The Domination Graphs of Complete Paired Comparison Digraphs

J. Richard Lundgren * K. B. Reid† Dustin J. Stewart*‡

Abstract

A complete paired comparison digraph D is a directed graph in which xy is an arc for all vertices x, y in D, and to each arc we assign a real number $0 \leq a \leq 1$ called a weight such that if xy has weight a then yx has weight $1 - a$. We say that two vertices x, y dominate a third z if the weights on xz and yz sum to at least 1. If x and y dominate all other vertices in a complete paired comparison digraph, then we say they are a dominant pair. We construct the domination graph of a complete paired comparison digraph D on the same vertices as D with an edge between x and y if x and y form a dominant pair in D. In this paper we characterize connected domination graphs of complete paired comparison digraphs. We also characterize the domination graphs of complete paired comparison digraphs with no arc weight of .5.

Introduction.

A *tournament* is an oriented complete graph. Let $V(D)$ and $A(D)$ denote the vertex and arc set of a digraph D respectively. If $xy \in A(D)$, we say that x beats y and write $x \rightarrow y$. Vertices x and y dominate a tournament T if for all vertices $z \neq x, y$, either $x \rightarrow z$ or $y \rightarrow z$. The domination graph of a tournament T, denoted dom(T), is the graph on the vertices $V(T)$ with $[x, y] \in E(\text{dom}(T))$ if and only if x and y dominate T. Domination graphs of tournaments were introduced by Merz et al. [7] in conjunction with competition graphs, and have been characterized in a series of papers (see [4], [7], [8]). This work was extended to what is called k−domination in a paper by McKenna et al.[6].

Recently, K. Factor [3] and J. Factor [2] considered domination when either ties are allowed in the tournament or when the digraph is a proper

*University of Colorado at Denver, Denver, CO 80217
†California State University San Marcos, San Marcos, CA 92096
‡Corresponding author, e-mail: dstewart@math.cudenver.edu

ARS COMBINATORIA 85(2007), pp. 107-127

subgraph of a tournament. Bergstrand and J. Friedler [1] have also considered this situation. Garth Isaak [5] suggested looking at domination graphs of complete paired comparison digraphs.

A *complete paired comparison digraph,* D, is a complete symmetric directed graph so that for each arc xy we associate a real number between 0 and 1, denoted w_{xy}, such that $w_{yx} = 1 - w_{xy}$. We also refer to a complete paired comparison digraph as a *PCD,* rather than the awkward CPCD.

We can think of a PCD as a model in which each vertex competes with all others, where if $x, y \in V(D)$, w_{xy} denotes the probability that x will beat y. Based on this we define a concept of domination in PCDs. If D is a PCD, and $x, y, z \in V(D)$ we say that x and y *dominate* z if $w_{xz} + w_{yz} \geq 1$. This is analogous to the situation in a tournament in which we require that either x or y beat z. Continuing the analogy to tournaments, we can go further still and ask the question as to which pairs of vertices $\{x, y\}$ dominate D. We say that vertices x and y form a *dominant pair* in a complete paired comparison digraph D if for all $z \in V(D) - \{x, y\}$ we have $w_{xz} + w_{yz} \geq 1$. Since a tournament can be considered a PCD with arc weights 0 or 1, observe that a dominant pair in a tournament is also a dominant pair in the tournament when considered as a PCD.

If D is a PCD, then we define the *domination graph* of D, denoted dom(D), on the same vertices of D with $[x, y]$ an edge of dom(D) if and only if $\{x, y\}$ is a dominant pair. In this paper we examine dominant pairs in PCDs by studying the structure of domination graphs. Surprisingly, there is a distinction between domination graphs of PCDs in which some competitors are equally matched (i.e., the arcs between them have weight .5) and domination graphs of PCDs having no equally matched competitors. We characterize domination graphs of PCDs in which $w_{xy} \neq .5$ for all $x, y \in V(D)$. We also characterize domination graphs of PCDs in which the domination graphs are connected graphs.

1 Preliminary Results.

In this section we give some preliminary results that will be used throughout the paper. A key to this study is the following lemma.

Lemma 1.1 *Let D be a complete paired comparison digraph. For any* 2 *vertex disjoint edges in dom(D), say $[r, s]$ and $[u, v]$, we have that in D*

$$w_{ru} = w_{us} = w_{sv} = w_{vr}.$$

Proof. Since $\{r, s\}$ and $\{u, v\}$ are dominant pairs in D, we know that $w_{vr} + w_{ur} \geq 1$, $w_{rv} + w_{sv} \geq 1$, $w_{su} + w_{ru} \geq 1$, and $w_{vs} + w_{us} \geq 1$. Also, by the definition of a PCD, $w_{ru} + w_{ur} = 1$, $w_{us} + w_{su} = 1$, $w_{sv} + w_{vs} = 1$, and $w_{vr} + w_{rv} = 1$. Thus,

$$w_{ru} = 1 - w_{ur} \leq w_{vr} = 1 - w_{rv} \leq w_{sv} = 1 - w_{vs} \leq w_{us} = 1 - w_{su} \leq w_{ru},$$

and so

$$w_{ru} = w_{us} = w_{sv} = w_{vr}.$$

□

Let n be an odd integer, and S a set of $\frac{n-1}{2}$ integers between 1 and $n-1$ such that if $x, y \in S$, then $x + y \not\equiv 0 \pmod{n}$. We define a *rotational tournament*, $T(S)$ with vertices $\{1, \ldots, n\}$ with $i \rightarrow j$ if and only if $j - i \equiv s \pmod{n}$ for some $s \in S$. We call S the *symbol* of $T(S)$. We are especially concerned with the rotational tournament of order n denoted U_n whose symbol is the set of odd numbers between $1, \ldots, n-2$. From U_n we can define an associated complete paired comparison digraph. Choose $0 \leq p \leq 1$. We define the PCD $U_{n,p}$ on vertex set $\{1, 2, \ldots, n\}$ by $w_{ij} = p$ if and only if $j - i$ is odd modulo n, and $w_{ij} = 1 - p$ otherwise. Let C_n denote the undirected $n-$cycle.

Lemma 1.2 *If n is odd, and C_n is an induced subgraph of the domination graph of some complete paired comparison digraph D, then the vertices which induce the cycle in dom(D) induce $U_{n,p}$ in D, for some p, $0 \leq p \leq 1$.*

Proof. We assume $V(C_n) = \{1, 2, \ldots, n\}$ and $E(C_n) = \{[i, i+1] : 1 \leq i \leq n-1\} \cup \{[n, 1]\}$. Suppose that $n \geq 5$. Consider the arc $1j$, $2 \leq j \leq n-2$. Assume that $j - 1$ is odd. Let $w_{1j} = p$. Apply Lemma 1.1 to edges $[n, 1]$ and $[j, j+1]$ to see that $w_{1j} = w_{jn} = w_{n(j+1)} = w_{(j+1)1} = p$. Apply Lemma 1.1 to edges $[1, 2]$ and $[j+1, j+2]$ to see that $w_{(j+2)1} = w_{1(j+1)} = 1 - w_{(j+1)1} = 1 - p$. Thus, $w_{1(j+2)} = p$, and $w_{1(j+1)} = 1 - p$. This implies that all arc weights w_{1j} are p when $j - 1$ is odd and $1 - p$ when $j - 1$ is even, $2 \leq j \leq n$. By an identical argument with i in place of 1, if $w_{ij} = q$ for $j - i$ odd, then all arc weights w_{ij} are q when $j - i$ is odd and $1 - q$ when $j - i$ is even. Thus, $p = w_{12} = 1 - w_{21} = 1 - q$. So, $p + q = 1$. Consequently, in D, $w_{ij} = p$ if and only if $j - i$ is odd modulo n, and $w_{ij} = 1 - p$ otherwise. Thus, $V(C_n)$ induces $U_{n,p}$ in D.

To complete the proof, suppose that $n = 3$. Then, since $\{1, 3\}$, $\{1, 2\}$ and $\{2, 3\}$ form dominant pairs, $w_{12} + w_{32} \geq 1$, $w_{13} + w_{23} \geq 1$, and $w_{21} + w_{31} \geq 1$. This means that $w_{12} + (1 - w_{23}) \geq 1$, $w_{23} + (1 - w_{31}) \geq 1$, and $w_{31} + (1 - w_{12}) \geq 1$. Thus,

$$w_{12} \geq w_{23} \geq w_{31} \geq w_{12},$$

and so the result follows.

□

Lemma 1.3 *Let D be a complete paired comparison digraph with at least 4 vertices. Then, dom(D) $= K_n$ if and only if $w_{xy} = .5$ for all $x \neq y \in V(D)$.*

Proof. Suppose dom$(D) = K_n$, and pick $x \neq y \in V(D)$. As $n \geq 4$, there are vertex disjoint edges $[x, x']$ and $[y, y']$ in dom(D). By Lemma 1.1, $w_{xy} = w_{yx'} = w_{x'y'} = w_{y'x}$. Now, apply Lemma 1.1 to edges $[x, y']$ and $[x'y]$ to see that $w_{yx} = w_{xx'} = w_{x'y'} = w_{y'y}$. Therefore, $w_{xy} = w_{x'y'} = w_{yx}$. But $w_{yx} = 1 - w_{xy}$, so $w_{xy} = w_{yx} = .5$, as desired. The converse is immediate. □

Note, it was shown in Lemma 1.2 that if D is a PCD with 3 vertices x, y, z so that $w_{xy} = w_{yz} = w_{zx} > .5$, then dom$(D) = K_3$.

Theorem 1.4 *Let D be a complete paired comparison digraph, and $S \subseteq V(D)$. Let D' be the PCD induced on S, then the subgraph of dom(D) induced on S is a subgraph of dom(D').*

Proof. Let $\{x, y\}$ be a dominant pair in D, with $x, y \in S$. Then, for all $v \in V(D) - \{x, y\}$, $w_{xv} + w_{yv} \geq 1$. In particular, this is true of all $v \in S$. Thus, $\{x, y\}$ is a dominant pair in D'. This proves our result. □

If D is a PCD, $v \in V(D)$, and $S \subseteq V(D)$, then we define the set $O_S^+(v)$ by

$$O_S^+(v) = \{x \in S : w_{vx} > .5\}.$$

If $S = V(D)$, then $O_{V(D)}^+(v)$ will be abbreviated as $O^+(v)$.

Lemma 1.5 *Let D be a complete paired comparison digraph, $v \in V(D)$, and $S \subseteq V(D)$. Then, $O_S^+(v)$ forms an independent set in dom(D).*

Proof. Let $x, y \in O_S^+(v)$. Then, $w_{xv} < .5$, and $w_{yv} < .5$, so $w_{xv} + w_{yv} < 1$. That is, $\{x, y\}$ does not form a dominant pair. □

The next proposition indicates why a characterization of domination graphs of PCDs is difficult. Consider the situation for tournaments. There are strict requirements on a graph G so that there exists a tournament T for which dom(T) contains G as an induced subgraph. This is not so in the context of PCDs since any graph will do, as seen in the next proposition.

Proposition 1.6 *Let G be a graph, then there exists a complete paired comparison digraph D for which dom(D) contains G as an induced subgraph.*

Proof. Let G be a graph on n vertices, and construct a PCD D in the following way. Start with $V(D) = V(G)$. For each $x, y \in V(G)$, let $w_{xy} = .5$. Now, for each pair $\{i, j\}$ of nonadjacent distinct vertices in G, add a vertex v_{ij} (same as v_{ji}) to $V(D)$, and set $w_{v_{ij}i} = w_{v_{ij}j} = 1$. Also, for each $z \in V(G)$ with $[i, z] \in E(G)$ or $[j, z] \in E(G)$, set $w_{zv_{ij}} = 1$. Set all other weights to .5. We now show the construction gives the desired result.

Consider two vertices $x, y \in V(G)$. Suppose that $[x, y] \in E(G)$. If $[x, y] \notin \text{dom}(D)$, then there is a $z \in V(D)$ so that $w_{xz} + w_{yz} < 1$. Because arc weights are 0, .5, or 1, this implies that at least one of w_{xz} or w_{yz} is 0. Suppose $w_{xz} = 0$; then by construction $z = v_{xk}$ for some vertex k. But, since $[x, y] \in E(G)$, the construction yields $w_{yz} = 1$. So, $w_{xz} + w_{yz} = 1$, a contradiction. Hence $[x, y] \in E(\text{dom}(D))$.

On the other hand, if $[x, y] \notin E(G)$, then using $z = v_{xy}$ we see that $w_{xz} + w_{yz} = 0 + 0 = 0$ so that $\{x, y\}$ is not a dominating pair in D. That is, $[x, y] \notin E(\text{dom}(D))$. So, G is an induced subgraph of dom(D). □

2 Complete Paired Comparison Digraphs With No Arc Weight .5.

As Proposition 1.6 shows, characterizing the domination graphs of PCDs is not an easy task. However, the problems seem to arise from equally matched competitors. So, in this section, we will assume that $w_{xy} \neq .5$ for every arc xy in our PCD. This is not such a bad assumption for it seems to be rare that two competitors are truly evenly matched. This assumption also gives a strong relation between tournaments and PCDs, as we shall see.

Let D be a PCD with no arc weight equal to .5. With D we associate a digraph T defined on the same vertices of D, where $x \to y$ in T if $w_{xy} > .5$. So for all $x, y \in V(T)$, we have exactly one of $x \to y$ or $y \to x$, i.e. T is a tournament.

Lemma 2.1 *Let D be a complete paired comparison digraph with no weight equal to .5, and T the associated tournament. Then dom(D) is a subgraph of dom(T). Furthermore, if D has only two weights, $a > .5$, and $1 - a$, then dom(D) =dom(T).*

Proof. Let $\{x, y\}$ be a dominant pair in D. Then for each $v \in V(D) - \{x, y\}$, $w_{xv} + w_{yv} \geq 1$. So, since $w_{xv} \neq .5$ and $w_{yv} \neq .5$, either $w_{xv} > .5$ or $w_{yv} > .5$. This implies that either $x \to v$ or $y \to v$ in T. Thus, $\{x, y\}$ is a dominant pair in T. So, dom(D) is a subgraph of dom(T).

Now, suppose that D has only two weights, $a > .5$ and $1 - a$. Let $\{x, y\}$ be a dominant pair in T. That is, for any other vertex v, $x \to v$ or $y \to v$, i.e., $w_{xv} > .5$ or $w_{yv} > .5$. As $a > .5$, this implies that $w_{xv} = a > 1 - a$ or

$w_{yv} = a > 1 - a$, so $w_{xv} + w_{yv} \geq a + (1 - a) = 1$. This means that $\{x, y\}$ is a dominant pair in D. Consequently, $\text{dom}(D) = \text{dom}(T)$. □

We point out that if the PCD D has more than two weights, we may have proper containment in the previous lemma.

Theorems 2.2 and 2.4 below are taken from [7]. These results, together with the previous lemma, are used in characterizing the domination graphs of PCDs with no arc weight .5. We now give a few constructions to obtain the main result of the section.

Recall that a *tree* is a connected acyclic graph. If T is a tree, and T has the property that the removal of all pendent vertices results in a path, then T is called a *caterpillar*. Each caterpillar has a path of maximum length called a *spine*. If G is a connected graph such that $V(G)$ can be partitioned into sets C and P such that C induces a cycle of length at least 3, and each vertex in P has degree 1, then we call G a *spiked cycle*. If the cycle in G is an odd cycle, then we call G a *spiked odd cycle*. Note, in the characterizations in this paper, we allow for the set of pendent vertices in a spiked cycle to be empty. That is, a spiked cycle could be just a cycle.

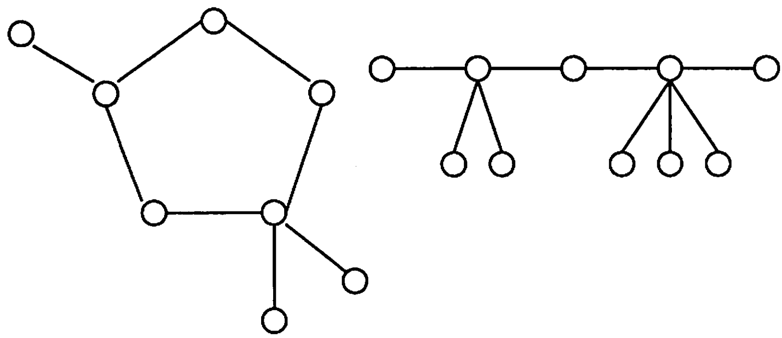

Figure 1: An example of a spiked cycle and a caterpillar

Theorem 2.2 *[7] Let T be a tournament on n vertices. Then dom(T) is either a spiked odd cycle with or without isolated vertices, or a forest of caterpillars.*

Theorem 2.3 *Let D be a complete paired comparison digraph in which no arc has weight equal to .5. Then dom(D) is a spiked odd cycle with or without isolated vertices, or a forest of caterpillars.*

Proof. Note that any subgraph of a spiked odd cycle is a either a spiked odd cycle, or a forest of caterpillars. So, the result follows from Lemma 2.1 and Theorem 2.2. □

Theorem 2.4 *[7] Any graph G consisting of a spiked odd cycle with possibly some isolated vertices is the domination graph of some tournament.*

Lemma 2.5 *If G is a caterpillar, then there exists a complete paired comparison digraph D in which no arc has weight .5 and dom(D) = G.*

Proof. Let C be a caterpillar with a spine $v_1, v_2, \ldots, v_m$. Construct the spiked odd cycle C' by adding to C the edge $[v_m, v_1]$ if m is odd, or the edge $[v_m, v_2]$ if m is even. From Theorem 2.4 we know that there exists a tournament with C' as its domination graph. In particular, the proof of Theorem 2.4 shows that the tournament T formed in the following way has C' as its domination graph. Define T on the same vertices as C', and orient the arcs on T so that U_m or U_{m-1}, whichever of m or $m-1$ is odd, is the subtournament on $v_1, \ldots, v_m$ or $v_2, \ldots, v_m$ respectively. Furthermore, if y is pendent to the cycle in C' and $[y, x] \in E(C')$, then let $x \to y$ in T, and for all z in $V(C') - \{y\}$, let $y \to z$ if $z \to x$ and let $z \to y$ if $x \to z$. It does not matter which direction the arcs between the pendant vertices have.

We now construct our PCD D from T. First, let $V(D) = V(T)$. Choose $.5 < a < 1$, and if $x \to y$ in T, then set $w_{xy} = a$, and $w_{yx} = 1 - a$. Finally, let $a < b < 1$ and change $w_{v_{m-1}v_m} = b$, and $w_{v_m v_{m-1}} = 1 - b$. Then for all $x, y \in V(D) - \{v_m\}$ we know from Lemma 2.1 that $[x, y] \in E(\text{dom}(D))$ if and only if $[x, y] \in E(\text{dom}(T))$ since all the arcs except $v_{m-1}v_m$ have weight a or $1 - a$, and $w_{v_{m-1}v_m} > a$. Furthermore, $\{v_m, v_{m-1}\}$ is a dominating pair, since it is one in T, and all arcs in $A(D) - \{v_m v_{m-1}, v_{m-1}v_m\}$ have weight either a or $1-a$. However, if $x \neq v_{m-1}$ then, $\{x, v_m\}$ does not form a dominant pair since $w_{xv_{m-1}} + w_{v_m v_{m-1}} < 1$. Since v_m is the end of the spine, it has no pendent vertices in C, and so $\text{dom}(D) = C$. □

Before we characterize the domination graphs of PCDs with no arc weight .5 we will first do it for the case in which the domination graph is a connected graph. Theorem 2.6 below is taken from [8]. Comparing this result to Theorem 2.7 below, we see that in PCDs, although the domination graph is a subgraph of the domination graph of some tournament, our characterization is less restrictive.

Theorem 2.6 *[8] A connected graph is the domination graph of a tournament if and only if it is a spiked odd cycle, a star, or a caterpillar with a triple end.*

Theorem 2.7 *If G is a connected graph, then G = dom(D) for some complete paired comparison digraph D with no arc having weight .5 if and only if G is a spiked odd cycle or a caterpillar.*

Proof. Let G be a spiked odd cycle. From Theorem 2.6, there exists a tournament T with $\text{dom}(T) = G$. Let D be any pcd with exactly two arc weights, $a \neq .5$ and $1-a$, such that T is the tournament associated with D. By Lemma 2.1, $\text{dom}(D) = \text{dom}(T) = G$. So, for any spiked odd cycle there exists a PCD with no arc weight .5 whose domination graph is that spiked odd cycle. From Lemma 2.5 we know that for any caterpillar, there exists a PCD with no arc weight .5 whose domination graph is that caterpillar. Also, Theorem 2.3 insures that these are the only possible connected domination graphs. Thus, the result follows. □

Theorem 2.8 *If G is a collection of isolated vertices, then there exists a complete paired comparison digraph D, with no arc having weight .5, such that $dom(D) = G$ if and only if G is not $2K_1$ or $3K_1$.*

Proof. First note that if a PCD has only two vertices, then they vacuously dominate all other vertices in the PCD. Thus $2K_1$ cannot be the domination graph of a PCD. Now suppose there exists a PCD D with $\text{dom}(D) = 3K_1$. Let $V(D) = \{1,2,3\}$. Then, $w_{12} + w_{32} < 1$, $w_{21} + w_{31} < 1$, $w_{13} + w_{23} < 1$. So,

$$w_{12} + w_{21} + w_{23} + w_{32} + w_{13} + w_{31} < 3.$$

But this contradicts the fact that

$$w_{12} + w_{21} + w_{23} + w_{32} + w_{13} + w_{31} = 3.$$

Thus, no such PCD exists.

Now, assume that G consists of 7 or more isolated vertices. It is straightforward to verify that the rotational tournament of order 7 with symbol $S = \{1,2,4\}$ (the so-called quadratic residue tournament of order 7) has a domination graph consisting of 7 isolated vertices. Consequently, any tournament of order $n \geq 7$ that contains such a subtournament of order 7, all of whose vertices dominate the remaining $n-7$ vertices, has a domination graph consisting of n isolated vertices. So, pick a tournament T with $\text{dom}(T) = G$, and let D be a PCD with no arc weight .5 whose associated tournament is T. Since $\text{dom}(D)$ is a subgraph of $\text{dom}(T)$ and $\text{dom}(T)$ has no edges, $\text{dom}(D) = \text{dom}(T) = G$.

We now consider the case where G is $4K_1$. Let $V(G) = \{1,2,3,v\}$. Construct a PCD D on the vertices of G in the following way. Let $\{1,2,3\}$ induce $U_{3,.7}$ in D and let $w_{vi} = .6$ for each $i \in \{1,2,3\}$. Then, no pair of vertices $\{i,j\}$ from $\{1,2,3\}$ can be dominant, since for each such pair $w_{iv} + w_{jv} = .8$. Also, $w_{v(i-1)} + w_{i(i-1)} = .6 + .3 = .9$ for each $i = 1,2,3$ so v is not part of any dominant pair. These are all possible combinations, so $\text{dom}(D) = 4K_1$.

Now, assume that $G = 5K_1$. Let D be a PCD with $V(D) = \{1,2,3,4,5\}$ and arc weights assigned as follows using addition modulo 5. For each

$i \in V(D)$ set $w_{i(i+1)} = .6$ and $w_{i(i+3)} = .7$ We now examine all possible pairs of vertices. Pick $i \in V(D)$. Since the pair $\{i, i+2\}$ is equivalent to the pair $\{i, i+3\}$ and the pair $\{i, i+1\}$ is equivalent to the pair $\{i, i+4\}$ we only need to look at the pairs $\{i, i+1\}$ and $\{i, i+2\}$. Then,

$$w_{i(i+2)} + w_{(i+1)(i+2)} = .3 + .6 = .9$$

and

$$w_{i(i+4)} + w_{(i+2)(i+4)} = .4 + .3 = .7.$$

Therefore no two vertices form a dominant pair, and so $\text{dom}(D) = 5K_1$.

Now suppose $G = 6K_1$, and let $V(G) = \{1, 2, 3, 4, 5, v\}$. Construct a PCD D on the vertices of G in the following way using addition modulo 5. Let $\{1, 2, 3, 4, 5\}$ induce $U_{5,.7}$ in D and set $w_{vi} = .6$ for each $i \in \{1, 2, 3, 4, 5\}$. So, if $i, j \in \{1, 2, 3, 4, 5\}$ then $w_{iv} + w_{jv} = .4 + .4$ so no pair of vertices not containing v is dominant. Now, if $i = 1, 2, 3, 4$, or 5, then $w_{v(i-1)} + w_{i(i-1)} = .6 + .3$. So, $\{v, i\}$ is not a dominant pair, as desired. Thus, $\text{dom}(D) = 6K_1$.

□

Lemma 2.9 *If G is a forest of caterpillars with at least one nontrivial component, then there exists a complete paired comparison digraph D with no arc weight of .5 for which $\text{dom}(D) = G$.*

Proof. Let $A_1, A_2, \ldots, A_k$ be the nontrivial caterpillars of G, and denote by $v_{i,1}, v_{i,2}, \ldots, v_{i,m_i}$, the spine of caterpillar A_i. Form the spiked odd cycle G' by adding to G the edges $[v_{i,m_i}, v_{(i+1),1}]$ for each $i = 1, \ldots, k-1$, and adding the edge $[v_{k,m_k}, v_{1,1}]$ if $\sum_{i=1}^{k} m_i$ is odd, and the edge $[v_{k,m_k}, v_{1,2}]$ if $\sum_{i=1}^{k} m_i$ is even. Now, let G'' be the graph which consists of G' and the isolated vertices of G. Note from Theorem 2.4, we know that there exists a tournament T such that $\text{dom}(T) = G''$. In particular, in [7], the authors show that the following tournament T satisfies $\text{dom}(T) = G$. Let $V(T) = V(G)$. Orient the arcs on the subtournament defined by the vertices on the cycle of G' to be the rotational tournament U_l, where l is the number of vertices in the cycle of G'. Now, if y is pendant to the cycle in G' at some vertex x, then in T let $x \rightarrow y$, and for all z in the cycle in G' if $x \rightarrow z$ in T then let $z \rightarrow y$ in T, and if $z \rightarrow x$ in T then let $y \rightarrow z$ in T. If v is isolated in G, then let $u \rightarrow v$ in T for all $u \in V(G')$. Orient arcs between isolated vertices arbitrarily.

We now construct our PCD D in the following way. Let $V(D) = V(T)$. Let $.5 < a < 1$, and if $x \rightarrow y$ in T, then set $w_{xy} = a$, and $w_{yx} = 1 - a$. Now, choose $a < b < 1$ and for all $i = 1, \ldots, k$ set $w_{v_{i,m_i-1}v_{i,m_i}} = b$ and $w_{v_{i,m_i}v_{i,m_i-1}} = 1 - b$. Then, for all $u, v \in V(D) - \{v_{i,m_i}\}$, $\{u, v\}$ is a dominant pair in T if and only if $\{u, v\}$ is a dominant pair in D, from

Lemma 2.1, since for all arcs incident with u or v, they either have weight a, $1-a$ or $b > a$. Also, $\{v_{i,m_i-1}, v_{i,m_i}\}$ is a dominant pair in D if and only if it is a dominant pair in T, for all $i = 1, \ldots, k$. This follows from Lemma 2.1 and the fact that for all arcs incident to v_{i,m_i} and v_{i,m_i-1}, other than $v_{i,m_i-1}v_{i,m_i}$ and $v_{i,m_i}v_{i,m_i-1}$ the weights are either a or $1-a$, for all $i = 1, \ldots, k$. Furthermore, for all $i = 1, \ldots, k$, if $x_i \neq v_{i,m_i-1}$ then, $\{x_i, v_{i,m_i-1}\}$ does not form a dominant pair in D since $w_{x_i,v_{i,m_i-1}} + w_{v_{i,m_i}v_{i,m_i-1}} < 1$. Since v_{i,m_i} is the end of a spine for all $i = 1, \ldots, k$, it is incident only to v_{i,m_i-1} in G. Thus, dom$(D) = G$.

□

Theorem 2.10 *A graph G is the domination graph of a complete paired comparison digraph with no arc having weight .5 if and only if G is a spiked odd cycle with or without isolated vertices, or a forest of caterpillars other than $2K_1$ or $3K_1$.*

Proof. From Theorem 2.3, we know that if G is the domination graph of a PCD with no arc having weight .5, then G must be a spiked odd cycle, with or without isolated vertices, or a forest of caterpillars. Given a graph G composed of a spiked odd cycle, with or without isolated vertices, Lemma 2.1 and Theorem 2.4 give the existence of a PCD with no arc weight of .5 which has G as its domination graph. From Lemma 2.9 and Theorem 2.8, for any forest of caterpillars other than $2K_1$ and $3K_1$, we can find a PCD with no arc weight of .5 which has that forest of caterpillars as its domination graph. Thus, the result follows. □

3 Connected Domination Graphs of Complete Paired Comparison Digraphs.

Recall that by Proposition 1.6 any graph we wish can be an induced subgraph of a domination graph of a PCD. So, in attempting to characterize domination graphs of PCDs, a first step is to see which connected graphs can be the domination graphs of PCDs. This characterization together with Theorem 1.4 is a major step towards characterizing the domination graphs of PCDs.

In the next theorem, we refer to the graph $NC7$. This is the smallest tree which is not a caterpillar and is shown in Figure 2. One should note that a tree is a caterpillar if and only if $NC7$ is not a subgraph.

Theorem 3.1 *If G is a connected graph, and G is the domination graph of a complete paired comparison digraph D, then $NC7$ is not an induced subgraph of G.*

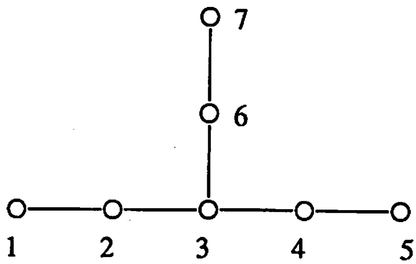

Figure 2: NC7

Proof. Suppose, to the contrary, that $NC7$ is an induced subgraph of G, and G is the domination graph of some PCD D. Let $\{1,2,\ldots,7\}$ denote the set of vertices which induce $NC7$, as in Figure 2. By applying Lemma 1.1 to the pairs of dominant pairs given in the following table, we force the situation in Figure 3, where each arc has weight $0 \le x \le 1$ in D.

step	pairs
1	$\{1,2\},\{6,7\}$
2	$\{2,3\},\{6,7\}$
3	$\{1,2\},\{3,6\}$
4	$\{3,4\},\{6,7\}$
5	$\{4,5\},\{6,7\}$
6	$\{4,5\},\{3,6\}$
7	$\{1,2\},\{3,4\}$
8	$\{1,2\},\{4,5\}$

Now, applying Lemma 1.1 to the pairs $\{2,3\}$ and $\{4,5\}$, we get that $x = w_{24} = w_{43} = 1 - w_{34} = 1 - x$. So $x = .5$. This implies that $w_{ij} = .5$ for each i,j except perhaps for $\{i,j\} = \{1,2\},\{4,5\},\{6,7\}$. As $\{2,3\}$ is a dominant pair, $w_{21} \ge .5$. Similarly, $w_{45} \ge .5$ (as $\{3,4\}$ is a dominant pair), and $w_{67} \ge .5$ (as $\{3,6\}$ is a dominant pair). This implies that $\{4,2\}$, $\{4,6\}$, and $\{2,6\}$ are dominant pairs in $D[\{1,2,\ldots,7\}]$. If G has only 7 vertices, then $G \ne \text{dom}(D)$, a contradiction.

Suppose that G has more than 7 vertices. Let $v \in V(G) - \{1,2,\ldots,7\}$. Since G is connected there exists a shortest path P from $\{1,2,\ldots,7\}$ to v, where P is given by $u_1,\ldots,u_m = v$ in G, and $u_1 \in \{1,2,\ldots,7\}$. We next show that $w_{2v} \ge .5$ and $w_{6v} \ge .5$.

Consider the case in which $u_1 \ne 3$. We show that $w_{2v} \ge .5$, $w_{6v} \ge .5$ and $w_{3v} \ge .5$ by induction on $m \ge 2$. Suppose that $m = 2$ (so $u_2 = v$). If

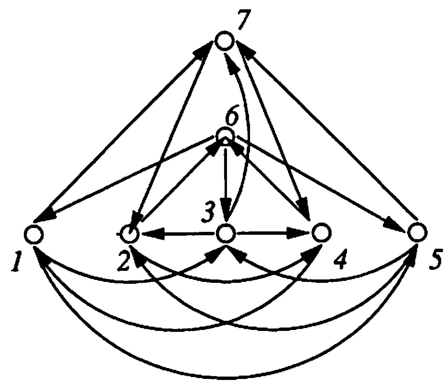

Figure 3: The arcs with weight x

$u_1 = 6$, use Lemma 1.1 on edges $[2,3]$ and $[u_1, u_2] = [6, v]$ to deduce that $w_{2v} = .5$ and $w_{3v} = .5$. Since $\{3,6\}$ is a dominating pair and $w_{3v} = .5$, $w_{6v} \geq .5$, as desired. If $u_1 = 2$, proceed similarly using edges $[3,6]$ and $[2,v]$. If $u_1 \neq 2,3,6$, use Lemma 1.1 on edges $[u_1, v]$ and $[2,3]$ to deduce $w_{2v} = w_{3v} = .5$, then use Lemma 1.1 on edges $[u_1, v]$ and $[3,6]$ to deduce that $w_{6v} = .5$, as desired.

Now, suppose that $m > 2$ and that $w_{2u_i} \geq .5$, $w_{6u_i} \geq .5$ and $w_{3u_i} \geq .5$ for $i = 2, \ldots, k-1$, where $k \leq m$. Use Lemma 1.1 on $[2,3]$ and $[u_{k-1}, u_k]$ to deduce $w_{2u_k} = w_{3u_{k-1}} \geq .5$. Again, use Lemma 1.1 on $[3,6]$ and $[u_{k-1}, u_k]$ to deduce $w_{6u_k} = w_{3u_{k-1}} \geq .5$ and $w_{3u_k} = w_{6u_{k-1}} \geq .5$, as desired. This completes the case in which $u_1 \neq 3$.

Now, consider the case in which $u_1 = 3$. We show that $w_{2v} = w_{6v} = w_{1v} = w_{7v} = .5$ by induction on $m \geq 2$. Suppose $m = 2$ (so $u_2 = v$). Use Lemma 1.1 on edges $[1,2]$ and $[3,v]$ and then on the edges $[6,7]$ and $[3,v]$ to deduce that $w_{2v} = w_{1v} = w_{6v} = w_{7v} = .5$, as required. Now suppose that $w_{2u_i} = w_{1u_i} = w_{6u_i} = w_{7u_i} = .5$ for $i = 2, \ldots, k-1$, where $k \leq m$. Use Lemma 1.1 on edges $[1,2]$ and $[u_{k-1}, u_k]$ and then on edges $[6,7]$ and $[u_{k-1}, u_k]$ to deduce that $w_{2u_k} = w_{1u_k} = .5$ (since $w_{1u_{k-1}} = .5$) and that $w_{6u_k} = w_{7u_k} = .5$ (since $w_{6u_{k-1}} = .5$). By induction, $w_{2v} = w_{6v} = w_{1v} = w_{7v} = .5$, as desired.

In particular, $w_{2v} + w_{6v} \geq 1$, for all $v \in V(G) - \{2,6\}$. Recall that $\{2,6\}$ is a dominant pair in $D[\{1,2,\ldots,7\}]$. Thus, $\{2,6\}$ is a dominant pair, but this means that $\text{dom}(D) \neq G$, a contradiction. □

This result, together with the fact that a tree is a caterpillar if and only if it has no $NC7$ as a subgraph, gives us the following corollary.

Corollary 3.2 *If a tree is the domination graph of a complete paired comparison digraph, then it is a caterpillar.*

Lemma 3.3 *Let D be a complete paired comparison digraph. Suppose that G is a subgraph of dom(D) such that for every $x \in V(G)$, $d_G(x) \geq 2$. Also, suppose that $w_{xy} = .5$ in D for all $x \neq y$ in $V(G)$. Let P denote a shortest path between $V(G)$ and a vertex v in $V(dom(D)) - V(G)$ given by $u_1, u_2, \ldots, u_m = v$, where $u_1 \in V(G)$ and $m \geq 2$. Then, for $m \geq 3$ and for all $x \in V(G)$, $w_{u_i x} = .5$ for all $i = 2, \ldots, m$, and for $m = 2$, $w_{u_2 x} = .5$ for all $x \in V(G) - \{u_1\}$ and $w_{u_1 u_2} \geq .5$.*

Proof. Suppose $m = 2$ (so $v = u_2$). Pick $x \in V(G) - \{u_1\}$. By assumption there is a vertex $y \in V(G) - \{u_1\}$ so that $[x, y] \in E(G)$. Use Lemma 1.1 on edges $[x, y]$ and $[u_1, u_2]$ to deduce $w_{vx} = w_{xu_1} = w_{u_1 y} = w_{yv}$. Since $w_{xu_1} = .5$, $w_{vx} = .5$. So, $w_{vx} = .5$ for all $x \in V(G) - \{u_1\}$. As $d_G(u_1) \geq 2$, u_1 is adjacent to some $z \in V(G)$. So, $\{u_1, z\}$ is a dominant pair and $w_{zv} = .5$, so $w_{u_1 v} \geq .5$.

Assume $m \geq 3$. Note, $[u_1, u_2]$ is a shortest path from u_2 to $V(G)$. So the case $m = 2$ above yields $w_{u_2 x} = .5$ for all $x \in V(G) - \{u_1\}$. Since $d_G(u_1) \geq 2$, we can choose $y \in V(G)$ such that $[u_1, y] \in E(G)$. Applying Lemma 1.1 to the edges $[u_1, y]$ and $[u_2, u_3]$ we deduce $w_{u_2 u_1} = w_{u_1 u_3} = w_{u_3 y} = w_{y u_2} = .5$, since $y \in V(G) - \{u_1\}$. Thus, $w_{u_2 z} = .5$ for all $z \in V(G)$. Inductively assume that $w_{u_i z} = .5$ for all $z \in V(G)$ and for $i = 2, \ldots, k-1$, where $3 \leq k \leq m$. Apply Lemma 1.1 to edges $[x, y]$ and $[u_{k-1}, u_k]$ to deduce $w_{u_k x} = w_{x u_{k-1}} = w_{u_{k-1} y} = w_{y u_k}$. Since $w_{u_{k-1} y} = .5$ by the induction hypothesis, $w_{u_k x} = .5$. That is, as x was arbitrary, $w_{u_k z} = .5$ for all $z \in V(G)$. So, by induction, for all $z \in V(G)$ and for all i, $2 \leq i \leq m$, $w_{u_i x} = .5$. □

Note that we can strengthen Lemma 3.3 slightly by weakening the hypothesis by replacing the phrase "for every $x \in V(G)$, $d_G(x) \geq 2$" with "for every $x \in V(G)$ and for every $y \in V(G) - x$, there is an edge of G incident with y but not incident with x, and dom(D) has no isolates."

Theorem 3.4 *If G is a connected graph and G contains an even cycle, C_{2k}, as an induced subgraph, then G is not the domination graph of any complete paired comparison digraph.*

Proof. Suppose, to the contrary, that some PCD D has G as its domination graph. Let $S = \{v_1, v_2, \ldots, v_{2k}\}$ be the set of vertices which induce the even cycle, and let the cycle be given by $[v_1, v_2], [v_2, v_3], \ldots, [v_{2k-1}, v_{2k}], [v_{2k}, v_1]$. We first do the case for $V(G) = S$. If $k = 2$ then two applications of Lemma 1.1 yields $w_{v_i v_j} = .5$ for all $i \neq j$ in $\{1, 2, 3, 4\}$. As $V(G) = S$,

$\mathrm{dom}(D) = K_4 \neq C_4 = G$, a contradiction. So, assume that $k \geq 3$. Fix a vertex v_j. We claim that

$$w_{v_j v_i} = w_{v_i v_{j-1}} = w_{v_{j-1} v_{i-1}} = w_{v_{i-1} v_j} \text{ when } i - j \pmod{2k} \text{ is even.} \quad (1)$$

Use induction on i. For $i - j = 2 \pmod{2k}$, apply Lemma 1.1 to edges $[v_{j-1}, v_j]$ and $[v_{j+1}, v_{j+2}]$ to deduce $w_{v_j v_{j+2}} = w_{v_{j+2} v_{j-1}} = w_{v_{j-1} v_{j+1}} = w_{v_{j+1} v_j}$ (arithmetic mod $2k$), as desired. Now, suppose that (1) holds for $i - j$ even where $2 \leq i - j < j - 2 \pmod{2k}$. Apply Lemma 1.1 to edges $[v_{j-1}, v_j]$ and $[v_i, v_{i+1}]$ to obtain

$$w_{v_j v_i} = w_{v_i v_{j-1}} = w_{v_{j-1} v_{i+1}} = w_{v_{i+1} v_j} (\text{arithmetic mod } 2k), \quad (2)$$

then apply Lemma 1.1 to edges $[v_{j-1}, v_j]$ and $[v_{i+1}, v_{i+2}]$ to obtain

$$w_{v_{j-1} v_{i+1}} = w_{v_{i+1} v_j} = w_{v_j v_{i+2}} = w_{v_{i+2} v_{j-1}} (\text{arithmetic mod } 2k). \quad (3)$$

Thus, as $w_{v_{i+1} v_j}$ appears in both (2) and (3),

$$w_{v_j v_{i+2}} = w_{v_{i+2} v_{j-1}} = w_{v_{j-1} v_{i+1}} = w_{v_{i+1} v_j}.$$

So, (1) holds by induction.

Statement (1) implies that

$$w_{v_j v_{j+2}} = w_{v_j v_{j+4}} = \cdots = w_{v_j v_{j-2}} (\text{arithmetic mod } 2k). \quad (4)$$

But, a similar statement is true for v_{j+2}, i.e.

$$w_{v_{j+2} v_{j+4}} = w_{v_{j+2} v_{j+6}} = \cdots = w_{v_{j+2} v_j} (\text{arithmetic mod } 2k). \quad (5)$$

Apply Lemma 1.1 to edges $[v_j, v_{j+1}]$ and $[v_{j+2}, v_{j+3}]$ to deduce

$$w_{v_j v_{j+2}} = w_{v_{j+2} v_{j+1}} = w_{v_{j+1} v_{j+3}} = w_{v_{j+3} v_j}. \quad (6)$$

Apply Lemma 1.1 to edges $[v_{j+1}, v_{j+2}]$ and $[v_{j+3}, v_{j+4}]$ to deduce

$$w_{v_{j+1} v_{j+3}} = w_{v_{j+3} v_{j+2}} = w_{v_{j+2} v_{j+4}} = w_{v_{j+4} v_{j+1}}. \quad (7)$$

Since $w_{v_{j+1} v_{j+3}}$ appears in both (6) and (7), $w_{v_j v_{j+2}} = w_{v_{j+2} v_{j+4}}$. This implies that all values in (4) and (5) are equal. In particular, $w_{v_j v_{j+2}} = w_{v_{j+2} v_j} = .5$. Thus, $w_{v_r v_s} = .5$ if $r - s$ (or $s - r$) is even. Further, if $r - s$ is odd, then by (1), $w_{v_r v_s} = w_{v_{r-1} v_s} = .5$ since $r - 1 - s$ is even. Similarly, if $s - r$ is odd, then $w_{v_r v_s} = .5$. In conclusion, $w_{xy} = .5$ for all $x, y \in S$. So, if $V(G) = S$, then all pairs of vertices are dominant in D, and $\mathrm{dom}(D) = K_{2k} \neq C_{2k} = G$, a contradiction.

So, assume that S is properly contained in $V(G)$. Let $z \in V(G) - S$. Since G is connected, there exists a shortest path P from S to z, given by $u_1, u_2, \ldots, u_m = z$ in G, where $u_1 \in S$. By Lemma 3.3, $w_{tz} = .5$ for all $t \in S - \{u_1\}$, and $w_{u_1 z} \geq .5$. Now, choose $s, t \in S$ such that $[s,t] \notin E(G)$. Since $w_{sz} + w_{tz} \geq .5 + .5 = 1$ for all $z \in V(D)$, $\{s,t\}$ must be a dominant pair in D. Thus, $G \neq \text{dom}(D)$, a contradiction. □

Lemma 3.5 *Let D be a complete paired comparison digraph. If dom(D) is connected, contains a 3−cycle C, and contains a vertex v of distance at least 2 from C, then C is contained in a larger clique in dom(D). Further, if P is a path from u to C, then some vertex of $V(P) - V(C)$ is contained in this clique.*

Proof. Let C be a 3−cycle in dom(D) given by $[1,2]$, $[2,3]$ and $[3,1]$, and suppose there is a vertex v not on C of distance at least 2 from C. Say, P is a path from C to v given by $u_1, u_2, \ldots, u_m = v$, where $m \geq 3$, $u_1 \in V(C)$, and $u_i \notin V(C)$ for all $i = 2, \ldots, m$. Use Lemma 1.1 with each edge of C and $[u_2, u_3]$ to deduce that $w_{xy} = .5$ for each $x \in \{1,2,3\}$ and $y \in \{u_2, u_3\}$. Then, use Lemma 1.1 with $[u_1, u_2]$ and $[a,b]$ where $a, b \in \{1,2,3\} - \{u_1\}$ to deduce that $w_{au_1} = w_{bu_1} = .5$. Also, note that $w_{ab} = w_{ba} = .5$. For if $w_{ab} > .5$, then $w_{ba} + w_{u_1 a} < 1$, but $[b, u_1] \in E(\text{dom}(D))$. Thus, $w_{xy} = .5$ for all $x, y \in \{1,2,3,u_2\}$. We show $w_{u_2 z} \geq .5$ and $w_{iz} \geq .5$ for all $z \in V(\text{dom}(D)) - \{u_2, i\}$ and $i = 1,2,3$. This will imply that $\{u_2, i\}$ is a dominant pair, so that u_2 is adjacent to each vertex of C in dom(D).

Choose $x \in V(D) - \{1,2,3,u_2\}$. From Lemma 3.3, $w_{ix} \geq .5$ for each $i = 1,2,3$. If $[x, u_2] \in E(\text{dom}(D))$, then from Lemma 3.3 $w_{ix} = .5$ for each $i = 1,2,3$. Also, since $\{u_1, u_2\}$ is a dominant pair and $w_{u_1 x} = .5$, $w_{u_2 x} \geq .5$. If $[u_1, x] \in E(\text{dom}(D))$, then apply Lemma 1.1 to $[x, u_1]$ and $[u_2, u_3]$ to see that $w_{u_2 x} = w_{u_1 u_2}$. Also, from Lemma 3.3 we know that $w_{u_1 u_2} = .5$ and so $w_{u_2 x} = .5$. If $[x, u_1], [x, u_2] \notin E(\text{dom}(D))$, then since dom($D$) is connected, there exists $a \in V(D)$, $a \neq u_1, u_2$ so that $[x, a] \in E(\text{dom}(D))$. Applying Lemma 1.1 to $[u_1, u_2]$ and $[x, a]$ we see that $w_{u_2 x} = w_{x u_1}$. By Lemma 3.3, $w_{x u_1} = .5$, so $w_{u_2 x} = .5$. Therefore, for any $x \in V(D)$, $w_{u_2 x} + w_{ix} \geq .5 + .5 = 1$ as desired. So, $\{1,2,3,u_2\}$ induces K_4 in dom(D).

□

The *chorded* 4−*cycle* is the only simple graph on 4 vertices with 5 edges (shown in Figure 4), and is sometimes referred to as a "kite." The *bowtie* is the graph on 5 vertices shown in Figure 5.

Theorem 3.6 *If D is a complete paired comparison digraph and dom(D) is connected and contains a 3−cycle C, then either dom(D) is a spiked 3−cycle or there is a clique of order at least 4 in dom(D) that contains C.*

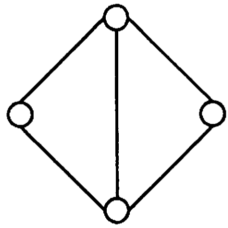

Figure 4: The chorded 4−cycle.

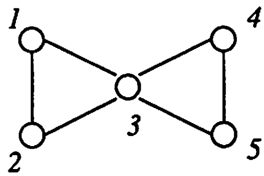

Figure 5: The Bowtie

Proof. Let C be given by $[1,2]$, $[2,3]$ and $[3,1]$. If dom(D) is not a spiked odd cycle, then in dom(D), either C is contained in a chorded 4−cycle, a bowtie, or there is a vertex v not on C but of distance at least 2 from C. Lemma 3.5 treats the latter case.

Suppose that C is contained in a bowtie and no vertex v is of distance at least 2 from C. Let $B = \{1,2,3,4,5\}$ be the set of vertices which induce the bowtie as shown in Figure 5. By applying Lemma 1.2 to $\{1,2,3\}$ we know that $w_{12} = w_{23} = w_{31}$. By applying Lemma 1.1 to $[1,2]$ and $[3,4]$, then to $[1,2]$ and $[3,5]$, and then to $[1,2]$ and $[4,5]$, we see that $w_{31} = w_{14} = w_{42} = w_{23}$, $w_{31} = w_{15} = w_{52} = w_{23}$, and $w_{42} = w_{25} = w_{51} = w_{14}$. But then, $w_{25} = w_{42} = w_{23} = w_{52}$. Thus, $w_{25} = w_{52} = .5$. So, applying Lemma 1.2 to $\{3,4,5\}$ and Lemma 1.1 to $[2,3]$ and $[4,5]$ gives us that $w_{xy} = .5$ for all $x,y \in B$. If $V(D) = B$, then dom(D) $= K_5$. So, assume there exists another vertex $v \in V(D)$. Then, since no vertex has distance 2 or more from C, v is adjacent to some vertex in B, and so by Lemma 3.3, $w_{xv} \geq .5$ for all $x \in B$. Thus, for each $x,y \in B$, $w_{xv} + w_{yv} \geq .5 + .5 = 1$. Since v was arbitrary we deduce that $\{x,y\}$ is a dominant pair for all $x,y \in B$. Thus, B induces a K_5 in dom(D), and C is contained in this K_5.

Now, assume that C is contained in a chorded 4−cycle C', and no vertex is of distance at least 2 from C. By applying Lemma 1.1 to the two vertex disjoint pairs of edges in the 4−cycle we deduce that $w_{xy} = .5$ for all $x, y \in C'$. So, if $V(D) = C'$, dom$(D) = K_4$. So, assume there exists another vertex $v \in V(D)$. Then, by Lemma 3.3, $w_{xv} \geq .5$ for all $x \in C'$. Thus, $w_{xv} + w_{yv} \geq .5 + .5 = 1$ for all $x, y \in C'$. Since v was arbitrarily chosen, we deduce that $\{x, y\}$ is a dominant pair for all pairs of vertices $x, y \in C'$, and so, the vertices of C' induce K_4 in dom(D). Thus, C is contained in a larger clique. □

Theorem 3.7 *Let G be a connected graph with an induced cycle C of odd order $n = 2k+1 \geq 5$. If G =dom(D) for some complete paired comparison digraph D, then G is a spiked odd cycle.*

Proof. Suppose, to the contrary, that there exists a PCD D with dom$(D) = G$, and G is not a spiked odd cycle. Then either G contains a vertex v adjacent to no vertex of C or it does not. If it does not, then either $G = C$ (and we are done) or G contains a vertex x not on C which is on a cycle which shares at least one vertex with C. Let C' be the smallest such cycle. If C' is a 3−cycle, then we use Lemma 3.5 to obtain a contradiction as follows.

Since C has at least 5 vertices, and shares at most two of them with C', there must be a vertex u on C of distance at least 2 from C'. Take C' to be the 3−cycle, u the vertex, and the path from C' to u contained in C to be the path in Lemma 3.5. This implies that C' must be contained in a larger clique, and this clique contains a vertex of $V(C) - V(C')$. Thus, there must be an edge between every vertex in C' and some vertex of C which is not in C'. Since $V(C) \cap V(C') \neq \emptyset$, some of these edges form chords in C contradicting it being an induced cycle.

Theorem 3.4 implies that C' is not an even cycle. So, suppose C' is an odd cycle of size 5 or greater. If x is the only vertex in C' not in C, then there must be at least three vertices in C which are not on C'. If there were only one, then this vertex together with x and the two vertices adjacent to x on C would form a 4−cycle contradicting the minimality of C'. If there were only two, then C would be an even cycle, contradicting the fact that C is an odd cycle. Thus, C' is an odd cycle of order at least 5 such that there is a vertex on C not adjacent to C'. Taking C' as C and this vertex as v we continue the proof.

We now treat the case where G contains a vertex v adjacent to no vertex of C. Let S be the set of vertices which induces C in G. From Lemma 1.2 we know that in D, S induces a $U_{n,p}$, for some $0 \leq p \leq 1$. We will prove that $p = .5$. We assume $S = \{v_1, \ldots, v_n\}$, where $w_{v_i v_j} = p$ if and only if $j - i$ is odd modulo n and $w_{v_i v_j} = 1 - p$ otherwise. Without loss of generality, assume that $p \geq .5$. Suppose $p > .5$. Since G is connected, there is a shortest path P from C to v, given by $u_1, u_2, \ldots, u_m = v$ in G, $m \geq 3$ with $u_1 \in S$. Without loss of generality we may assume that $u_1 = v_1$.

Apply Lemma 1.1 to edges $[u_1, u_2]$ and $[v_r, v_{r+1}]$ $(2 \leq r \leq n-1)$ to deduce that $w_{u_1 v_r} = w_{u_2 v_{r+1}}$. Since $\{v_i : i \text{ is even}, 2 \leq i \leq 2k\} \subseteq O_S^+(u_1)$, we see that $\{v_i : i \text{ is odd}, 3 \leq i \leq 2k+1\} \subseteq O_S^+(u_2)$. That is, for all i, $2 \leq i \leq 2k+1$,

$$w_{u_2 v_i} = \begin{cases} p & \text{if } i \text{ is odd,} \\ 1-p & \text{if } i \text{ is even.} \end{cases} \tag{8}$$

Since $\{v_{2k+1}, v_1\}$ is a dominant pair, and since $w_{v_{2k+1} u_2} = 1-p$, we deduce that $w_{v_1 u_2} \geq p$. Apply Lemma 1.1 to edges $[u_2, u_3]$ and $[v_r, v_{r+1}]$ $(2 \leq r \leq n$, arithmetic modulo $n)$ to deduce that $w_{u_2 v_r} = w_{u_3 v_{r+1}}$. By (8) this implies that for all i, $2 \leq i \leq 2k+1$,

$$w_{u_3 v_i} = \begin{cases} p & \text{if } i \text{ is even,} \\ 1-p & \text{if } i \text{ is odd,} \end{cases} \tag{9}$$

and (when $i = n$) $w_{u_3 v_1} = p$. Thus, $\{v_1, v_2, v_4, \ldots, v_{2k}\} \subseteq O_S^+(u_3)$. By Lemma 1.5, $O_S^+(u_3)$ is an independent set in G that contains edge $[v_1, v_2]$, a contradiction. Thus, $p = .5$.

As $n \geq 5$, there exist vertices $x, y \in S$ so that x and y are not adjacent in C. By Lemma 3.3, $w_{xv} + w_{yv} = .5 + .5 = 1$, for all such vertices v. Also, by the above argument, $w_{xw} + w_{yw} = .5 + .5 = 1$ for all $w \in S - \{x, y\}$. To show that $\{x, y\}$ is a dominant pair in D it remains to show that $w_{xz} + w_{yz} \geq 1$ for all vertices z adjacent to a vertex of C. Let z be a vertex not on C, but adjacent to, say, v_i on C. If $v_i \neq x, y$, then by Lemma 3.3 (where $m = 2, u_2 = z, x \neq u_1 \neq y$), $w_{xz} + w_{yz} = .5 + .5 = 1$. If $v_i = x$, then by Lemma 3.3 (where $m = 2, u_2 = z, u_1 = x, y \neq u_1$) $w_{xz} + w_{yz} \geq .5 + .5 = 1$. So, $\{x, y\}$ is a dominant pair in D. But, then $[x, y] \in E(\text{dom}(D))$, contradicting the choice of x and y. Consequently, there is no such vertex v, i.e. G is a spiked odd cycle. □

A connected graph G of order $m + n$ is called a *spiked clique* if $V(G)$ can be written as $V_1 \cup V_2$, where $|V_1| = n$, $|V_2| = m$, $G[V_1]$ is complete of order at least 4, and each vertex in V_2 has degree 1. A spiked clique with $n = 4$ and $m = 5$ is shown in Figure 6.

Theorem 3.8 *If G is a connected graph which contains a maximal clique K of order $m \geq 4$ and $G = \text{dom}(D)$ for some complete paired comparison digraph D, then G is a spiked clique.*

Proof. Suppose, to the contrary, that there exists a PCD D with $\text{dom}(D) = G$, and G is not a spiked clique. By Lemma 1.3, $w_{xy} = .5$ for each $x, y \in V(K)$. Since G is not a spiked clique, either G contains a vertex v which is not adjacent to any vertex in K or it does not. If it does not, then there must be some cycle C containing at most 2 vertices of K in G.

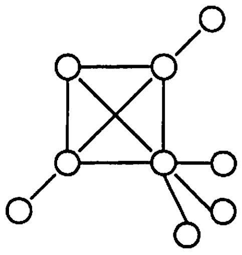

Figure 6: An example of a spiked clique.

First, suppose this cycle has order 3. If C and K share exactly 1 vertex then C together with any pair of vertices of K form a bowtie. If C and K share exactly 2 vertices, then C together with any vertex of K forms a chorded 4−cycle. From the proof of Theorem 3.6 we see that any bowtie or chorded 4−cycle must be contained in a clique. However, since the pair or single vertex which forms the bowtie or chorded 4−cycle were chosen arbitrarily, each vertex of C is adjacent to every vertex in K, a contradiction to K being maximal.

Now, the cycle cannot be of even length since this contradicts Theorem 3.4. If the cycle is odd of order at least 5, then Theorem 3.7 assures us that G is a spiked odd cycle, contradicting the existence of K in G. So, assume that there exists $v \in V(G)$ with v not adjacent to any vertex of K. Then, since G is connected, there exists a shortest path P from K to v given by $u_1, u_2, \ldots, u_m = v$ where $m \geq 3$ and $u_1 \in V(K)$. We now show that $\{u_2, x\}$ is a dominant pair for every $x \in K$ to draw a contradiction.

Pick $x \in V(K)$. First note, by Lemma 3.3, that $w_{xy} + w_{u_2y} = .5 + .5 = 1$ for all $y \in V(K) - \{x\}$. Now select $z \in V(G) - V(K)$. By Lemma 3.3, $w_{xz} \geq .5$. We show $w_{u_2z} \geq .5$. If $[z, u_2] \in E(G)$, then since $w_{u_1z} = .5$, Lemma 3.3 and the fact that $\{u_1, u_2\}$ is a dominant pair implies that $w_{u_2z} \geq .5$. If $[u_1, z] \in E(G)$, then apply Lemma 1.1 to $[z, u_1]$ and $[u_2, u_3]$ to see that $w_{u_2z} = w_{u_1u_2}$. Lemma 3.3 yields $w_{u_1u_2} = .5$, so $w_{u_2z} = .5$ as desired. Now, if z is not adjacent to u_1 or u_2 in G, then since G is connected, there exists $a \in V(G)$ such that $[z, a]$ is in G. By applying Lemma 1.1 to the edges $[u_1, u_2]$ and $[z, a]$ we see that $w_{u_2z} = w_{zu_1}$. By Lemma 3.3, $w_{zu_1} = .5$ and so $w_{u_2z} = .5$ as desired. Thus, for each $z \in V(G) - V(K)$ $w_{xz} + w_{u_2z} \geq .5 + .5 = 1$. Thus, $\{x, u_2\}$ is a dominant pair, a contradiction.

□

The previous two theorems show us that if D is a PCD such that dom(D) is connected and contains an induced cycle or clique, then dom(D) is a spiked odd cycle or a spiked clique. This together with the following lemma and some results from the previous section will yield a classification of connected domination graphs of PCDs in Theorem 3.10.

Lemma 3.9 *If G is a spiked clique, then there exists a complete paired comparison digraph D for which $dom(D) = G$.*

Proof. Take the vertex set of D to be $V(G)$. Let S denote the set of vertices in $V(G)$ that induces the clique in G. If $[x, y] \in E(G)$ with $x \in S$ and $y \in V(G) - S$, define $w_{xy} = b$, where $.5 < b \leq 1$ (so, $w_{yx} = 1 - b$). For all other pairs of distinct vertices u and v in $V(G)$ define $w_{uv} = .5$. Pick $x \neq y$ in $V(G)$. We check that $w_{xz} + w_{yz} \geq 1$ for all $z \in V(G) - \{x, y\}$ if and only if $[x, y] \in E(G)$. Suppose that $[x, y] \in E(G)$. If $x, y \in S$, then for all $z \in V(G) - \{x, y\}$,

$$w_{xz} + w_{yz} = \begin{cases} b + .5 & \text{, if } z \in S \text{ and } z \text{ is adjacent to } x \text{ or } y, \\ .5 + .5 & \text{, otherwise.} \end{cases}$$

If exactly one of x, y is in S, say $x \in S$ (and $y \notin S$), then for all $z \in V(G) - \{x, y\}$,

$$w_{xz} + w_{yz} = \begin{cases} b + .5 & \text{, if } [z, x] \in E(G), z \notin S, \\ .5 + .5 & \text{, otherwise.} \end{cases}$$

Since $b > .5$, $w_{xz} + w_{yz} \geq 1$ for all $z \in V(G) - \{x, y\}$, as desired.

On the other hand, suppose that $[x, y] \notin E(G)$. Since G is a spiked clique, at least one of x, y is not in S, say $y \notin S$. Now, choose $z \in S$ such that $[y, z] \in E(G)$. Then $w_{yz} = 1 - b$, and since $z \in S$, $w_{xz} \leq .5$. So, $w_{xz} + w_{yz} \leq .5 + (1 - b) < .5 + .5 = 1$. Thus, $\{x, y\}$ is not a dominant pair, as desired.

□

Together, Lemma 3.9, and Theorems 2.7, 3.1, 3.4, 3.6, 3.7 and 3.8, give us the following theorem classifying which connected graphs are the domination graphs of complete paired comparison digraphs. In comparison to the analogous result for tournaments, Theorem 2.6, we see that there are several new connected graphs which are the domination graphs of complete paired comparison digraphs, namely caterpillars without triple ends and spiked cliques.

Theorem 3.10 *Let G be a connected graph. Then there exists a complete paired comparison digraph D such that $dom(D) = G$ if and only if G is a spiked odd cycle, a caterpillar, or a spiked clique.*

Acknowledgements.

The authors would like to thank the referee for several helpful comments.

References

[1] Deborah Bergstrand and Louis Friedler, *Domination graphs of tournaments and other digraphs*, preprint.

[2] Jim Factor and Kim Factor, *Partial domination graphs of extended tournaments*, Congr. Numer. **158** (2002), 119–130.

[3] Kim Factor, *Domination graphs of compressed tournaments*, Congr. Numer. **157** (2002), 63–78.

[4] David Guichard, Sarah K. Merz, David C. Fisher, J. Richard Lundgren, and K. B. Reid, *Domination graphs with nontrivial components*, Graphs Combin. **17** (2001), 227–236.

[5] Garth Isaak, personal communication, 2002.

[6] Patricia McKenna, Margaret Morton, and Jamie Sneddon, *New domination conditions for tournaments*, Australas. J. Combin. **26** (2002), 171–182.

[7] Sarah K. Merz, David C. Fisher, J. Richard Lundgren, and K. B. Reid, *The domination and competition graphs of a tournament*, J. Graph Theory **29** (1998), 103–110.

[8] ———, *Connected domination graphs of tournaments*, J. Combin. Math. Combin. Comput. **31** (1999), 169–176.