shall consider the case where $c$ is an even positive integer. Let an even positive integer $c$ be given. From Lemma 2 it follows that

$$r(c,2) \geq \left\lceil \frac{2\left\lceil \frac{2+c}{2} \right\rceil + c}{2} \right\rceil = c + 1 = |c| + 1.$$

Next we shall show that

$$r(c,2) \leq |c| + 1$$

by showing that for every coloring of the set $\{1,2,\ldots,|c|+1\}$ with two colors there exists a monochromatic solution to $L(c,2)$. Let a coloring

$$\Delta : \{1,2,\ldots,|c|+1\} \to \{0,1\}$$

be given. Without loss of generality we may assume that

$$\Delta(1) = 0.$$

If $\Delta\left(\frac{2+c}{2}\right) = 0$, then

$$\left(1, 1, \frac{2+c}{2}\right)$$

is a monochromatic solution to $L(c, 2)$ and we are done. Hence, we may assume that

$$\Delta\left(\frac{2+c}{2}\right) = 1.$$

If $\Delta(c+1) = 0$, then

$$(1, c+1, c+1)$$

is a monochromatic solution to $L(c, 2)$. If $\Delta(c+1) = 1$, then

$$\left(\frac{2+c}{2}, \frac{2+c}{2}, c+1\right)$$

is a monochromatic solution to $L(c, 2)$. Hence, there exists a monochromatic solution to $L(c, 2)$ for both possible values of $\Delta(c+1)$, and it follows that

$$r(c, 2) \leq |c| + 1.$$

Therefore,

$$r(c, 2) = |c| + 1$$

for every positive even integer $c$.

Finally, we shall consider the case where $c$ is a negative even integer. Let a negative even integer $c$ be given. From Lemma 3 it follows that

$$r(c, 2) \geq \left\lceil \frac{2\left\lceil \frac{2-c}{2} \right\rceil - c}{2} \right\rceil = -c + 1 = |c| + 1.$$

Next we shall show that

$$r(c, 2) \leq |c| + 1$$

by showing that for every coloring of the set $\{1, 2, \ldots, |c| + 1\}$ with two colors there exists a monochromatic solution to $L(c, 2)$. Let a coloring

$$\Delta : \{1, 2, \ldots, |c| + 1\} \to \{0, 1\}$$

be given. Without loss of generality we may assume that

$$\Delta(|c| + 1) = 0.$$

If $\Delta\left(\frac{2+|c|}{2}\right) = 0$, then

$$\left(|c| + 1, |c| + 1, \frac{2+|c|}{2}\right)$$

is a monochromatic solution to $L(c,2)$ and we are done. Hence we may assume that

$$\Delta\left(\frac{2+|c|}{2}\right)=1.$$

If $\Delta(1)=0$, then

$$(1,|c|+1,1)$$

is a monochromatic solution to $L(c,2)$. If $\Delta(1)=1$, then

$$\left(\frac{2+|c|}{2},\frac{2+|c|}{2},1\right)$$

is a monochromatic solution to $L(c,2)$. Hence there exists a monochromatic solution to $L(c,2)$ for both possible values of $\Delta(c+1)$, and it follows that

$$r(c,2)\leq|c|+1.$$

Therefore,

$$r(c,2)=|c|+1$$

for every negative even integer $c$ and the proof of Theorem 3 is complete. □

We shall now prove Theorem 4, which gives linear upper and lower bounds for $r(c,3)$ for every positive integer $c$.

**Theorem 4**: *For every positive integer $c$,*

$$\frac{5c+4}{9}\leq r(c,3)\leq c.$$

**Proof of Theorem 4**: Let a positive integer $c$ be given. Since $(c,c,c)$ is a solution to $L(c,3)$, it follows immediately that

$$r(c,3)\leq c.$$

Now, from Lemma 2 it follows that

$$r(c,3)\geq\left\lceil\frac{2\lceil\frac{2+c}{3}\rceil+c}{3}\right\rceil\geq\left(\frac{2(\frac{2+c}{3})+c}{3}\right)=\frac{5c+4}{9},$$

and the proof of Theorem 4 is complete. □

## Results of Computer Experiments

The following table shows the results of computer experiments where the exact values of $r(c,3)$ were determined for the first twenty positive integers and the first twenty negative integers. The lower bounds on $r(c,3)$ given in Lemma 2 and Lemma 3 are also shown.

| $c$ | $r(c,3)$ | Lower bound from Lemma 2 |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 2 |
| 3 | 3 | 3 |
| 4 | 4 | 3 |
| 5 | 5 | 4 |
| 6 | 6 | 4 |
| 7 | 7 | 5 |
| 8 | 8 | 6 |
| 9 | 9 | 6 |
| 10 | 7 | 6 |
| 11 | 8 | 7 |
| 12 | 9 | 8 |
| 13 | 8 | 8 |
| 14 | 9 | 9 |
| 15 | 9 | 9 |
| 16 | 10 | 10 |
| 17 | 11 | 11 |
| 18 | 11 | 11 |
| 19 | 11 | 11 |
| 20 | 12 | 12 |

| $c$ | $r(c,3)$ | Lower bound from Lemma 3 |
|---|---|---|
| − 1 | 8 | 4 |
| − 2 | 7 | 6 |
| − 3 | 9 | 6 |
| − 4 | 8 | 8 |
| − 5 | 9 | 9 |
| − 6 | 11 | 11 |
| − 7 | 11 | 11 |
| − 8 | 13 | 13 |
| − 9 | 14 | 14 |
| − 10 | 16 | 16 |
| − 11 | 16 | 16 |
| − 12 | 18 | 18 |
| − 13 | 19 | 19 |
| − 14 | 21 | 21 |
| − 15 | 21 | 21 |
| − 16 | 23 | 23 |
| − 17 | 24 | 24 |
| − 18 | 26 | 26 |
| − 19 | 26 | 26 |
| − 20 | 28 | 28 |

It should be noted that the lower bound for $r(c,3)$ given in Lemma 2 is equal to $r(c,3)$ for $c \in \{13, 14, \ldots, 20\}$, and that the lower bound for $r(c,3)$ given in Lemma 3 is equal to $r(c,3)$ for $c \in \{-4, -5, \ldots, -20\}$. This fact leads the authors to the following conjecture.

**Conjecture**: For every integer $c \geq 13$

$$r(c,3) = \left\lceil \frac{2\left\lceil \frac{2+c}{3} \right\rceil + c}{3} \right\rceil,$$

and for every integer $c \leq -4$

$$r(c,3) \geq \left\lceil \frac{3\left\lceil \frac{3-c}{2} \right\rceil - c}{2} \right\rceil.$$

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