On Relatively Short Sides of Convex Hexagons *

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Abstract

Let C be a plane convex body, and let $l(ab)$ be the Euclidean length of a longest chord of C parallel to the segment ab in C. By the relative length of ab in a convex body C, we mean the ratio of the Euclidean length of ab to $\frac{l(ab)}{2}$. We say that a side ab of a convex n-gon is relatively short if the relative length of ab is not greater than the relative length of a side of the regular n-gon. In this article, we provide a significant sufficient condition for a convex hexagon to have a relatively short side.
Keywords: affine diameter, convex body, relatively short side.

We need some definitions from [1]. Let ab be the closed segment with endpoints a and b in the Euclidean plane $\mathbb{R}^2$, and $|ab|$ be the Euclidean length of ab. Let $C \subset \mathbb{R}^2$ be a convex body. A chord pq of C is called an *affine diameter* of C if there is no longer parallel chord in C. Let $l(ab)$ denote the Euclidean length of an affine diameter of C parallel to the segment ab in C. The ratio of $|ab|$ to $\frac{1}{2}l(ab)$, namely $\frac{|ab|}{\frac{1}{2}l(ab)}$, is called the *$C$- distance* between a and b, or *C-length* of ab, denoted by $\operatorname{dist}_C(a,b)$. If there is no doubt about C, we may just say *relative distance* between a and b, or *relative length* of ab. Let λ_n denote the relative length of a side of the regular n-gon. By simple computation, we obtain $\lambda_3 = \lambda_4 = 2$, $\lambda_5 = \sqrt{5} - 1$ and $\lambda_6 = 1$. A side ab of a convex n-gon C is called *relatively short* if $\operatorname{dist}_C(a,b) \le \lambda_n$, and it is called *relatively long* if $\operatorname{dist}_C(a,b) \ge \lambda_n$.

Karol Doliwka and Marek Lassak proved that for $n \le 5$ every convex n-gon has a relatively long side and a relatively short side (see [1]). In

*This research was supported by National Natural Science Foundation of China (10571042), NSF of Hebei (A2005000144, A2007000226, A2007000002) and the SF of Hebei Normal University (L2004202).

answering the conjectures suggested by [1], the articles [2] and [3] proved independently by different methods that every convex hexagon has a side of relative length at most $8-4\sqrt{3}$.

In [1] the authors provided a hexagon which has no relatively short side. [4] proved that every convex hexagon has a relatively long side and claimed without proof a significant sufficient condition for a convex hexagon to have a relatively short side. In this article, we give the proof. By a pair of *main parallel sides* of a convex hexagon H, we refer to two parallel sides S_1 and S_2 of H such that each component of $\partial H \setminus (S_1 \cup S_2)$ is the union of two consecutive sides of H. For example, in a regular hexagon, any pair of parallel sides is a pair of main parallel sides. In this note we prove the following result:

Theorem. *Every convex hexagon with at least one pair of main parallel sides has a relatively short side.*

Let H denote a convex hexagon with vertices a_1, a_2, a_3, a_4, a_5 and a_6 in cyclic order. Since $\lambda_6 = 1$, we should prove that for every convex hexagon H with at least one pair of main parallel sides there exists a side xy of H with $\mathrm{dist}_H(x,y) \le 1$, namely $|xy| \le \frac{l(xy)}{2}$. For two segments vs and vt emanating from the same point v, we say *vs follows vt*, or equivalently, *vt precedes vs*, if the rotation from vs to vt around v with rotation angle less than π is clockwise. Let P be a parallelogram such that one or two pairs of main parallel sides of H lie on the boundary (two pairs of parallel sides) of P.

We define S^* as the set of sides of H such that a side xy of H is a member of S^* if and only if there is a side uv of P such that $xy \subset uv$ and $2|xy| \ge |uv|$.

For simplicity, if two segments pq and rs are parallel, we write $pq \parallel rs$.

First, we prove the following two lemmas.

Lemma 1. *Every convex hexagon H with at least two pairs of main parallel sides has a relatively short side.*

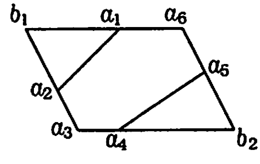

Figure 1: H has at least two pairs of main parallel sides.

Proof. Let P be a parallelogram such that two pairs of main parallel sides of H lie on the boundary of P, as shown in Figure 1. Let b_1, a_3, b_2, a_6 denote the vertices of P in cyclic order.

Case 1: $|S^*| \geq 3$. Then there exist two elements of S^*, say a_3a_4 and a_5a_6, and a point f such that either $f \in a_5a_6$ with $a_3f \parallel a_4a_5$, or $f \in a_3a_4$ with $a_6f \parallel a_4a_5$. Thus, a_3f or a_6f is an affine diameter of H parallel to a_4a_5, and $l(a_4a_5) \geq 2|a_4a_5|$. Hence a_4a_5 is relatively short.

Case 2: $|S^*| = 2$. If $S^* = \{a_3a_4, a_5a_6\}$, then obviously a_4a_5 is relatively short. Similarly, if $S^* = \{a_2a_3, a_5a_6\}$, $S^* = \{a_1a_6, a_2a_3\}$ or $S^* = \{a_1a_6, a_3a_4\}$, H has a relatively short side.

Next, we assume $S^* = \{a_1a_6, a_5a_6\}$. The argument for $S^* = \{a_2a_3, a_3a_4\}$ is similar.

Let $c_1 \in b_2a_6$ be the point such that $a_2c_1 \parallel a_3a_4$. If $c_1 \in a_5a_6$, then a_3a_4 is relatively short; otherwise $c_1 \in b_2a_5$. Consider the point $c_2 \in a_6b_1$ with $a_4c_2 \parallel a_2a_3$. If $c_2 \in a_6a_1$, then a_2a_3 is relatively short; otherwise $c_2 \in a_1b_1$. Consider the point $c_3 \in a_1a_6$ with $a_3c_3 \parallel a_1a_2$ and the point $c_4 \in a_5a_6$ with $a_3c_4 \parallel a_4a_5$. Let x denote the point of intersection of a_2c_1 and a_4c_2.

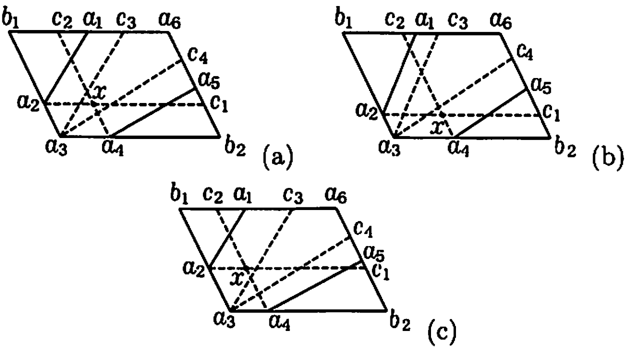

Figure 2: (a) x lies between a_3c_3 and a_3c_4; (b) a_3x precedes a_3c_4; (c) a_3x follows a_3c_3.

If a_3x lies between a_3c_3 and a_3c_4 (see Figure 2 (a)), then $l(a_2a_3) \geq 2|a_2a_3|$, $l(a_3a_4) \geq 2|a_3a_4|$, and both a_2a_3 and a_3a_4 are relatively short. If a_3x precedes a_3c_4 or lies on a_3c_4 (see Figure 2 (b)), then $l(a_2a_3) \geq 2|a_2a_3|$, and a_2a_3 is a relatively short side. If a_3x follows a_3c_3 or lies on a_3c_3 (see Figure 2 (c)), then $l(a_3a_4) \geq 2|a_3a_4|$, and a_3a_4 is relatively short.

Case 3: $|S^*| = 1$. Without loss of generality, we may assume that $S^* = \{a_1a_6\}$.

Subcase 3.1 $|a_1b_1| \leq |a_3a_4|$. Let $c_1 \in a_3a_4$ be the point such that $a_1c_1 \parallel a_2a_3$. Then clearly a_5a_6 is relatively short.

Subcase 3.2 $|a_1b_1| > |a_3a_4|$. Since $S^* = \{a_1a_6\}$, we have $|a_2a_3| < \frac{1}{2}|a_3b_1|$ and $|a_5a_6| < \frac{1}{2}|a_3b_1|$. Let $c_1 \in a_5b_2$, $c_2 \in a_1b_1$, $c_3 \in a_1a_6$ and $c_4 \in a_5a_6 \cup a_1a_6$ be the points such that $a_2c_1 \parallel a_3a_4$, $a_4c_2 \parallel a_2a_3$, $a_3c_3 \parallel a_1a_2$, and $a_3c_4 \parallel a_4a_5$ respectively.

If $c_4 \in a_5a_6$, then an argument, similar to the one used in Case 2, yields that a_2a_3 or a_3a_4 is a relatively short side of H. If $c_4 \in a_1a_6$, then $l(a_5a_6) \geq |a_2a_3| + |a_5a_6|$ and $l(a_2a_3) \geq |a_2a_3| + |a_5a_6|$. Therefore $|a_2a_3| \geq |a_5a_6|$ implies $l(a_5a_6) \geq 2|a_5a_6|$ and that a_5a_6 is relatively short; $|a_2a_3| < |a_5a_6|$ implies $l(a_2a_3) \geq 2|a_2a_3|$ and that a_2a_3 is relatively short.

Case 4: $|S^*| = 0$. Let $c_1 \in a_5b_2$ be the point such that $a_2c_1 \parallel a_3a_4$, $c_2 \in a_1b_1$ be the point such that $a_4c_2 \parallel a_2a_3$. Now consider the point $c_3 \in a_1a_6 \cup b_2a_6$ with $a_3c_3 \parallel a_1a_2$, and the point $c_4 \in a_5a_6 \cup b_1a_6$ with $a_3c_4 \parallel a_4a_5$.

If a_3c_3 follows a_3c_4, then H has a relatively short side a_2a_3 or a_3a_4, the argument is similar to that in Case 2. If a_3c_3 precedes a_3c_4, as shown in Figure 3, then let $d_1 \in a_2b_1$ be the point such that $a_5d_1 \parallel a_3a_4$, $d_2 \in a_4b_2$ be the point such that $a_1d_2 \parallel a_2a_3$. Now consider the point $d_3 \in a_2a_3 \cup b_2a_3$ with $a_6d_3 \parallel a_1a_2$, and the point $d_4 \in a_3a_4 \cup b_1a_3$ with $a_6d_4 \parallel a_4a_5$. Clearly a_6d_4 follows a_6d_3. By an argument similar to that in Case 2, we know that H has a relatively short side a_1a_6 or a_5a_6. □

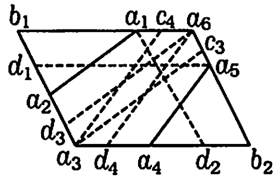

Figure 3: a_3c_3 precedes a_3c_4, a_6d_4 follows a_6d_3.

Lemma 2. *Every convex hexagon H with only one pair of main parallel sides has a relatively short side.*

Proof. Let P be a parallelogram such that the only pair of main parallel sides and one of the remaining sides of H lies on the boundary of P, as shown in Figure 4. Here we have two cases: (a) all the vertices of H lie on the boundary of P; (b) one vertex of H lies in the interior of P and all the other vertices lie on the boundary of P.

First, consider (b). If $|a_1a_6| \leq \frac{1}{2}|a_3a_4|$ or $|a_2a_3| \leq \frac{1}{2}|a_3b_1|$, then a_1a_6 or a_2a_3 is relatively short. Otherwise, there is a point c such that either $c \in a_2a_3$ with $a_6c \parallel a_1a_2$, or $c \in a_1a_6$ with $a_3c \parallel a_1a_2$. Thus, $l(a_1a_2) \geq 2|a_1a_2|$, and a_1a_2 is relatively short.

Now consider (a). Let b_1, b_0, b_2, a_6 denote the vertices of P in cyclic order.

Case 1: $|S^*| = 3$. Then $S^* = \{a_1a_6, a_5a_6, a_3a_4\}$. Consider $c_1 \in a_5a_6$ with $a_3c_1 \parallel a_4a_5$ or $c_2 \in a_3a_4$ with $a_6c_2 \parallel a_4a_5$. It is easy to see that

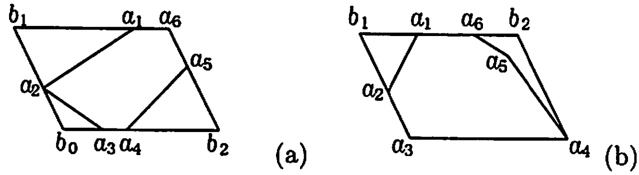

Figure 4: H has only one pair of main parallel sides.

$l(a_4a_5) \geq 2|a_4a_5|$ and that a_4a_5 is relatively short.

Case 2: $|S^*| = 2$. If $S^* = \{a_5a_6, a_3a_4\}$ or $S^* = \{a_1a_6, a_3a_4\}$, clearly H has a relatively short side. Now consider the only remaining case $S^* = \{a_1a_6, a_5a_6\}$.

Let $c_1 \in b_2a_6$ be the point such that $a_2c_1 \parallel a_3a_4$. If $c_1 \in a_5a_6$, then a_3a_4 is relatively short; otherwise $c_1 \in b_2a_5$. Consider the point $c_2 \in a_6b_1 \cup a_2b_1$ with $a_4c_2 \parallel a_2a_3$. If $c_2 \in a_6a_1$, then a_2a_3 is relatively short; otherwise $c_2 \in a_1b_1 \cup a_2b_1$. Consider the point $c_3 \in a_1a_6 \cup b_2a_6$ with $a_3c_3 \parallel a_1a_2$, and the point $c_4 \in a_5a_6 \cup b_1a_6$ with $a_3c_4 \parallel a_4a_5$. Assume that $c_4 \in b_1a_6$. Consider the point $d_1 \in a_3a_4$ with $a_6d_1 \parallel a_4a_5$, and we obtain that a_4a_5 is relatively short. Now assume $c_4 \in a_5a_6$. If $c_3 \in b_2a_6$, then a_3a_4 is relatively short; if $c_3 \in a_1a_6$, then a_3c_3 follows a_3c_4, by an argument similar to that in Case 2 of Lemma 1, we obtain that a_2a_3 or a_3a_4 is relatively short.

Case 3: $|S^*| = 1$.

Subcase 3.1: $S^* = \{a_3a_4\}$. Let $c_1 \in b_0b_1$ be the point such that $a_5c_1 \parallel a_1a_6$. If $c_1 \in a_2b_0$, then a_1a_6 is relatively short; otherwise $c_1 \in b_1a_2$. Consider the point $c_2 \in b_2a_3$ with $a_1c_2 \parallel a_5a_6$. If $c_2 \in a_3a_4$, then a_5a_6 is relatively short; otherwise $c_2 \in a_4b_2$. Consider the point $c_3 \in a_2b_0 \cup b_0a_4$ with $a_6c_3 \parallel a_1a_2$, and the point $c_4 \in a_3a_4$ with $a_6c_4 \parallel a_4a_5$.

If a_6c_4 follows a_6c_3, then, by an argument similar to that in Case 2 of Lemma 1, we obtain that a_1a_6 or a_5a_6 is a relatively short side of H. Now it remains to consider the case that a_6c_4 precedes a_6c_3. Let $d_1 \in a_5b_2$ be the point such that $a_2d_1 \parallel a_3a_4$, $d_2 \in a_1b_1 \cup a_2b_1$ be the point such that $a_4d_2 \parallel a_2a_3$. Consider the point $d_3 \in a_1a_6 \cup b_2a_6$ with $a_3d_3 \parallel a_1a_2$, and the point $d_4 \in a_5a_6 \cup b_1a_6$ with $a_3d_4 \parallel a_4a_5$. Clearly a_3d_3 follows a_3d_4. We obtain similarly that a_2a_3 a relatively short side of H.

Subcase 3.2: $S^* = \{a_5a_6\}$. Let $c_1 \in b_2a_6$ be the point such that $a_2c_1 \parallel a_3a_4$. If $c_1 \in a_5a_6$, then a_3a_4 is relatively short; otherwise $c_1 \in b_2a_5$. Consider the point $c_2 \in a_6b_1 \cup a_2b_1$ with $a_4c_2 \parallel a_2a_3$. If $c_2 \in a_6a_1$, then a_2a_3 is relatively short; otherwise $c_2 \in a_1b_1 \cup a_2b_1$. Consider the point $c_3 \in a_1a_6 \cup b_2a_6$ with $a_3c_3 \parallel a_1a_2$, and the point $c_4 \in a_5a_6 \cup b_1a_6$ with $a_3c_4 \parallel a_4a_5$. If $c_4 \in b_1a_6$, then a_4a_5 is relatively short; otherwise $c_4 \in a_5a_6$. If a_3c_3 precedes a_3c_4, then a_1a_6 or a_3a_4 is relatively short. If a_3c_3 follows a_3c_4, then, by a similar argument as before, we obtain that a_2a_3 or a_3a_4 is a relatively short side of H.

Subcase 3.3: $S^* = \{a_1a_6\}$. The conclusion can be obtained by an argument similar to that in the case $S^* = \{a_3a_4\}$ mentioned above.

Case 4: $|S^*| = 0$. The conclusion can be obtained by an argument similar to that in the case $S^* = \{a_3a_4\}$ mentioned above. □

Combining Lemmas 1 and 2, we obtain our Theorem.

Acknowledgements

We are very grateful to the anonymous referee for having generously given time in evaluating our manuscript, for the helpful comments and expert suggestions on our manuscript.

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