

Some equitably 2-colourable cycle decompositions

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Abstract

Let G be a graph in which each vertex has been coloured using one of k colours, say c_1, c_2, \dots, c_k . If an m -cycle C in G has n_i vertices coloured c_i , $i = 1, 2, \dots, k$, and $|n_i - n_j| \leq 1$ for any $i, j \in \{1, 2, \dots, k\}$, then C is equitably k -coloured. An m -cycle decomposition \mathcal{C} of a graph G is equitably k -colourable if the vertices of G can be coloured so that every m -cycle in \mathcal{C} is equitably k -coloured. For $m = 4, 5$ and 6 , we completely settle the existence problem for equitably 2-colourable m -cycle decompositions of complete graphs and complete graphs with the edges of a 1-factor removed.

1 Introduction

Let G and H be graphs. A G -decomposition of H is a set $\mathcal{G} = \{G_1, G_2, \dots, G_p\}$ such that G_i is isomorphic to G for $1 \leq i \leq p$ and \mathcal{G} partitions the edge set of H . Most commonly, $H = K_v$, the complete graph on v vertices. Another popular choice for H is $K_v - F$, the complete graph with the edges of a 1-factor removed. The problem of determining all values of v for which there exists a G -decomposition of K_v is called the *spectrum problem* for G .

An m -cycle, denoted by (x_1, x_2, \dots, x_m) , is the graph with vertex set $\{x_1, x_2, \dots, x_m\}$ and edge set $\{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_m, x_1\}\}$. The spectrum problem for m -cycles has recently been solved; see [3] and [9].

A *colouring* of an m -cycle decomposition \mathcal{C} of a graph G is an assignment of colours to the vertices of G . A k -*colouring* of \mathcal{C} is a colouring in which

k distinct colours are used. A k -colouring of an m -cycle decomposition \mathcal{C} induces a colouring of each m -cycle in \mathcal{C} . If n_i is the number of vertices coloured c_i in an m -cycle $C \in \mathcal{C}$, then C is *equitably k -coloured* if $|n_i - n_j| \leq 1$ for any $i, j \in \{1, 2, \dots, k\}$ and an m -cycle decomposition \mathcal{C} is equitably k -coloured if every $C \in \mathcal{C}$ is equitably k -coloured. An m -cycle decomposition is *equitably k -colourable* if it can be equitably k -coloured.

A 3-cycle decomposition of K_v is just a Steiner triple system of order v . There is extensive literature on colourings of Steiner triple systems and a fine survey can be found in [5]. Much of the literature on vertex colourings of Steiner triple systems has focused on colourings in which each triple is assigned at least two distinct colours.

In [4], Colbourn, Dinitz and Rosa consider colouring the vertices of a Steiner triple system using 3, 4 or 5 colours such that, in each triple of the system, exactly two colours are represented. In [8], Milici, Rosa and Voloshin consider colouring Steiner triple systems and also Steiner systems of the form $S(2, 4, v)$, with the restriction that the blocks of each design must have specified colour patterns.

The existence question for equitably 3-colourable m -cycle decompositions of K_v and $K_v - F$, where $m \in \{4, 5, 6\}$, has been completely settled in [2]. Quattrocchi has also considered certain colourings of 4-cycle decompositions of K_v in [10]. Quattrocchi's paper differs from both this paper and [2] as it is a condition that each system must have at least one 4-cycle with 3 vertices of one colour.

We consider a natural extension of these colouring problems by investigating the existence of equitably 2-colourable m -cycle decompositions. (In this paper we use colours black and white and, unless otherwise stated, b and w are used to denote the number of black and white vertices in K_v or $K_v - F$ respectively. Furthermore, an edge which connects two black (white) vertices is said to be a one-coloured black (white) edge, and an edge which connects two differently coloured vertices is said to be a two-coloured edge.) A simple counting argument shows that the only equitably 2-colourable Steiner triple system is the trivial system of order 3. Moreover, it is easy to see that for $k \geq 3$, there exists an equitably k -colourable Steiner triple system of order v if and only if $v = k$ and $v \equiv 1$ or $3 \pmod{6}$.

As we shall see, equitable k -colourings of m -cycle decompositions of K_v and $K_v - F$ are no longer trivial when $m \geq 4$. This paper is primarily concerned with equitably 2-colourable m -cycle decompositions of K_v and $K_v - F$ for $m = 4, 5$ and 6 . We make frequent use of the following important result.

Lemma 1.1 ([3], [9]) *An m -cycle decomposition of K_v ($K_v - F$) exists for all admissible v , that is, for all odd (even) v such that $3 \leq m \leq v$ and m divides the number of edges in K_v ($K_v - F$).*

Our main result is the following theorem which completely settles the existence question for equitably 2-coloured m -cycle decompositions of K_v and $K_v - F$ for $m = 4, 5$ and 6 .

Main Theorem

- No equitably 2-colourable m -cycle decomposition of K_v exists for m even.
- For all admissible v , there exists at least one 5-cycle decomposition of K_v which is equitably 2-colourable.
- For all admissible v , $v > 5$, there exists at least one 5-cycle decomposition of K_v which is *not* equitably 2-colourable.
- There exists an equitably 2-colourable m -cycle decomposition of $K_v - F$, with $m \in \{4, 5, 6\}$, for all admissible values of v .

Finally, we introduce some notation to be used throughout the paper. We use $K_{p(n)}$ to denote the multipartite graph with p parts with n vertices in each part. Let G and H be graphs. The join of G and H , denoted $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{u, v\} \mid u \in V(G) \text{ and } v \in V(H)\}$.

2 Equitably 2-colourable even-length cycle decompositions of K_v

We start with a trivial result.

Lemma 2.1 *No equitably 2-colourable m -cycle decomposition of K_v exists for m even.*

Proof. Every equitably 2-coloured cycle of even length contains an equal number of vertices of each colour. Hence, v must be even for an equitably 2-coloured even-length cycle decomposition of K_v to exist. However, it is not possible to decompose a complete graph of even order into cycles as each vertex is of odd degree and cycles have even degree. \square

3 Equitably 2-colourable 4-cycle decompositions

Every equitably 2-coloured 4-cycle must contain two black and two white vertices, which may only be arranged in two formations; see Figure 1. From

the previous section we know that no equitably 2-colourable 4-cycle decomposition of K_v exists. This prompts us to consider instead the decomposition of $K_v - F$ since this graph has the required even number of vertices and even degree.

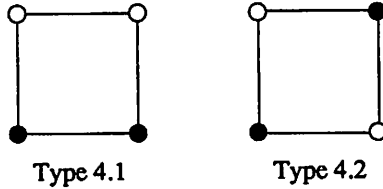


Figure 1: Possible equitable 2-colourings of 4-cycles.

We prove the following:

Theorem 3.1 *There exist equitably 2-colourable 4-cycle decompositions of $K_v - F$ if and only if $v \equiv 0 \pmod{2}$, $v \geq 4$.*

Proof. From Lemma 1.1, a 4-cycle decomposition of $K_v - F$ exists if and only if $v \equiv 0 \pmod{2}$, $v \geq 4$. As v is even, we colour $v/2$ vertices black and $v/2$ vertices white. The proof that these conditions are also sufficient requires the consideration of two cases. Note that there are $\frac{1}{8}v(v-2)$ 4-cycles in any 4-cycle decomposition of $K_v - F$.

Case 1: $v \equiv 0 \pmod{4}$.

Let the vertex set of $K_v - F$ be $\bigcup_{i=1,2} \left\{ 0_i, 1_i, \dots, \frac{1}{2}(v-2)_i \right\}$. Colour the vertices with subscript 1 black and colour the the vertices with subscript 2 white. Let the edges in F be $\left\{ j_k, \left(j + \frac{v}{4} \right)_k \right\}$, where $j = 0, 1, \dots, (v-4)/4$ and $k = 1, 2$. We obtain $v/4$ cycles of Type 4.2 (see Figure 1) by forming the 4-cycles

$$\left(j_1, j_2, \left(j + \frac{v}{4} \right)_1, \left(j + \frac{v}{4} \right)_2 \right)$$

for $j = 0, 1, \dots, (v-4)/4$. We generate the remaining $v(v-4)/8$ cycles of Type 4.1 (see Figure 1) by working modulo $v/2$ and forming the 4-cycles

$$(j_1, (j+l)_1, j_2, (j+l)_2)$$

for $j = 0, 1, \dots, (v-2)/2$ and $l = 1, 2, \dots, (v-4)/4$.

Case 2: $v \equiv 2 \pmod{4}$.

Label and colour the vertices as for Case 1. Let the edges in F be $\{j_1, j_2\}$, where $j = 0, 1, \dots, (v-2)/2$. Working modulo $v/2$, we obtain $v(v-2)/8$ cycles of Type 4.1 by forming the 4-cycles

$$(j_1, (j+l)_1, j_2, (j+l)_2)$$

for $j = 0, 1, \dots, (v-2)/2$ and $l = 1, 2, \dots, (v-2)/4$. □

4 Equitably 2-colourable 5-cycle decompositions

When considering cycles of odd length, if b vertices of a cycle are coloured black and w white, we cannot have $b = w$. Instead, each cycle within the decomposition must satisfy $|b - w| = 1$. The only ways of equitably 2-colouring a 5-cycle are shown in Figure 2. In this case, as we do not require equal numbers of black and white vertices, it is feasible to consider equitably 2-colourable decompositions of K_v , as well as decompositions of $K_v - F$.

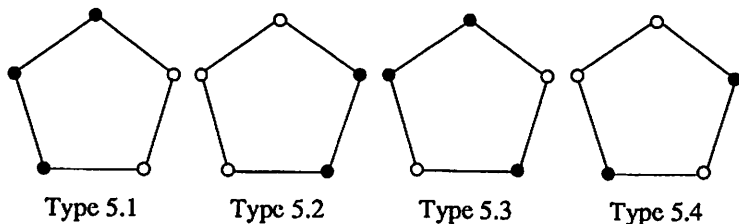


Figure 2: Possible equitable 2-colourings of 5-cycles.

4.1 Equitably 2-colourable 5-cycle decompositions of K_v

We begin with some definitions.

Definition A *pairwise balanced design* of order v , with block sizes in the set K and index λ , denoted $\text{PBD}(v, K, \lambda)$, is a pair (V, \mathcal{B}) . Here V is a set of elements such that $|V| = v$ and \mathcal{B} is a collection of subsets of V , called blocks, such that for each $B \in \mathcal{B}$, $|B| \in K$ and any two elements in V occur together in precisely λ blocks.

Definition A *group divisible design* of order v , with block sizes in the set K , group sizes in the set M and index λ , denoted $\text{GDD}[K, \lambda, M; v]$, is a triple

(V, Γ, \mathcal{B}) . Here V is a set of elements such that $|V| = v$, $\Gamma = \{G_1, G_2, \dots\}$ is a partition of X with each class of the partition called a group, such that for each $G_i \in \Gamma$, $|G_i| \in M$ and \mathcal{B} is a collection of subsets of V , called blocks, such that for each $B \in \mathcal{B}$, $|B| \in K$. The triple also satisfies the following properties:

1. for all $x, y \in V$ belonging to distinct groups, x and y occur together in precisely λ blocks; and
2. for all $x, y \in V$ belonging to the same group, x and y do not occur together in any blocks.

The following result is well-known, and is useful in proving Theorem 4.8 below.

Lemma 4.1 [6] *For all positive integers x there exists a $PBD(2x + 1, 3, 1)$ or a $PBD(2x + 1, \{3, 5^*\}, 1)$.*

Corollary 4.2 *For all positive integers x there exists a $GDD[3, 1, 2; 2x]$ or a $GDD[3, 1, \{2, 4^*\}; 2x]$.*

Proof: Take a $PBD(2x + 1, 3, 1)$ or a $PBD(2x + 1, \{3, 5^*\}, 1)$ (see Lemma 4.1), and remove an element e (if there is a block of size 5, remove an element occurring in this block). The result is a $GDD[3, 1, 2; 2x]$ or a $GDD[3, 1, \{2, 4^*\}; 2x]$ respectively, where the old blocks not containing e are blocks in the design and the old blocks containing e are now groups within the design. \square

We also make use of the following existence results. Most of these and other designs in this paper were obtained using computational techniques.

Lemma 4.3 *There exists an equitably 2-coloured 5-cycle decomposition of K_5 .*

Proof. Let the vertex set of K_5 be \mathbb{Z}_5 . Colour the vertices 0, 1 and 2 black and colour the vertices 3 and 4 white. A suitable decomposition of K_5 is given by: $(0, 1, 2, 3, 4)$, $(0, 2, 4, 1, 3)$. \square

Lemma 4.4 *There exists an equitably 2-coloured 5-cycle decomposition of K_{11} .*

Proof. Let the vertex set of K_{11} be \mathbb{Z}_{11} . Colour the vertices 1, 2, 4, 7, 9 and 10 black and colour the remaining vertices white. Working modulo 11, the starter cycle $(0, 1, 10, 2, 6)$ generates an equitably 2-coloured 5-cycle decomposition of K_{11} . \square

Lemma 4.5 *There exists an equitably 2-coloured 5-cycle decomposition of K_{21} .*

Proof. Let the vertex set of K_{21} be $\bigcup_{i=1,\dots,7} \{0_i, 1_i, 2_i\}$. Colour the vertices with subscripts 4, 5, 6 and 7 black and colour the remaining vertices white. Fourteen equitably 2-coloured starter cycles are given by:

$$\begin{array}{cccc} (0_1, 1_1, 0_2, 0_4, 1_4), & (0_1, 0_2, 2_1, 1_4, 0_5), & (0_1, 0_3, 0_4, 0_5, 1_5), & (0_1, 1_3, 0_4, 1_5, 0_6), \\ (0_1, 2_3, 0_4, 0_6, 2_5), & (0_1, 0_4, 1_2, 2_4, 1_6), & (0_1, 2_6, 0_2, 0_5, 0_7), & (0_1, 1_7, 0_2, 1_5, 2_7), \\ (0_2, 1_2, 0_5, 0_6, 1_6), & (0_2, 0_3, 0_5, 2_7, 0_6), & (0_2, 1_3, 0_6, 2_4, 0_7), & (0_2, 2_3, 0_6, 0_7, 2_7), \\ (0_3, 1_3, 0_7, 1_4, 1_7), & (0_3, 1_5, 2_3, 2_6, 0_7). & & \end{array}$$

Each starter cycle is developed modulo 3 such that the subscripts remains fixed, thus ensuring that every cycle is equitably 2-coloured. For example, the first starter cycle develops to produce:

$$(0_1, 1_1, 0_2, 0_4, 1_4), (1_1, 2_1, 1_2, 1_4, 2_4), (2_1, 0_1, 2_2, 2_4, 0_4).$$

□

Lemma 4.6 *There exists an equitably 2-coloured 5-cycle decomposition of $K_{3(5)}$.*

Proof. Let the vertex set of $K_{3(5)}$ be $\bigcup_{i=1,2,3} \{0_i, 1_i, \dots, 4_i\}$, with the obvious vertex partition. Colour the vertices $0_i, 2_i$ and 4_i black for $i = 1, 2, 3$, and colour the remaining vertices white. A suitable decomposition of $K_{3(5)}$ is given by:

$$\begin{array}{cccc} (1_1, 1_2, 0_1, 0_2, 0_3), & (1_1, 3_2, 0_1, 2_2, 2_3), & (1_1, 1_3, 0_1, 4_2, 4_3), & (1_1, 3_3, 0_1, 0_3, 2_2), \\ (3_1, 1_2, 2_1, 0_2, 2_3), & (3_1, 3_2, 2_3, 0_1, 4_3), & (3_1, 1_3, 2_1, 4_2, 0_3), & (3_1, 3_3, 2_1, 2_3, 4_2), \\ (1_2, 1_3, 0_2, 4_3, 4_1), & (1_2, 3_3, 2_2, 2_1, 0_3), & (3_2, 1_3, 2_2, 4_3, 2_1), & (3_2, 3_3, 4_2, 4_1, 0_3), \\ (1_1, 4_2, 1_3, 4_1, 0_2), & (3_1, 0_2, 3_3, 4_1, 2_2), & (1_2, 4_3, 3_2, 4_1, 2_3). & \end{array}$$

□

Lemma 4.7 *There exists an equitably 2-coloured 5-cycle decomposition of $K_{5(5)}$.*

Proof. Let the vertex set of $K_{5(5)}$ be $\bigcup_{i=1,\dots,5} \{0_i, 1_i, \dots, 4_i\}$, with the obvious vertex partition. Colour the vertices $0_i, 2_i$ and 4_i black for $i = 1, \dots, 5$, and colour the remaining vertices white. A suitable decomposition is given by developing the following ten starter cycles:

$$\begin{array}{cccc} (1_1, 1_2, 2_3, 0_4, 4_5), & (1_1, 3_2, 4_3, 4_4, 2_5), & (3_4, 3_1, 2_3, 4_5, 0_2), & (0_1, 0_2, 1_3, 4_4, 3_5), \\ (1_4, 1_1, 2_3, 0_5, 4_2), & (1_4, 3_1, 4_3, 4_5, 2_2), & (2_1, 2_2, 3_3, 1_4, 0_5), & (0_4, 0_1, 1_3, 4_5, 3_2), \\ (3_1, 3_2, 2_3, 4_4, 0_5), & (2_4, 2_1, 3_3, 1_5, 0_2). & & \end{array}$$

Working modulo 5, each starter cycle is developed by incrementing the subscript by one. So, for example, the first starter cycle develops to produce the following 5-cycles:

$$(11, 12, 23, 04, 45), (12, 13, 24, 05, 41), (13, 14, 25, 01, 42), (14, 15, 21, 02, 43), \\ (15, 11, 22, 03, 44).$$

□

Theorem 4.8 *There exist equitably 2-colourable 5-cycle decompositions of K_v if and only if $v \equiv 1, 5 \pmod{10}$, $v \geq 5$.*

Proof. By Lemma 1.1, a 5-cycle decomposition of K_v exists if and only if $v \equiv 1, 5 \pmod{10}$, $v \geq 5$. We need to consider these two cases separately.

Case 1: $v \equiv 1 \pmod{10}$.

Let $v = 10x + 1$, $x \geq 1$. By Corollary 4.2, we can take either a GDD[3, 1, 2; 2x] or a GDD[3, 1, {2, 4*}; 2x] and simultaneously construct the complete graph K_v and its equitably 2-coloured 5-cycle decomposition as follows. Replace each element of the design with five vertices, colouring three vertices black and two vertices white. Adjoin a new vertex ∞ , coloured white, which is adjacent to all other vertices.

By Lemma 4.6, we can place an equitably 2-coloured 5-cycle decomposition of $K_{3(5)}$ on each set of vertices arising from a block of the design. Furthermore, by Lemmas 4.4 and 4.5, we can place an equitably 2-coloured 5-cycle decomposition of K_{11} or K_{21} on $\{\infty\} \cup g$ for each set of vertices g arising from a group of the design of size 2 or 4 respectively. It is not difficult to check that the result is an equitably 2-coloured 5-cycle decomposition of K_v .

Case 2: $v \equiv 5 \pmod{10}$.

Let $v = 10x + 5$, $x \geq 0$. By Lemma 4.1, we can take either a PBD(2x + 1, 3, 1) or a PBD(2x + 1, {3, 5*}, 1) and, proceeding as for Case 1, replace each element of the design with five vertices, colouring three vertices black and two vertices white.

By Lemma 4.3, we can place an equitably 2-coloured 5-cycle decomposition of K_5 on each set of five new vertices. By Lemmas 4.6 and 4.7, we can place an equitably 2-coloured 5-cycle decomposition of $K_{3(5)}$ or $K_{5(5)}$ on each set of vertices arising from a block of the design of size 3 or 5 respectively. It is not difficult to check that the result is an equitably 2-coloured 5-cycle decomposition of K_v . □

4.2 Equitably 2-colourable 5-cycle decompositions of $K_v - F$

When proving Theorem 4.13, we again make use of Corollary 4.2 as well as the existence results in Lemma 4.6 and Lemmas 4.9 to 4.12.

Lemma 4.9 *There exists an equitably 2-coloured 5-cycle decomposition of $K_{10} - F$.*

Proof. Let the vertex set of $K_{10} - F$ be \mathbb{Z}_{10} . Colour the vertices $0, 1, \dots, 5$ black and colour the vertices $6, 7, 8$ and 9 white. Let the edges in F be $\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}$ and $\{8, 9\}$. A suitable decomposition of $K_{10} - F$ is given by:

$$\begin{array}{cccc} (4, 0, 3, 6, 8), & (0, 2, 4, 6, 9), & (1, 3, 4, 7, 8), & (4, 1, 2, 7, 9), \\ (5, 2, 6, 0, 7), & (5, 3, 7, 1, 6), & (5, 0, 8, 2, 9), & (5, 1, 9, 3, 8). \end{array}$$

□

Lemma 4.10 *There exists an equitably 2-coloured 5-cycle decomposition of $K_{12} - F$.*

Proof. Let the vertex set of $K_{12} - F$ be \mathbb{Z}_{12} . Colour the vertices $0, 1, \dots, 5$ black and colour the remaining vertices white. Let the edges in F be $\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}$ and $\{10, 11\}$. A suitable decomposition of $K_{12} - F$ is given by:

$$\begin{array}{ccccc} (0, 2, 1, 8, 6), & (3, 0, 4, 9, 10), & (1, 5, 0, 7, 10), & (11, 8, 10, 2, 4), & (9, 7, 11, 2, 5), \\ (9, 6, 11, 5, 3), & (1, 3, 8, 0, 9), & (3, 4, 10, 0, 11), & (4, 1, 6, 2, 8), & (11, 9, 2, 7, 1), \\ (8, 7, 3, 6, 5), & (10, 6, 4, 7, 5). & & & \end{array}$$

□

Lemma 4.11 *There exists an equitably 2-coloured 5-cycle decomposition of $K_{20} - F$.*

Proof. Let the vertex set of $K_{20} - F$ be \mathbb{Z}_{20} . Colour the vertices $0, 1, \dots, 11$ black and colour the vertices $12, 13, \dots, 19$ white. Let the edges in F be $\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 11\}, \{12, 13\}, \{14, 15\}, \{16, 17\}$ and $\{18, 19\}$. A suitable decomposition of $K_{20} - F$ is given by:

$$\begin{array}{cccc} (12, 14, 3, 1, 2), & (13, 15, 0, 4, 6), & (16, 18, 5, 7, 8), & (17, 19, 10, 9, 11), \\ (12, 15, 1, 4, 3), & (13, 14, 0, 2, 5), & (16, 19, 6, 8, 10), & (17, 18, 11, 7, 9), \\ (12, 16, 0, 3, 5), & (13, 17, 1, 6, 9), & (14, 18, 2, 7, 10), & (15, 19, 4, 8, 11), \\ (12, 17, 0, 5, 1), & (13, 16, 2, 6, 10), & (14, 19, 7, 4, 11), & (15, 18, 8, 3, 9), \\ (12, 18, 0, 6, 11), & (13, 19, 1, 7, 3), & (14, 16, 4, 2, 9), & (15, 17, 8, 5, 10), \\ (12, 19, 2, 10, 4), & (13, 18, 3, 11, 1), & (14, 17, 7, 0, 8), & (15, 16, 9, 5, 6), \\ (12, 6, 14, 1, 10), & (13, 7, 18, 9, 0), & (15, 4, 17, 2, 8), & (17, 6, 18, 10, 3), \\ (12, 7, 14, 4, 9), & (13, 2, 15, 5, 11), & (16, 3, 19, 9, 1), & (17, 5, 19, 0, 10), \\ (12, 8, 19, 11, 0), & (13, 4, 18, 1, 8), & (14, 5, 16, 11, 2), & (15, 7, 16, 6, 3). \end{array}$$

□

Lemma 4.12 *There exists an equitably 2-coloured 5-cycle decomposition of $K_{22} - F$.*

Proof. Let the vertex set of $K_{22} - F$ be \mathbb{Z}_{22} . Colour the vertices $0, 1, \dots, 11$ black and colour the vertices $12, 13, \dots, 21$ white. Let the edges in F be $\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 11\}, \{12, 13\}, \{14, 15\}, \{16, 17\}, \{18, 19\}$ and $\{20, 21\}$. A suitable decomposition of $K_{22} - F$ is given by:

(12, 14, 13, 0, 2),	(12, 15, 13, 1, 3),	(12, 16, 13, 2, 1),	(12, 17, 13, 3, 0),
(12, 18, 13, 4, 6),	(12, 19, 13, 5, 7),	(12, 21, 13, 7, 4),	(14, 16, 15, 0, 4),
(14, 17, 15, 1, 5),	(14, 18, 15, 2, 6),	(14, 19, 15, 3, 7),	(14, 20, 0, 5, 2),
(14, 21, 0, 6, 1),	(15, 20, 1, 4, 8),	(15, 21, 1, 7, 9),	(16, 18, 0, 7, 2),
(16, 19, 0, 8, 1),	(16, 20, 2, 4, 3),	(16, 21, 2, 8, 5),	(17, 18, 1, 9, 0),
(17, 19, 1, 10, 2),	(17, 20, 3, 5, 9),	(17, 21, 3, 6, 8),	(18, 20, 4, 9, 2),
(18, 21, 4, 10, 3),	(19, 20, 5, 10, 6),	(19, 21, 5, 11, 2),	(12, 8, 13, 9, 10),
(12, 9, 14, 0, 11),	(13, 10, 14, 3, 11),	(14, 8, 16, 4, 11),	(15, 4, 17, 1, 11),
(15, 5, 17, 7, 10),	(15, 6, 16, 11, 7),	(16, 7, 18, 8, 10),	(16, 9, 18, 10, 0),
(17, 6, 20, 8, 3),	(17, 10, 19, 8, 11),	(18, 4, 19, 9, 11),	(18, 5, 19, 11, 6),
(19, 7, 21, 9, 3),	(20, 10, 21, 6, 9),	(20, 11, 21, 8, 7).	

□

Theorem 4.13 *There exist equitably 2-colourable 5-cycle decompositions of $K_v - F$ if and only if $v \equiv 0, 2 \pmod{10}$, $v \geq 10$.*

Proof. By Lemma 1.1, a 5-cycle decomposition of $K_v - F$ exists if and only if $v \equiv 0, 2 \pmod{10}$, $v \geq 10$. We need to consider these two cases separately.

Case 1: $v \equiv 0 \pmod{10}$.

Let $v = 10x$, $x \geq 1$. By Corollary 4.2, we can take either a $\text{GDD}[3, 1, 2; 2x]$ or a $\text{GDD}[3, 1, \{2, 4^*\}; 2x]$ and simultaneously construct $K_v - F$ and its equitably 2-coloured 5-cycle decomposition as follows. Replace each element of the design with five vertices, colouring three vertices black and two vertices white. Within each set of vertices arising from a group of the design of size 2, let there be three one-coloured black edges and two one-coloured white edges in F . Similarly, within any set of vertices arising from a group of size 4, let there be six one-coloured black edges and four one-coloured white edges in F .

By Lemma 4.6, we can place an equitably 2-coloured 5-cycle decomposition of $K_{3(5)}$ on each set of vertices arising from a block of the design. Furthermore, by Lemmas 4.9 and 4.11, we can place an equitably 2-coloured 5-cycle decomposition of $K_{10} - F$ or $K_{20} - F$ on g for each set of vertices g arising from a group of the design of size 2 or 4 respectively. It is not difficult to check that the result is an equitably 2-coloured 5-cycle decomposition of $K_v - F$.

Case 2: $v \equiv 2 \pmod{10}$.

Let $v = 10x + 2$, $x \geq 1$. Proceeding as for Case 1, we replace each element of either a $\text{GDD}[3, 1, 2; 2x]$ or $\text{GDD}[3, 1, \{2, 4^*\}; 2x]$ with five vertices, colouring three vertices black and two vertices white. Let the edges in F be as described in Case 1. Adjoin two new independent vertices ∞_1 and ∞_2 , both coloured white, such that ∞_1 and ∞_2 are adjacent to all other vertices except each other; that is, $\{\infty_1, \infty_2\}$ is an edge of F .

We can again place an equitably 2-coloured 5-cycle decomposition of $K_{3(5)}$ on each set of vertices arising from a block of the design. Furthermore, by Lemmas 4.10 and 4.12, we can place an equitably 2-coloured 5-cycle decomposition of $K_{12} - F$ or $K_{22} - F$ on $\{\infty_1, \infty_2\} \cup g$ for each set of vertices g arising from a group of the design of size 2 or 4 respectively. It is not difficult to check that the result is an equitably 2-coloured 5-cycle decomposition of $K_v - F$. \square

5 Equitably 2-colourable 6-cycle decompositions

For 6-cycles, we proceed in much the same manner as for 4-cycles. In this case, the only possible equitable 2-colourings of a 6-cycle are shown in Figure 3.

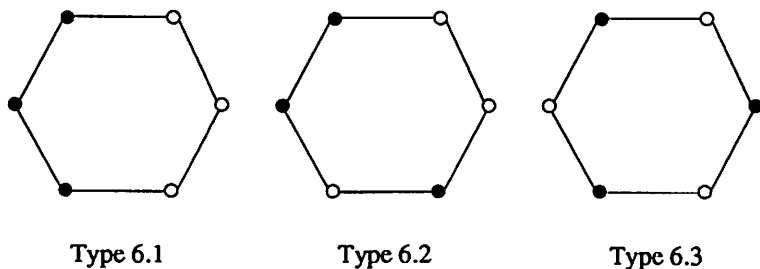


Figure 3: Possible equitable 2-colourings of 6-cycles.

We use the following existence results when proving Theorem 5.4.

Lemma 5.1 *There exists an equitably 2-coloured 6-cycle decomposition of $K_6 - F$.*

Proof. Let the vertex set of $K_6 - F$ be \mathbb{Z}_6 . Colour the vertices 0, 1 and 2 black and colour the vertices 3, 4 and 5 white. Let the edges in F be

$\{0, 4\}$, $\{1, 5\}$ and $\{2, 3\}$. A suitable decomposition of $K_6 - F$ is given by: $(0, 1, 2, 5, 4, 3)$, $(0, 2, 4, 1, 3, 5)$. \square

Lemma 5.2 *There exists an equitably 2-coloured 6-cycle decomposition of $K_8 - F$.*

Proof. Let the vertex set of $K_8 - F$ be \mathbb{Z}_8 . Colour the vertices 0, 1, 2 and 3 black and colour the vertices 4, 5, 6 and 7 white. Let the edges in F be $\{i, i + 4\}$, for $i = 0, 1, 2, 3$. A suitable decomposition of $K_8 - F$ is given by: $(0, 1, 3, 4, 5, 7)$, $(1, 2, 3, 5, 6, 7)$, $(2, 0, 6, 1, 4, 7)$, $(3, 0, 5, 2, 4, 6)$. \square

Lemma 5.3 *There exists an equitably 2-coloured 6-cycle decomposition of $K_{6,6}$.*

Proof. Let the vertex set of $K_{6,6}$ be $\bigcup_{i=1,2} \{0_i, 1_i, \dots, 5_i\}$, with the obvious vertex partition. Colour the vertices 0_i , 2_i and 4_i black, for $i = 1, 2$, and colour the remaining vertices white. A suitable decomposition of $K_{6,6}$ is given by:

$$\begin{array}{lll} (0_1, 0_2, 4_1, 5_2, 1_1, 3_2), & (2_1, 2_2, 0_1, 1_2, 3_1, 5_2), & (4_1, 4_2, 2_1, 3_2, 5_1, 1_2), \\ (4_1, 2_2, 1_1, 0_2, 3_1, 3_2), & (0_1, 4_2, 3_1, 2_2, 5_1, 5_2), & (2_1, 0_2, 5_1, 4_2, 1_1, 1_2). \end{array}$$

\square

We prove the following:

Theorem 5.4 *There exist equitably 2-colourable 6-cycle decompositions of $K_v - F$ if and only if $v \equiv 0, 2 \pmod{6}$, $v \geq 6$.*

Proof. By Lemma 1.1, a 6-cycle decomposition of $K_v - F$ exists if and only if $v \equiv 0, 2 \pmod{6}$, $v \geq 6$. We need to consider these two cases separately.

Case 1: $v \equiv 0 \pmod{6}$.

Let $v = 6x$, $x \geq 1$. Let the vertex set of $K_v - F$ be $\bigcup_{i=1, \dots, x} V_i$, where $V_i = \{0_i, 1_i, \dots, 5_i\}$. Colour the vertices 0_i , 1_i and 2_i black for $i = 1, 2, \dots, x$, and colour the remaining vertices white. Let the edges in F be $\{0_i, 3_i\}$, $\{1_i, 4_i\}$ and $\{2_i, 5_i\}$, for $i = 1, 2, \dots, x$.

By Lemma 5.3 we can place an equitably 2-coloured 6-cycle decomposition of $K_{6,6}$ on $V_i \vee V_j$, for $1 \leq i < j \leq x$. By Lemma 5.1, we can place an equitably 2-coloured 6-cycle decomposition of $K_6 - F$ on V_i for $1 \leq i \leq x$.

Case 2: $v \equiv 2 \pmod{6}$.

Let $v = 6x + 2$, $x \geq 1$. Label and colour the vertices of a copy of $K_{6x} - F$ as for Case 1. Also, let the edges in F be as described for Case 1. Create the

graph $K_{6x+2} - F$ by adjoining two new independent vertices, ∞_1 , coloured black, and ∞_2 , coloured white, such that ∞_1 and ∞_2 are adjacent to all other vertices except each other; that is, $\{\infty_1, \infty_2\}$ is an edge in F .

By Lemma 5.3 we can place an equitably 2-coloured 6-cycle decomposition of $K_{6,6}$ on $V_i \vee V_j$, for $1 \leq i < j \leq x$. By Lemma 5.2, we can place an equitably 2-coloured 6-cycle decomposition of $K_8 - F$ on $V_i \cup \{\infty_1, \infty_2\}$ for $1 \leq i \leq x$. \square

6 Equitably 2-colourable 2-perfect cycle decompositions

In the previous sections, we have shown how to construct equitably 2-coloured m -cycle decompositions for $m = 4, 5$ and 6 . In this section, we show that not every cycle decomposition which satisfies the obvious necessary conditions can be equitably 2-coloured. That is, we show that there exist cycle decompositions which are not equitably 2-colourable.

If C is a cycle of length m , the *distance i graph of C* , denoted $C(i)$, is formed by connecting the vertices in C that are at distance i . If $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ forms a cycle decomposition of G , and the collection of graphs $\{C_1(i), C_2(i), \dots, C_n(i)\}$ also forms a decomposition of G , then we say that \mathcal{C} is an *i -perfect cycle decomposition of G* . The spectrum problem for i -perfect m -cycle systems has been considered for a number of cycle lengths; see [1]. We proceed by briefly considering the existence of 2-perfect, equitably 2-coloured m -cycle systems for $m = 5$ and 7 .

Theorem 6.1 [7] *There exists a 2-perfect 5-cycle decomposition of K_v if and only if*
 $v \equiv 1, 5 \pmod{10}$, $v \geq 5$ and $v \neq 15$.

Theorem 6.2 *No 2-perfect 5-cycle system of order $v > 5$ can be equitably 2-coloured.*

Proof. Consider the four possible equitable 2-colourings of a 5-cycle using two colours as shown in Figure 2. Suppose that the vertices of K_v are coloured with a total of b black and w white vertices, and that there exists a 2-perfect 5-cycle decomposition of K_v which contains: α 5-cycles of Type 5.1; β of Type 5.2; γ of Type 5.3; and δ of Type 5.4. We now establish equations relating the number of vertices of each colour in K_v to the required number of 5-cycles of each type.

The number of one-coloured white edges in K_v is $\frac{1}{2}w(w-1)$. Since 5-cycles of Types 5.1 and 5.4 each contain one one-coloured white edge, those

of Type 5.2 contain two such edges and those of Type 5.3 contain none, we deduce that:

$$\alpha + 2\beta + \delta = \frac{1}{2}w(w - 1). \quad (1)$$

Since every vertex in K_v appears in $\frac{1}{2}(v - 1)$ 5-cycles of the decomposition the cycles in the decomposition contain a total of $\frac{1}{2}(v - 1)b$ black vertices and $\frac{1}{2}(v - 1)w$ white vertices. Noting that our decomposition has $\frac{1}{10}v(v - 1)$ 5-cycles, let x denote the number of 5-cycles with three white vertices (Types 5.2 and 5.4) and y denote the number with two white vertices (Types 5.1 and 5.3). Then

$$\begin{aligned} x + y &= \frac{1}{10}v(v - 1), \text{ and} \\ 3x + 2y &= \frac{1}{2}(v - 1)w. \end{aligned}$$

Solving the system for x and y , we find that

$$x = \frac{1}{10}(v - 1)(5w - 2v) = \beta + \delta, \text{ and} \quad (2)$$

$$y = \frac{1}{10}(v - 1)(3v - 5w) = \alpha + \gamma. \quad (3)$$

Taking the distance 2 graph of a 5-cycle we again obtain a 5-cycle. Clearly, the distance 2 graph of a cycle of Type 5.1 is a cycle of Type 5.3 and vice versa, and the distance 2 graph of a cycle of Type 5.2 is a cycle of Type 5.4 and vice versa. As the system is 2-perfect, the collection of distance 2 graphs must also be a 5-cycle decomposition of K_v , this time containing: γ 5-cycles of Type 5.1; δ of Type 5.2; α of Type 5.3; and β of Type 5.4. Based on the distance 2 decomposition, we can derive a new equation by following the reasoning used to derive Equation 1. We have

$$\gamma + 2\delta + \beta = \frac{1}{2}w(w - 1). \quad (4)$$

Adding Equations (1) and (4), we obtain

$$w(w - 1) = (\alpha + \gamma) + 3(\beta + \delta). \quad (5)$$

Substituting our expressions for $(\alpha + \gamma)$ and $(\beta + \delta)$ as given by Equations 2 and 3, we can manipulate equation Equation 5 to give

$$w^2 - vw + \frac{3}{10}v(v - 1) = 0. \quad (6)$$

This is a quadratic equation in w . For a real solution to this quadratic to exist, it is required that $v(6 - v) \geq 0$, which implies that $v \leq 6$. Combining

this result with Theorem 4.8, we deduce that no 2-perfect 5-cycle system of order $v > 5$ can be equitably 2-coloured. \square

Lemma 6.3 *There exists a 5-cycle decomposition of K_{15} which is not equitably 2-colourable.*

Proof. Let the vertex set of K_{15} be \mathbb{Z}_{15} . A computational search has shown that the following 5-cycle decomposition of K_{15} cannot be equitably 2-coloured.

(14, 4, 12, 13, 8),	(9, 2, 5, 4, 6),	(6, 5, 12, 7, 3),	(13, 9, 0, 4, 2),
(10, 8, 1, 6, 0),	(3, 4, 11, 8, 0),	(0, 1, 2, 3, 5),	(0, 2, 6, 7, 11),
(0, 7, 1, 3, 12),	(0, 13, 1, 5, 14),	(1, 4, 7, 2, 10),	(1, 9, 3, 8, 12),
(1, 11, 2, 12, 14),	(2, 8, 4, 9, 14),	(3, 10, 4, 13, 11),	(3, 13, 5, 7, 14),
(5, 8, 6, 10, 9),	(5, 10, 13, 6, 11),	(6, 12, 11, 10, 14),	(7, 8, 9, 12, 10),
(7, 9, 11, 14, 13).			

\square

Theorem 6.4 *For all $v \equiv 1, 5 \pmod{10}$, $v > 5$, there exists a 5-cycle decomposition of K_v which is not equitably 2-colourable.*

Proof. This follows immediately from Theorem 6.2 and Lemma 6.3. \square

However, Theorem 6.2 does not generalise to all 2-perfect m -cycle decompositions for $m > 5$; see the following example.

Example 6.5 There exists a 2-perfect 7-cycle decomposition of K_{15} which can be equitably 2-coloured.

Proof. Let the vertex set of K_{15} be \mathbb{Z}_{15} . Colour the vertices 0, 1, 3, 5, 7, 9, 11 and 13 black and colour the remaining vertices white. Then the starter cycle (0, 1, 3, 10, 7, 2, 6) can be developed modulo 15 to produce a 2-perfect 7-cycle decomposition. It is easy to check that the decomposition is equitably 2-coloured. \square

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