Graphic Sequences with a Realization Containing a Friendship Graph

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Abstract

For any simple graph H, let $\sigma(H,n)$ be the minimum m so that for any realizable degree sequence $\pi = (d_1, d_2, \ldots, d_n)$ with sum of degrees at least m, there exists an n-vertex graph G witnessing π that contains H as a weak subgraph. Let F_k denote the friendship graph on $2k+1$ vertices, that is, the graph of k triangles intersecting in a single vertex. In this paper, for n sufficiently large, $\sigma(F_k, n)$ is determine precisely.
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1 Introduction

Let G be a simple undirected graph, without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. For a

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vertex $v \in V(G)$, let $N(v)$ denote the set of neighbors (or neighborhood) of v, and $d(v)$ the degree of v, that is the order of $N(v)$. We let $\overline{G}$ denote the complement of G. Denote the complete graph on t vertices by K_t, and the *friendship graph* by F_k, where F_k is the graph of k triangles intersecting in a single vertex.

A sequence of nonincreasing, nonnegative integers

$$\pi = (d_1, d_2, \dots, d_n)$$

is called *graphic* if there is a (simple) graph G of order n having degree sequence π. In this case, G is said to *realize* π, and we will write $\pi = \pi(G)$. If a sequence π consists of the terms $d_1, \dots, d_t$ having multiplicities $m_1, \dots, m_t$, we may write $\pi = (d_1^{m_1}, \dots, d_t^{m_t})$. There are numerous elementary methods to check if a given sequence is graphic (for example, see [3, 7, 8]).

Define $\sigma(H, n)$ to be the smallest integer m so that for every n-term graphic degree sequence with degree sum at least m there exists a realization containing H as a weak subgraph. Such sequences are said to be *potentially H-graphic.* Note that in the definition of this function one only needs to replace the quantifier 'there exists a' with 'for every' to obtain a value that is two more than twice the Turán number, $ex(n, H)$. In this paper we determine the value of $\sigma(F_k, n)$.

For a survey of similar results we refer the reader to [18], and for any undefined terms to [1]

2 Useful Known Results

In [4] Erdős, Jacobson and Lehel conjectured that

$$\sigma(K_t, n) = (t-2)(2n-t+1) + 2.$$

The conjecture rises from consideration of the graph $K_{(t-2)} + \overline{K}_{(n-t+2)}$, where $+$ denotes the join. It is easy to observe that this graph contains no K_t, is the unique realization of the sequence

$$((n-1)^{t-2}, (t-2)^{n-t+2}),$$

and has degree sum $(t-2)(2n-t+1)$. Erdős *et al.* proved the conjecture for $t = 3$ and $n \geq 6$. The cases $t = 4$ and 5 were proved separately (see [6] and [10], and [11]). For $t \geq 6$ and $n \geq \binom{t}{2} + 3$, Li, Song & Luo [12] proved the conjecture true via linear algebraic techniques. Later, the present authors proved all cases of the conjecture via induction on t using graph theoretic techniques [5].

The following summarizes these results.

Theorem 1 *For $t \geq 3$ and $n > n_0(t)$,*

$$\sigma(K_t, n) = (t-2)(2n-t+1)+2.$$

The following results will be used in the proof of our main result.

Theorem 2 (Erdős-Gallai [3]) *A nonincreasing sequence of nonnegative integers*

$$\pi = (d_1, d_2, \ldots, d_n)$$

$(n \geq 2)$ is graphic if, and only if, the sum of the degrees is even and for each integer k, $1 \leq k \leq n-1$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min\{k, d_i\}.$$

The following is an extension of a theorem of Rao [17].

Theorem 3 ([6]) *If π is a graphic sequence with a realization G containing H as a subgraph, then there is a realization G' of π containing H with the vertices of H having the $|V(H)|$ largest degrees of π.*

Theorem 4 ([13], [14]) *Let $\pi = (d_1, d_2, \ldots d_n)$ be a non-increasing sequence of non-negative integers, where $d_1 = m$ and the degree sum is even. If there exists an integer $n_1 \leq n$ such that $d_{n_1} \geq h \geq 1$ and $n_1 \geq \frac{1}{h}\left[\frac{(m+h+1)^2}{4}\right]$, then π is graphic.*

Theorem 5 ([15]) *Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \ldots d_n)$ be graphic with $d_{r+1} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially K_{r+1}-graphic.*

The value of $\sigma(kK_2, n)$ was determined in [6].

Theorem 6 ([6]) $\sigma(kK_2, n) = (k-1)(2n-k)+2$.

The lower bound for $\sigma(kK_2, n)$ is easy to obtain by considering the graph $G' = K_{k-1} + \overline{K}_{n-k+1}$. This graph is the unique realization of the degree sequence $\pi = ((n-1)^{k-1}, (k-1)^{n-k+1})$, contains no matching of size k, and has degree sum $(k-1)(2n-k)$.

3 The Main Theorem

Erdős *et al.* [2], showed that any graph on n vertices having at least

$$\left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k + 1 & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k + 1 & \text{if } k \text{ is even} \end{cases}$$

edges contains a copy of F_k. The following is an analogue to this result. Our proof utilizes a technique developed in [16].

Theorem 7 *For $k \geq 1$ and $n \geq \frac{9}{2}k^2 + \frac{7}{2}k - \frac{1}{2}$,*

$$\sigma(F_k, n) = k(2n - k - 1) + 2. \tag{1}$$

As F_1 is isomorphic to K_3, (1) is established for $k = 1$ by Theorem 1. Equation (1) was established for $k = 2$ by Lai in [9]. Our proof of Theorem 7 holds for all $k \geq 1$.

PROOF: To see that $\sigma(F_k, n) \geq k(2n - k - 1) + 2$, consider the graph $G = K_1 + G'$, where G' is any graph on $n-1$ vertices where no realization of the degree sequence given by G' contains k disjoint edges. We may choose G' to be the graph $K_{k-1} + \overline{K}_{n-k}$ as in Theorem 6. Thus G is the graph $K_k + \overline{K}_{n-k}$. The graph G is the unique realization of the degree sequence $\pi = ((n-1)^k, (k)^{n-k})$ and has degree sum equal to $k(n-1) + (n-k)k = k(2n - k - 1)$. To see that G contains no copy of F_k first notice that any $k + 1$ vertices of F_k must contain at least one edge. Now if G were to contain a copy of F_k it must contain at least $k + 1$ of its vertices from the subgraph $\overline{K}_{n-k}$ of G, however this subgraph does not contain an edge. This establishes the lower bound.

We now establish the upper bound through a sequence of lemmas.

The following establishes that there are sufficiently many vertices of sufficiently large degree in any graph with the degree sum at least that given by (1).

Lemma 1 *Let $S = (d_1, \ldots, d_n)$ be a non-increasing graphic degree sequence with with degree sum at least $k(2n - k - 1) + 2$ and $n > k^2 + k - 2$, then $d_1 \geq 2k$ and $d_{2k+1} \geq 2$.*

PROOF: To see that $d_1 \geq 2k$, suppose otherwise, so S contains no term larger than $2k - 1$. Then the degree sum of S is at most $n(2k - 1)$, a contradiction.

Suppose now that $d_{2k+1} \leq 1$. Then, by Theorem 2,

$$\begin{aligned}
\sum_{i=1}^{n} d_i &= \sum_{i=1}^{2k} d_i + \sum_{i=2k+1}^{n} d_i \\
&\leq (2k)(2k-1) + \sum_{i=2k+1}^{n} min\{2k, d_i\} + \sum_{i=2k+1}^{n} d_i \\
&= 4k^2 - 2k + 2 \sum_{i=2k+1}^{n} 1 \\
&\leq 4k^2 - 2k + 2(n-2k) \\
&= 2n + 4k^2 - 6k.
\end{aligned}$$

This is a contradiction. □

Let $\pi = (d_1, \ldots, d_n)$ be a non-increasing, n-term graphic sequence with degree sum at least $k(2n-k-1)+2$. We will now recursively define a sequence $\pi_1, \ldots, \pi_{2k+1}$ of degree sequences. We begin by constructing the sequence π_1', on $n-1$ terms, by deleting d_1 from π and subtracting 1 from the first d_1 remaining terms. That is,

$$\pi_1' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n).$$

We then obtain the sequence π_1 from π_1' by subtracting one from each of the first $2k$ terms in π_1' and arranging the first $2k$ terms in non-increasing order and then arranging the last $n-2k-1$ terms in non-increasing order. (As Lemma 1 guarantees that $d_{2k+1} \geq 2$ we are assured that this step is feasible.) Let

$$\pi_1 = (d_2^{(1)}, d_3^{(1)}, \ldots, d_n^{(1)}).$$

For $2 \leq i \leq 2k+1$, we obtain the sequence

$$\pi_i = (d_{i+1}^{(i)}, \ldots, d_n^{(i)})$$

of length $n-i$ from

$$\pi_{i-1} = (d_i^{(i-1)}, \ldots, d_n^{(i-1)})$$

by deleting $d_i^{(i-1)}$ from π_{i-1}, subtracting one from the largest $d_i^{(i-1)}$ non-negative remaining terms and arranging the first $2k+1-i$ terms in non-increasing order and then arranging the last $n-2k-1$ terms in non-increasing order.

Lemma 2 *If π_{2k+1} is graphic then π is potentially F_k-graphic.*

PROOF: Clearly, if π_{2k+1} is graphic, then π_1 is graphic. As π is graphic, the Havel-Hakimi algorithm [7, 8] implies that π_1' is graphic. If we can show that there is a realization of π_1' that has a matching on those vertices of degree $d_2 - 1, \ldots, d_{2k+1} - 1$, then clearly π is potentially F_k-graphic. Let G_1' be a realization of π_1' and let G_1 be a realization of π_1 such that $V_1 = V(G_1) = V(G_1') = \{v_2, \ldots, v_n\}$ with $d_{G_1}(v_i) = d_{G_1'}(v_i) - \delta_i$ where δ_i=1 for $2 \leq i \leq 2k+1$ and $\delta_i = 0$ otherwise.

Let H be a copy of K_{n-1} on V_1, and consider the function $W : E(H) \rightarrow \{-1, 0, 1\}$ defined by

$$W(v_i v_j) = \begin{cases} -1 & v_i v_j \in E(G_1) \setminus E(G_1') \\ 1 & v_i v_j \in E(G_1') \setminus E(G_1) \\ 0 & \text{otherwise.} \end{cases}$$

The function W induces a weighting $w : V_1 \rightarrow \mathbb{Z}$, where the weight of a vertex v is the sum of the weights of the edges incident to v in H. If we let $X = \{v_2, \ldots, v_{2k+1}\}$, then one can see that $w(v) = 1$ if v is a member of X and $w(v) = 0$ otherwise.

It will be shown that there exists a collection of trails $T_1, \ldots, T_k$ in H that satisfy the following four properties.

(1) $T_1, \ldots, T_k$ are edge disjoint.

(2) The end-vertices of $T_1, \ldots, T_k$ are distinct vertices in X, and hence cover X.

(3) The first edge, and last edge, in each trail has weight 1 under W.

(4) If $T_j = e_1 e_2 \ldots e_p$ then $W(e_{i+1}) = -W(e_i)$ for $1 \leq i \leq p-1$.

If v lies on T_i, let w_i denote the vertex weighting induced by $W|_{E(T_i)}$. Note that if v is an end-vertex of T_i then $w_i(v) = 1$ and if v is an internal vertex of T_i, then $w_i(v) = 0$.

We begin by showing that T_1 exists. Select v_2 as an end-vertex of T_1. Note that as v_2 is in X, $w(v_2) = 1$ so there is some edge e in H incident to v_2 with $W(e) = 1$. If there is such an edge between v_2 and some other vertex x in X, let T_1 consist of the edge $v_2 x$. Otherwise, there is an edge $v_2 y$ such that $W(v_2 y) = 1$ and y is not in X. Include the edge $v_2 y$ in T_1. As $w(y) = 0$, there is some edge incident to y having weight -1, which is then included in T_1. Continue this process, and construct an alternating $+1/-1$ trail in H. If at any point there exists an edge e with $W(e) = 1$ satisfying (1) - (4) above then include e in T_1. As this process clearly terminates, we wish to show that it must terminate with such a choice. Assume not, so that T_1 is an alternating $+1/-1$ trail that violates (2) or (3) above. We show that such a trail can be extended. Assume first that (2) is violated. If the end-vertex of this trail is v_2, then as $w(v_2) = 1$, our choice for the initial edge of T_1 implies that we can clearly continue the trail regardless of the weight of the final edge. If the end-vertex of the trail is some v in $V \setminus X$ then we note that $w(v) = 0$, and each time, if any, that v appears previously in the trail, it is adjacent to one edge of weight $+1$ and one edge of weight -1. Thus, if the last edge e on the trail has weight $W(e)$ (which is necessarily $+1$ or -1), there is some edge not already in the trail which is adjacent to v and has weight $-W(e)$ and the trail can be extended. If we assume that (2) is satisfied, but (3) is violated then the last vertex on the trail is some x in $X \setminus \{v_2\}$ but the last edge e added to the trail has weight $W(e) = -1$. However, $w(x) = 1$, which implies that we can extend the trail. Hence, T_1 exists.

Assume that trails $T_1, \ldots, T_j$ exist satisfying (1) - (4) and without loss of generality, let the end vertices of T_i be v_{2i}, v_{2i+1}. Note that if v is in $\{v_2, \ldots, v_{2j+1}\}$ then

$$\sum_{i=1}^{j} w_i(v) = 1$$

and otherwise,

$$\sum_{i=1}^{j} w_i(v) = 0.$$

To show trail T_{j+1} exists, begin with v_{2j+2} as an end-vertex. As $w(v_{2j+2}) = 1$ and

$$\sum_{i=1}^{j} w_i(v_{2j+2}) = 0,$$

there is some edge e in H adjacent to v_{2j+2} with $W(e) = 1$ that does not lie in any of $T_1, \ldots, T_j$. If there is such an edge between v_{2j+2} and some other vertex x in $X \setminus \{v_2, \ldots, v_{2j+2}\}$, let T_{j+1} consist of the edge $v_{2j+2}x$. Otherwise, we will proceed in a manner similar to the construction of T_1, described above. That is, it can be shown that T_{j+1} is an alternating $+1/-1$ trail, which is edge disjoint from $T_1 \ldots, T_j$. If at any point T_{j+1} can be extended by an edge e of weight $W(e) = 1$ to a vertex in $X \setminus \{v_2, \ldots, v_{2j+2}\}$ the edge e will be added to T_{j+1}. Otherwise, we will assume that T_{j+1} is an alternating trail that violates either (2) or (3). Then, as above, we can use the induced weights from the previous trails to extend T_{j+1}. As the process of extending T_{j+1} must terminate, we can see that T_{j+1} exists satisfying (1) - (4).

Thus there exists trails $T_1, \ldots, T_k$ satisfying (1) - (4), and assume without loss of generality that the end-vertices of T_i are v_{2i} and v_{2i+1} for all $1 \le i \le k$. Note that if an edge in H has weight 1 then it is in G_1' and an edge in H having weight -1 is not in G_1'. For each trail T_i, if $v_{2i}v_{2i+1}$ is an edge in G_1' do nothing. If $v_{2i}v_{2i+1}$ is not an edge in G_1' add this edge and all edges of weight -1 on T_i to G_1' and remove all edges of weight 1 on T_i from G_1'. In the event that $W(v_{2i}v_{2i+1}) = -1$ and $v_{2i}v_{2i+1}$ lies in some T_j, we examine $e_j = v_{2j}v_{2j+1}$. If e_j is in G_1', then we will proceed as above to add $v_{2i}v_{2i+1}$ to G_1'. If e_j is not in G_1', we will add e_j to G_1' and "switch" the edges in T_j. This will also serve to add the edge $v_{2i}v_{2i+1}$ to G_1'. Note that it is not possible for $v_{2i}v_{2i+1}$ to lie in some T_j with $j \ne i$ if $W(v_{2i}v_{2i+1}) = +1$. Thus we can create a realization of π_1' that contains the matching $v_2v_3, \ldots, v_{2k}v_{2k+1}$, implying that π is potentially F_k-graphic.□

Lemma 3 *If $n \ge 4k+2$, and $d_{4k+2} \ge 2k-1$ then π is potentially F_k-graphic.*

PROOF: If $d_{2k+1} \ge 2k$ then π is potentially K_{2k+1}-graphic by Theorem 5, and thus obviously F_k-graphic.

Otherwise $d_{2k+1} \le 2k-1$, which together with the hypothesis implies that $d_{2k+1} = d_{2k+2} = \ldots = d_{4k+2} = 2k-1$. Thus, for $i = 0, 1, \ldots, 2k+1$ the values of $d_{2k+2}^{(i)}, \ldots, d_{4k+2}^{(i)}$ differ by at most 1. Hence π_{2k+1} satisfies, for some $m \ge 1$,

$$2k-1 \ge m = d_{2k+2}^{(2k+1)} \ge \ldots \ge d_{4k+2}^{(2k+1)} \ge m-1.$$

If $m = 1$, π_{2k+1} must be graphic as the degree sum of π_{2k+1} is even. If $m \ge 2$, then

$$\frac{1}{m-1}\left\lfloor\frac{(m+(m-1)+1)^2}{4}\right\rfloor \le m+2 \le 2k+1.$$

By Theorem 4, π_{2k+1} is graphic, and hence, by Lemma 2, π is F_k-graphic.□

Lemma 4 *Let π be an n-term graphic degree sequence with $n \geq \frac{9}{2}k^2 + \frac{7}{2}k - \frac{1}{2}$ and degree sum at least $k(2n-k-1)+2$. If $d_{4k+2} \leq 2k-2$ then π is potentially F_k-graphic.*

PROOF: First, we claim that $d_1 \geq 4k$. If not, then the degree sum of π is at most $(4k-1)(4k+1)+(n-4k-1)(2k-2)$, which is less than $k(2n-k-1)+2$ for the given values of n.

If $d_1 = n-1$ then the degree sum of π_1' is at least $\sigma(kK_2, n-1)$. Therefore, there exists a realization of π_1' that contains a copy of kK_2 and thus a realization of π that contains a copy of F_k.

Now suppose there exists an r such that $2k+1 \leq r \leq d_1+1$ such that $d_{r+1} < d_r$. As the degree sum of (π_1') is at least $\sigma(kK_2, n-1)$ there exists a graph realizing π_1' that contains a copy of kK_2. Furthermore, by Theorem 3 there exists a realization of π_1' with kK_2 on those vertices having degree $d_2-1, \ldots d_{2k+1}-1$. This implies that π is potentially F_k-graphic.

Otherwise, $n-2 \geq d_1 \geq d_2 \geq \ldots \geq d_{2k+1} = d_{2k+2} = \ldots d_{4k+2} = \ldots = d_{d_1+2}$.

We may conclude that there exists an m such that

$$2k-2 \geq m = d_{2k+2}^{(2k+1)} \geq \ldots \geq d_{4k+2}^{(2k+1)} \geq m-1.$$

We may then complete the proof as in the previous lemma.□

Together, Lemma 3 and Lemma 4 imply that $\sigma(F_k, n) \leq k(2n-k-1)+2$, completing the proof of Theorem 7. □

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References

[1] Bollobás, B., *Extremal Graph Theory*, Academic Press Inc. (1978).

[2] Erdős, P., Füredi, Z., Gould, R.J., Gunderson, D.S., Extremal Graphs for Intersecting Triangles, *J. Combin. Th., Ser. B* **64**, (1995), 89-100.

[3] Erdős, P. & Gallai, T., Graphs with prescribed degrees (in Hungarian) *Matemoutiki Lapor* **11** (1960), 264-274.

[4] Erdős, P., Jacobson, M.S., Lehel, J., Graphs Realizing the Same Degree Sequence and their Respective Clique Numbers, *Graph Theory, Combinatorics and Applications*, Vol. I, 1991, ed. Alavi, Chartrand, Oellerman and Schwenk, 439-449.

[5] Ferrara, M., Gould, R., Schmitt, J., Potentially K_s^t-graphic degree sequences, submitted.

[6] Gould, R.J., Jacobson, M.S., Lehel, J., Potentially G-graphic degree sequences, *Combinatorics, Graph Theory, and Algorithms* (eds. Alavi, Lick and Schwenk), Vol. I, New York: Wiley & Sons, Inc., 1999, 387-400.

[7] Hakimi, S.L., On the realizability of a set of integers as degrees of vertices of a graph, *J. SIAM Appl. Math*, **10** (1962), 496-506.

[8] Havel, V., A remark on the existence of finite graphs (Czech.), *Časopis Pěst. Mat.* **80** (1955), 477-480.

[9] Lai, C., An extremal problem on potentially $K_m - C_4$-graphic sequences, submitted.

[10] Li, J., Song, Z., An extremal problem on the potentially P_k-graphic sequences, *The International Symposium on Combinatorics and Applications*, June 28-30, 1996 (W.Y.C. Chen et. al., eds.) Tanjin, Nankai University 1996, 269-276.

[11] Li, J., Song, Z., The smallest degree sum that yields potentially P_k-graphical sequences, *J. Graph Theory* **29** (1998), no.2, 63-72.

[12] Li, J., Song, Z., Luo, R., The Erdős-Jacobson-Lehel conjecture on potentially P_k-graphic sequences is true, *Science in China, Ser. A*, **41** 1998, (5):510-520.

[13] Li, J., Yin, J., The smallest degree sum that yields potentially $K_{r,r}$-graphic sequences, *Science in China, Ser. A*, **45** (June 2002),(6):694-705.

[14] Li, J., Yin, J., An extremal problem on potentially $K_{r,s}$-graphic sequences, *Discrete Math.*, **260** (2003), 295-305.

[15] Li, J., Yin, J., Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size, submitted.

[16] Yin, J., Chen, G., On Potentially $K_{r_1,r_2,\ldots,r_m}$-graphic Sequences, preprint.

[17] Rao, A.R., The clique number of a graph with a given degree sequence, Proc. Symposium on Graph Theory (A.R. Rao ed.), MacMillan and Co. India Ltd., *Indian Statistical Institute Lecture Notes Series* 4 (1979), 251-267.

[18] Rao, S.B., A survey of the theory of potentially P-graphic and forcibly P-graphic degree sequences, *Lecture Notes in Math.*, No. **855**, Springer Verlag, 1981, 417-440.