The Binet Formulas for the Pell and Pell-Lucas $p-$Numbers

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ABSTRACT

In this paper, we define the Pell and Pell-Lucas $p-$numbers and derive the analytical formulas for these numbers. These formulas are similar to Binet's formula for the classical Pell numbers.

Keywords: Pell numbers, Pell-Lucas numbers, Recurrence relation, Binet's formula

1. Introduction

In [2], the authors define the Fibonacci and Lucas $p-$numbers and give the analytic formulas for these numbers. Afterwards, they show that these formulas are similar to Binet's formulas for the Fibonacci and Lucas numbers. The purpose of this article is to define Pell and Pell−Lucas $p-$numbers and derive the analytical formulas for these numbers.

Now, we define the recurrence relation for the Pell $p-$ numbers. For $p = 0, 1, 2, \dots$, the recurrence relation is given as follows

$$P_p(n) = 2P_p(n-1) + P_p(n-p-1) \tag{1}$$

for initial conditions

$$P_p(1) = a_1,\ P_p(2) = a_2, \dots, P_p(p+1) = a_{p+1} \tag{2}$$

where $a_1,\ a_2,\ \ldots,\ a_{p+1}$ are integers, real, or complex numbers.

In particular, we can take these initial conditions as follows

$$P_p(n) = 2^{n-1},\ n = 1, 2, \ldots, p+1. \tag{3}$$

For different values of p, the recurrence relation generates different numerical sequences. For example, in the case $p = 0$, the recurrence relation is given as follows

$$P_0(n) = 3P_0(n-1)$$

for the given initial condition

$$P_0(1) = 1,$$

which generates the sequences $1, 3, 9, 27, 81 \ldots$.

If we take $p = 1$, we obtain

$$P_1(n) = 2P_1(n-1) + P_1(n-2), \tag{4}$$

for initial conditions $P_1(1) = 1,\ P_1(2) = 2$. This recurrence relation generates Pell numbers

$$P_1(n) = \{1, 2, 5, 12, 29, 70, 169, \ldots\}$$

Taking initial conditions $P_1(1) = 2$ and $P_1(2) = 6$, we obtain Pell–Lucas sequence

$$Q_1(n) = \{2, 6, 14, 34, 82, 198, 478, \ldots\}.$$

It's known that the characteristic equation for the classical Pell numbers is given as follows

$$x^2 - 2x - 1 = 0. \tag{5}$$

This equation has two real roots;

$$x_1 = \alpha_1 = 1 + \sqrt{2}, \quad x_2 = -\frac{1}{\alpha_1} = 1 - \sqrt{2}.$$

Binet's formula allows all Pell numbers $P_1(n)$ and Pell–Lucas numbers $Q_1(n)$ to be represented by the roots x_1 and x_2 of equation (5);

$$\begin{aligned} P_1(n) &= \frac{\alpha_1^n - \left(\frac{-1}{\alpha_1}\right)^n}{2\sqrt{2}} \\ Q_1(n) &= \alpha_1^n + \left(\frac{-1}{\alpha_1}\right)^n \end{aligned}$$

where $n = 0,\ \pm 1,\ \pm 2,\ \ldots$.

2. Some Properties of the Pell p–sequences

2.1. Pell and Pell–Lucas Numbers

For positive and negative values of n, we show the Pell and Pell–Lucas numbers in the following table.

n	0	1	2	3	4	5	6	...
$P_1(n)$	0	1	2	5	12	29	70	...
$P_1(-n)$	0	1	−2	5	−12	29	−70	...
$Q_1(n)$	2	2	6	14	34	82	198	...
$Q_1(-n)$	2	−2	6	−14	34	−82	198	...

From this table we see that for all the even and odd values of n, we have the following correlations for the Pell and Pell–Lucas numbers

$$\begin{aligned} P_1(2k) &= -P_1(-2k), \quad P_1(2k+1) = P_1(-2k-1) \\ Q_1(2k) &= Q_1(-2k), \quad Q_1(2k+1) = -Q_1(-2k-1). \end{aligned}$$

2.2. Pell and Pell–Lucas p–Numbers

Let us consider Pell p–numbers that are given by (1) at initial conditions 3. For a given set of Pell p–numbers

$$P_p(0),\ P_p(-1),\ P_p(-2),\ \ldots,\ P_p(-p),\ \ldots,\ P_p(-2p+1),\ldots$$

we will use recurrence relations and initial conditions.

$$P_p(p+1) = 2P_p(p) + P_p(0).$$

According to (3), $P_p(p+1) = 2^p$ and $P_p(p) = 2^{p-1}$, thus $P_p(0) = 0$. Continuing in this way, we obtain

$$P_p(-1) = P_p(-2) = \ldots = P_p(-p+1) = 0.$$

Let us write the Pell p–number $P_p(1)$ in the form

$$P_p(1) = 2P_p(0) + P_p(-p),$$

and we get $P_p(-p) = 1$. Also we have

$$P_p(-p-1) = P_p(-p-2) = \ldots = P_p(-2p+1) = 0.$$

The values of the Pell $p-$numbers for negative and positive values of n are given in the following table.

n	...	-5	-4	-3	-2	-1	0	1	2	3	4	5	...
$P_1(n)$	...	29	-12	5	-2	1	0	1	2	5	12	29	...
$P_2(n)$	...	1	-2	0	1	0	0	1	2	4	9	20	...
$P_3(n)$	...	0	0	1	0	0	0	1	2	4	8	17	...
$P_4(n)$	...	0	1	0	0	0	0	1	2	4	8	16	...

The characteristic equation of the Pell $p-$numbers is

$$x^{p+1} - 2x^p - 1 = 0. \tag{6}$$

We note that this equation has $(p+1)$ roots $x_1, x_2, x_3, \ldots, x_{p+1}$. The positive root of the equation is α_p, and let $x_1 = \alpha_p$.

Now, we give the following theorem associated with the properties of the characteristic equation of the Pell $p-$numbers.

Theorem 1 *(2)For the given integer $p > 0$, the following relationship for the roots of the characteristic equation $x^{p+1} - 2x^p - 1 = 0$ is valid;*

$$\begin{aligned}
x_1 + x_2 + \cdots + x_p + x_{p+1} &= 2 \\
x_1 x_2 \cdots x_p x_{p+1} &= (-1)^p \\
x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_1x_p + x_1x_{p+1} + x_2x_3 + x_2x_4 & \\
+ \cdots + x_2x_p + x_2x_{p+1} + \cdots + x_{p-1}x_p + x_{p-1}x_{p+1} + x_px_{p+1} &= 0 \\
x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_{p-2}x_{p-1}x_px_{p+1} &= 0 \\
&\vdots \\
x_1x_2x_3\cdots x_{p-2}x_{p-1}x_p + x_1x_3x_4\cdots x_{p-1}x_px_{p+1} & \\
+ \cdots + x_2x_3x_4\cdots x_{p-1}x_px_{p+1} &= 0
\end{aligned}$$

Proof. The characteristic equation $x^{p+1} - 2x^p - 1 = 0$ has $p+1$ roots $x_1, x_2, x_3, \ldots, x_{p+1}$. Therefore, we write

$$x^{p+1} - 2x^p - 1 = (x - x_1)(x - x_2)(x - x_3)\cdots(x - x_p)(x - x_{p+1}) = 0.$$

For the even values of p, we obtain

$$
\begin{aligned}
x^{p+1} - 2x^p - 1 = \;& (x - x_1)(x - x_2)(x - x_3)\cdots(x - x_p)(x - x_{p+1}) \\
= \;& x^{p+1} - (x_1 + x_2 + \cdots + x_p + x_{p+1})\, x^p \\
& + (x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_1x_p \\
& + x_1x_{p+1} + x_2x_3 + x_2x_4 + \cdots + x_2x_p + x_2x_{p+1} \\
& + \cdots + x_{p-1}x_p + x_{p-1}x_{p+1} + x_px_{p+1})x^{p-1} \\
& - (x_1x_2x_3 + x_1x_3x_4 + \cdots + x_1x_px_{p+1} + x_2x_3x_4 \\
& + x_2x_3x_5 + \cdots + x_2x_px_{p+1} + \cdots + x_{p-1}x_px_{p+1})x^{p-2} \\
& + (x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_{p-2}x_{p-1}x_px_{p+1})\, x^{p-3} \\
& + \cdots + (x_1x_2x_3x_4\cdots x_{p-2}x_{p-1}x_p + x_1x_3x_4\cdots x_{p-1}x_px_{p+1} \\
& + \cdots + x_2x_3x_4\cdots x_{p-1}x_px_{p+1})x - x_1x_2x_3x_4\cdots x_px_{p+1} \\
= \;& 0.
\end{aligned}
$$

Thus

$$
\begin{aligned}
x_1 + x_2 + \cdots + x_p + x_{p+1} &= 2 \\
x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_1x_p + x_1x_{p+1} + x_2x_3 + x_2x_4 & \\
+ \cdots + x_2x_p + x_2x_{p+1} + \cdots + x_{p-1}x_p + x_{p-1}x_{p+1} + x_px_{p+1} &= 0 \\
x_1x_2x_3 + x_1x_3x_4 + \cdots + x_1x_px_{p+1} + x_2x_3x_4 & \\
+ x_2x_3x_5 + \cdots + x_2x_px_{p+1} + \cdots + x_{p-1}x_px_{p+1} &= 0 \\
x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_{p-2}x_{p-1}x_px_{p+1} &= 0 \\
x_1x_2x_3\cdots x_{p-2}x_{p-1}x_p + x_1x_3x_4\cdots x_{p-1}x_px_{p+1} & \\
+ \cdots + x_2x_3x_4\cdots x_{p-1}x_px_{p+1} &= 0 \\
x_1x_2x_3x_4\cdots x_px_{p+1} &= 1
\end{aligned}
$$

For the odd values of p, we have

$$
\begin{aligned}
x^{p+1} - 2x^p - 1 = \;& (x - x_1)(x - x_2)(x - x_3)\cdots(x - x_p)(x - x_{p+1}) \\
= \;& x^{p+1} - (x_1 + x_2 + \cdots + x_p + x_{p+1})\, x^p \\
& + (x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_1x_p + x_1x_{p+1} \\
& + x_2x_3 + x_2x_4 + \cdots + x_2x_p + x_2x_{p+1} + \cdots \\
& + x_{p-1}x_p + x_{p-1}x_{p+1} + x_px_{p+1})x^{p-1} - (x_1x_2x_3 \\
& + x_1x_3x_4 + \cdots + x_1x_px_{p+1} + x_2x_3x_4 + x_2x_3x_5 \\
& + \cdots + x_2x_px_{p+1} + \cdots + x_{p-1}x_px_{p+1})x^{p-2} \\
& + (x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_{p-2}x_{p-1}x_px_{p+1})\, x^{p-3} \\
& + \cdots - (x_1x_2x_3x_4\cdots x_{p-2}x_{p-1}x_p + x_1x_3x_4\cdots x_{p-1}x_px_{p+1} \\
& + \cdots + x_2x_3x_4\cdots x_{p-1}x_px_{p+1})x + x_1x_2x_3x_4\cdots x_px_{p+1} \\
= \;& 0.
\end{aligned}
$$

Then

$$\begin{aligned}
x_1+x_2+\cdots+x_p+x_{p+1} &= 2\\
x_1x_2+x_1x_3+x_1x_4+\cdots+x_1x_p+x_1x_{p+1}+x_2x_3+x_2x_4 &\\
+\cdots+x_2x_p+x_2x_{p+1}+\cdots+x_{p-1}x_p+x_{p-1}x_{p+1}+x_px_{p+1} &= 0\\
x_1x_2x_3+x_1x_3x_4+\cdots+x_1x_px_{p+1}+x_2x_3x_4+x_2x_3x_5 &\\
+\cdots+x_2x_px_{p+1}+\cdots+x_{p-1}x_px_{p+1} &= 0\\
x_1x_2x_3x_4+x_1x_2x_3x_5+\cdots+x_{p-2}x_{p-1}x_px_{p+1} &= 0\\
x_1x_2x_3\cdots x_{p-2}x_{p-1}x_p+x_1x_3x_4\cdots x_{p-1}x_px_{p+1} &\\
+\cdots+x_2x_3x_4\cdots x_{p-1}x_px_{p+1} &= 0\\
x_1x_2x_3x_4\cdots x_px_{p+1} &= -1
\end{aligned}$$

Thus, the proof is clear. ■

Let us consider the following expression for the roots of characteristic equation (6)

$$(x_1+x_2+\cdots+x_p+x_{p+1})^k$$

where $k=1,\ 2,\ \ldots,\ p$. From Therem 1, we can write

$$(x_1+x_2+\cdots+x_p+x_{p+1})^k=2^k.$$

On the other hand, this expression can be factorized. If we use the binomial, trinomial and multinomial formulas, for the given k this formula will include the sum of all the k^{th} powers of the characteristic equation that are taken with the coefficient 1, that is, $x_1^k+x_2^k+x_3^k+x_4^k+\cdots+x_p^k+x_{p+1}^k$.

Now we give the following theorem.

Theorem 2 *(2)The following identity is true for the roots of the characteristic equation* $x^{p+1}-2x^p-1=0$

$$(x_1+x_2+\cdots+x_p+x_{p+1})^k=x_1^k+x_2^k+x_3^k+x_4^k+\cdots+x_p^k+x_{p+1}^k=2^k$$

where $p=1,\ 2,\ 3,\ \ldots$ *and* $k=1,\ 2,\ 3,\ \ldots,\ p$.

3. The Binet formulas for the Pell and Pell-Lucas p–numbers

Let us consider Binet's formula for the classical Pell and Pell–Lucas numbers. For a given $p>0$, using Binet's formula for the classical Pell and

Pell–Lucas numbers, we derive the following Binet's formula that gives Pell p–numbers.

$$P_p(n) = k_1 (x_1)^n + k_2 (x_2)^n + \cdots + k_{p+1} (x_{p+1})^n \tag{7}$$

where $x_1, x_2, \ldots, x_p, x_{p+1}$ are roots of characteristic equation, and $k_1, k_2, \ldots, k_{p+1}$ are constant coefficients that depend on the initial terms of the Pell p–numbers.

We will consider the Pell p–numbers given by the recurrence relation

$$P_p(n) = 2P_p(n-1) + P_p(n-p-1)$$

with initial conditions

$$P_p(0) = 0,\ P_p(n) = 2^{n-1},\ n = 1, 2, \ldots p+1.$$

Since we calculate the numerical values of the coefficients $k_1, k_2, \ldots, k_{p+1}$, consider solutions of the following system of the equations

$$\begin{cases} P_p(0) = k_1 + k_2 + \ldots + k_{p+1} = 0 \\ P_p(1) = k_1 x_1 + k_2 x_2 + \ldots + k_{p+1} x_{p+1} = 1 \\ P_p(2) = k_1 x_1^2 + k_2 x_2^2 + \ldots + k_{p+1} x_{p+1}^2 = 2 \\ \quad\vdots \\ P_p(p) = k_1 x_1^p + k_2 x_2^p + \ldots + k_{p+1} x_{p+1}^p = 2^{p-1} \end{cases} \tag{8}$$

3.1. The Binet formula for the Pell and Pell–Lucas numbers

Taking $p = 1$, we have the characteristic equation $x^2 - 2x - 1 = 0$, and the roots of this equation are $x_1 = \alpha_1 = 1 + \sqrt{2}$ and $x_2 = -\frac{1}{\alpha_1} = 1 - \sqrt{2}$. Thus, the formula (7) for the case $p = 1$, takes the following form

$$P_1(n) = k_1 (\alpha_1)^n + k_2 \left(-\frac{1}{\alpha_1}\right)^n$$

with the system of algebraic equations

$$\begin{aligned} P_1(0) &= k_1 + k_2 \\ P_1(1) &= k_1 \alpha_1 + k_2 \left(-\frac{1}{\alpha_1}\right) \end{aligned}$$

where $P_1(0) = 0$ and $P_1(1) = 1$. The solutions of system are $k_1 = \frac{\sqrt{2}}{4}$, $k_2 = -\frac{\sqrt{2}}{4}$. Therefore, we obtain the well-known Binet formula for the classical Pell numbers. Taking $k_1 = k_2 = 1$, we get Binet's formula for the classical Pell–Lucas numbers.

3.2. The Binet formulas for the Pell and Pell–Lucas 2–numbers

For the case $p = 2$, the characteristic equation, recurrence relation, and initial conditions are given as follows.

$$x^3 - 2x^2 - 1 = 0$$

$$\begin{aligned} P_2(n) &= 2P_2(n-1) + P_2(n-3) \\ P_2(0) &= 0,\ P_2(1) = 1,\ P_2(2) = 2. \end{aligned}$$

The roots of the characteristic equation are

$$\begin{aligned} x_1 &= \frac{k}{6} + \frac{8}{3k} + \frac{2}{3} = 2.205569431 \\ x_2 &= -\frac{k}{12} - \frac{4}{3k} + \frac{2}{3} + i\frac{\sqrt{3}}{2}\left(\frac{k}{6} - \frac{8}{3k}\right) \\ &= -0.1027847152 + i0.6654569515 \\ x_3 &= -\frac{k}{12} - \frac{4}{3k} + \frac{2}{3} - i\frac{\sqrt{3}}{2}\left(\frac{k}{6} - \frac{8}{3k}\right) \\ &= -0.1027847152 - i0.6654569515 \end{aligned}$$

where $k = \sqrt[3]{172 + 12\sqrt{177}}$.

Binet's formula for the Pell 2–numbers is

$$P_2(n) = k_1 (x_1)^n + k_2 (x_2)^n + k_3 (x_3)^n.$$

The numerical values of k_1, k_2, k_3 are solutions of the following system

$$\begin{aligned} P_2(0) &= k_1 + k_2 + k_3 \\ P_2(1) &= k_1 x_1 + k_2 x_2 + k_3 x_3 \\ P_2(2) &= k_1 (x_1)^2 + k_2 (x_2)^2 + k_3 (x_3)^2 \end{aligned}$$

where $P_2(0) = 0$ and $P_2(1) = 1$, $P_2(2) = 2$. Solving the system, we obtain

$$\begin{aligned} k_1 &= 0.382\,16 \\ k_2 &= -0.191\,08 - i0.088\,541 \\ k_3 &= -0.191\,08 + i0.088\,541 \end{aligned}$$

Therefore, the Binet's formula for the Pell 2−numbers is

$$\begin{aligned} P_2(n) &= (0.382\,16)\left(\frac{k}{6}+\frac{8}{3k}+\frac{2}{3}\right)^n \\ &\quad +(-0.191\,08-i0.088\,541)\left(-\tfrac{k}{12}-\tfrac{4}{3k}+\tfrac{2}{3}+i\tfrac{\sqrt{3}}{2}\left(\tfrac{k}{6}-\tfrac{8}{3k}\right)\right)^n \\ &\quad +(-0.191\,08+i0.088\,541)\left(-\tfrac{k}{12}-\tfrac{4}{3k}+\tfrac{2}{3}-i\tfrac{\sqrt{3}}{2}\left(\tfrac{k}{6}-\tfrac{8}{3k}\right)\right)^n. \end{aligned}$$

Taking $k_1 = k_2 = k_3 = 1$, we obtain the Binet's formula for the Pell-Lucas 2−numbers

$$\begin{aligned} Q_2(n) &= \left(\frac{k}{6}+\frac{8}{3k}+\frac{2}{3}\right)^n+\left(-\frac{k}{12}-\frac{4}{3k}+\frac{2}{3}+i\frac{\sqrt{3}}{2}\left(\frac{k}{6}-\frac{8}{3k}\right)\right)^n \\ &\quad +\left(-\frac{k}{12}-\frac{4}{3k}+\frac{2}{3}-i\frac{\sqrt{3}}{2}\left(\frac{k}{6}-\frac{8}{3k}\right)\right)^n. \end{aligned}$$

For $n = 0$, we obtain $Q_2(0) = 3$. According to the following recurrence relation

$$Q_2(n) = 2Q_2(n-1) + Q_2(n-3)$$

and initial conditions $Q_2(0) = 3,\ Q_2(1) = 2,\ Q_2(2) = 4$, the Pell-Lucas 2−sequence is

$$3, 2, 4, 11, 24, 52, 115\ldots.$$

3.3. The Binet formulas for the Pell and Pell−Lucas 3−numbers

For the case $p = 3$, the characteristic equation, recurrence relation, and initial conditions are given as follows

$$x^4 - 2x^3 - 1 = 0$$

$$\begin{aligned} P_3(n) &= 2P_3(n-1) + P_3(n-4) \\ P_3(0) &= 0,\ P_3(1) = 1,\ P_3(2) = 2,\ P_3(3) = 4. \end{aligned}$$

The numerical values of characteristic equation are

$$\begin{aligned} x_1 &= 2.106919340 \\ x_2 &= -0.7166727493 \\ x_3 &= 0.3048767045 + i0.7545291731 \\ x_4 &= 0.3048767045 - i0.7545291731 \end{aligned}$$

Formula (7) for the Pell 3−numbers takes the following form

$$P_3(n) = k_1(x_1)^n + k_2(x_2)^n + k_3(x_3)^n + k_4(x_4)^n.$$

The values of k_1, k_2, k_3, k_4 are solutions of the system

$$\begin{aligned} P_3(0) &= k_1 + k_2 + k_3 + k_4 \\ P_3(1) &= k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 \\ P_3(2) &= k_1(x_1)^2 + k_2(x_2)^2 + k_3(x_3)^2 + k_4(x_4)^2 \\ P_3(3) &= k_1(x_1)^3 + k_2(x_2)^3 + k_3(x_3)^3 + k_4(x_4)^3 \end{aligned}$$

where

$$P_3(0) = 0,\ P_3(1) = 1,\ P_3(2) = 2,\ P_3(3) = 4.$$

These solutions are

$$\begin{aligned} k_1 &= 0.411\,92 \\ k_2 &= -0.112\,78 \\ k_3 &= -0.149\,57 - i0.09\,4428 \\ k_4 &= -0.149\,57 + i0.09\,4428 \end{aligned}$$

Thus, we obtain Binet's formula for the Pell 3−numbers in the following numerical form

$$\begin{aligned} P_3(n) &= (0.411\,92)(2.106919340)^n \\ &\quad + (-0.112\,78)(-0.7166727493)^n \\ &\quad + (-0.149\,57 - i0.09\,4428)(0.3048767045 + i0.7545291731)^n \\ &\quad + (-0.149\,57 + i0.09\,4428)(0.3048767045 - i0.7545291731)^n. \end{aligned}$$

Taking $k_1 = k_2 = k_3 = k_4 = 1$, we have Binet's formula for the Pell−Lucas 3−numbers

$$\begin{aligned} Q_3(n) &= (2.106919340)^n + (-0.7166727493)^n \\ &\quad + (0.3048767045 + i0.7545291731)^n \\ &\quad + (0.3048767045 - i0.7545291731)^n. \end{aligned}$$

Using Binet's formula, we obtain the initial terms of the Pell–Lucas 3–numbers. Hence $Q_3(0) = 4,\ Q_3(1) = 2,\ Q_3(2) = 4,\ Q_3(3) = 8$, and the Pell–Lucas 3–sequence is

$$4, 2, 4, 8, 20, 42, 88, 184 \ldots.$$

3.4. The Binet formulas for the Pell and Pell–Lucas 4–numbers

For the case $p = 4$, the characteristic equation, recurrence relation, and initial conditions are given as follows

$$x^5 - 2x^4 - 1 = 0$$

$$\begin{aligned} P_4(n) &= 2P_4(n-1) + P_4(n-5) \\ P_4(0) &= 0,\ P_4(1) = 1,\ P_4(2) = 2,\ P_4(3) = 4,\ P_4(4) = 8. \end{aligned}$$

The numerical values of the characteristic equation are

$$\begin{aligned} x_1 &= 2.055967397 \\ x_2 &= 0.5541864024 + i0.6945926546 \\ x_3 &= 0.5541864024 - i0.6945926546 \\ x_4 &= -0.5821701008 + i0.5263901664 \\ x_5 &= -0.5821701008 - i0.5263901664. \end{aligned}$$

Binet's formula for the Pell 4–numbers is

$$P_4(n) = k_1(x_1)^n + k_2(x_2)^n + k_3(x_3)^n + k_4(x_4)^n + k_5(x_5)^n.$$

The values of k_1, k_2, k_3, k_4 are solutions of the following system

$$\begin{aligned} P_4(0) &= k_1 + k_2 + k_3 + k_4 + k_5 \\ P_4(1) &= k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5 \\ P_4(2) &= k_1(x_1)^2 + k_2(x_2)^2 + k_3(x_3)^2 + k_4(x_4)^2 + k_5(x_5)^2 \\ P_4(3) &= k_1(x_1)^3 + k_2(x_2)^3 + k_3(x_3)^3 + k_4(x_4)^3 + k_5(x_5)^3 \\ P_4(4) &= k_1(x_1)^4 + k_2(x_2)^4 + k_3(x_3)^4 + k_4(x_4)^4 + k_5(x_5)^4 \end{aligned}$$

where $P_4(0) = 0,\ P_4(1) = 1,\ P_4(2) = 2,\ P_4(3) = 4,\ P_4(4) = 8$. The numerical values are

$$\begin{aligned} k_1 &= 0.43863 \\ k_2 &= -0.1327 - i0.088136 \\ k_3 &= -0.1327 + i0.088136 \\ k_4 &= -0.086612 - i0.020893 \\ k_5 &= -0.086612 + i0.020893 \end{aligned}$$

Binet's formulas for the Pell and Pell–Lucas 4–numbers are

$$\begin{aligned} P_4(n) = & (0.43863)(2.055967397)^n \\ & + (-0.1327 - i0.088136)(0.5541864024 + i0.6945926546)^n \\ & + (-0.1327 + i0.088136)(0.5541864024 - i0.6945926546)^n \\ & + (-0.086612 - i0.020893)(-0.5821701008 + i0.5263901664)^n \\ & + (-0.086612 + i0.020893)(-0.5821701008 - i0.5263901664)^n \end{aligned}$$

and

$$\begin{aligned} Q_4(n) = & (2.055967397)^n + (0.5541864024 + i0.6945926546)^n \\ & + (0.5541864024 - i0.6945926546)^n \\ & + (-0.5821701008 + i0.5263901664)^n \\ & + (-0.5821701008 - i0.5263901664)^n . \end{aligned}$$

The initial terms of the Pell–Lucas 4–numbers are

$$Q_4(0) = 5,\ Q_4(1) = 2,\ Q_4(2) = 4,\ Q_4(3) = 8,\ Q_4(4) = 16.$$

Hence, the Pell–Lucas 4– sequence is

$$5, 2, 4, 8, 16, 37, 76, 156, 320 \ldots$$

and the recurrence relation is

$$Q_4(n) = 2Q_4(n-1) + Q_4(n-5).$$

3.5. The Binet formulas for the Pell and Pell–Lucas p–numbers (a general case)

Theorem 3 *For the integer $p > 0$, any Pell p–number $P_p(n)$ can be represented as follows*

$$P_p(n) = k_1(x_1)^n + k_2(x_2)^n + \cdots + k_{p+1}(x_{p+1})^n \qquad (9)$$

where $x_1, x_2, \ldots, x_{p+1}$ are the roots of the characteristic equation $x^{p+1} - 2x^p - 1 = 0$ and $k_1, k_2, \ldots, k_{p+1}$ are constant coefficients of the system in (8).

Proof. From equation (7), we have

$$P_p(p+1) = k_1(x_1)^{p+1} + k_2(x_2)^{p+1} + \cdots + k_{p+1}(x_{p+1})^{p+1}.$$

The roots, $x_1, x_2, \ldots, x_{p+1}$, of the characteristic equation have the following property.

$$x_k^n = 2x_k^{n-1} + x_k^{n-p-1}$$

where $k = 1, 2, \ldots, p+1$ and $n = 0, \pm1, \pm2, \ldots$. From this property, we have

$$\begin{aligned} P_p(p+1) &= 2\left[k_1(x_1)^p + k_2(x_2)^p + \cdots + k_{p+1}(x_{p+1})^p\right] \\ &\quad + \left[k_1(x_1)^0 + k_2(x_2)^0 + \cdots + k_{p+1}(x_{p+1})^0\right]. \end{aligned}$$

Therefore, $P_p(p+1) = 2P_p(p) + P_p(0)$. The basic recurrence relation is true for $P_p(p+1)$. Moreover, we can easily prove that this formula is valid for all positive values of n. Now, we prove that the formula is true for negative values of n. Taking $n = -1$ in equation(7), we obtain

$$P_p(-1) = k_1(x_1)^{-1} + k_2(x_2)^{-1} + \cdots + k_{p+1}(x_{p+1})^{-1}.$$

Since $x_k^n - 2x_k^{n-1} = x_k^{n-p-1}$, for the case $n = p$, we obtain

$$x_k^p - 2x_k^{p-1} = x_k^{-1}.$$

Therefore,

$$\begin{aligned} P_p(-1) &= \left[k_1(x_1)^p + k_2(x_2)^p + \cdots + k_{p+1}(x_{p+1})^p\right] \\ &\quad -2\left[k_1(x_1)^{p-1} + k_2(x_2)^{p-1} + \cdots + k_{p+1}(x_{p+1})^{p-1}\right] \end{aligned}$$

and then

$$P_p(-1) = P_p(p) - 2P_p(p-1) = 0.$$

Similarly, it is easy to prove that formula (9) is valid for all negative values of n. ■

Theorem 4 *For a given integer $p > 0$, the Binet's formula*

$$Q_p(n) = (x_1)^n + (x_2)^n + \cdots + (x_{p+1})^n \tag{10}$$

where $x_1, x_2, \ldots, x_{p+1}$ *are the roots of the characteristic equation, gives the Pell–Lucas p–sequences* $Q_p(n)$ *which can be expressed by the recurrence relation*

$$Q_p(n) = 2Q_p(n-1) + Q_p(n-p-1)$$

with the following initial conditions

$$Q_p(0) = p+1,\ Q_p(n) = 2^n \qquad \textit{for } n = 1,2,3,\ldots,p.$$

Proof. For the case $n = 0$, we rewrite the formula (10)

$$Q_p(n) = (x_1)^0 + (x_2)^0 + \cdots + (x_{p+1})^0 = 1 + 1 + \cdots + 1 = p+1.$$

For the cases $n = 1,2,3,\ldots,p$, we write

$$\begin{aligned} Q_p(1) &= x_1 + x_2 + \cdots + x_{p+1} \\ Q_p(2) &= (x_1)^2 + (x_2)^2 + \cdots + (x_{p+1})^2 \\ Q_p(3) &= (x_1)^3 + (x_2)^3 + \cdots + (x_{p+1})^3 \\ &\vdots \\ Q_p(p) &= (x_1)^p + (x_2)^p + \cdots + (x_{p+1})^p. \end{aligned}$$

From Theorem 2, the expressions we previously considered are respectively equal to $2, 2^2, 2^3, \ldots, 2^p$. This proves that formula (10) is true for the cases $n = 1,2,3,\ldots,p$.

The validity of formula (10) is proved similarly to Theorem 3.

We calculate the initial conditions of Pell–Lucas p–sequences by using (10) and Theorem 2. We have for $n = 1,2,3,\ldots,p$

$$\begin{aligned} Q_p(0) &= p+1 \\ Q_p(n) &= 2^n. \end{aligned}$$

Therefore, the Pell–Lucas p–sequence is determined by the recurrence relation

$$Q_p(n) = 2Q_p(n-1) + Q_p(n-p-1).$$

■

REFERENCES

[1] Koshy, T., Fibonacci and Lucas Numbers with Applications, A Wiley-Interscience Publication, 2001.

[2] Stakhov,A., Rozin, B., Theory of Binet's formulas for Fibonacci and Lucas $p-$numbers, Chaos, Solutions&Fractals, 27:1162−1177, 2006.

[3] Vajda, S., Fibonacci and Lucas Numbers, and the Golden Section Theory and Applications, Ellis Harwood Limited, 1989.

[4] Vorobiev, N.N., Fibonacci Numbers, Birkhauser Verlag, 1992.