USE OF COMPLEMENTARY PROPERTY OF BLOCK DESIGNS IN PBIB DESIGNS

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Abstract: Some results on combinatorial aspects of block designs using the complementary property have been obtained. The results pertain to non-existence of partially balanced incomplete block (PBIB) designs and identification of new 2-associate and 3-associate PBIB designs. A method of construction of extended group divisible (EGD) designs with three factors using self-complementary rectangular designs has also been given. Some rectangular designs have also been obtained using self-complementary balanced incomplete block designs. Catalogues of EGD designs and rectangular designs obtainable from these methods of construction, with number of replications $r \leq 10$ and block size $k \leq 10$ have been prepared.

Key Words: Complementary design, block design, BIB design, PBIB design, EGD design.

1. Introduction

Consider a block design represented as D (v, b, **r**, **k**, **N**), where v treatments are arranged in b blocks such that $\mathbf{r} = (r_1, \ldots, r_v)'$ is a column vector of treatment replications and $\mathbf{k} = (k_1, \ldots, k_b)'$ is a column vector of block sizes, $\mathbf{N} = ((n_{ij}))$ is a $v \times b$ treatments versus blocks incidence matrix. For $i = 1, \ldots, v; j = 1, \ldots, b$, non-negative integer n_{ij} denotes the number of times i^{th} treatment appears in j^{th} block. Also $r_i = \sum_{j=1}^{b} n_{ij}$ and $k_j = \sum_{i=1}^{v} n_{ij}$. A block design is said to be binary if $n_{ij} = 0$ or 1, for all i, j. If all r_i's are equal then the design is said to be equireplicated. The design is said to be proper if all k_j's are equal. For details on block designs, one may refer to Dey (1986), Nigam, Puri and Gupta (1988) and Raghavarao (1971).

Complementary design of a binary block design D (v, b, **r**, **k**, **N**) is a binary block design $\overline{\mathrm{D}}$ (v, b, $b\mathbf{1}_v - \mathbf{r}$, $v\mathbf{1}_b - \mathbf{k}$, $\overline{\mathbf{N}} = \mathbf{1}_v\mathbf{1}_b' - \mathbf{N}$) that is obtained by taking those treatments in the j^{th} block which don't appear in the j^{th} block of the original design D for all $j = 1, \ldots, b$, where $\mathbf{1}_t$ is a $t \times 1$ column vector of ones.

The complementary design of a binary block design with v treatments in which $s(\geq 1)$ treatments appear exactly once in all the blocks will have $v-s$ treatments. If the binary block design D (v, b, **r**, **k**, **N**) has c complete blocks, i.e., every treatment appears exactly once in these c blocks, then the number of effective blocks in the complementary design will be b - c. Using the complementary property of binary block designs, we have the following lemma:

Lemma 1.1: Barring trivial exceptions, the complementary design of any equireplicate, pairwise balanced binary block design D (v, b, r, **k**, **N**; λ) is also an equireplicate, pairwise balanced binary block design $\overline{\text{D}}$ (v, b, $b-r$, $v\mathbf{1}_b-\mathbf{k}$, $\mathbf{1}_v\mathbf{1}_b'-\mathbf{N}$; $b-2r+\lambda$), where $\lambda=\sum_{j=1}^{b} n_{ij}n_{i'j}$ for all $i\neq i'(=1,2,\ldots,v)$ in $\mathbf{N}=((n_{ij}))$. Further, if the design D is binary equireplicate and proper, then its complementary is also a binary, equireplicate, proper block design denoted by $\overline{\text{D}}$ (v, b, $b-r$, $v-k$, $\mathbf{1}_v\mathbf{1}_b'-\mathbf{N}$). If the original design D is variance balanced (partially balanced), then $\overline{\text{D}}$ is also variance balanced (partially balanced with same association scheme).

Exploiting Lemma 1.1, necessary conditions for the non-existence of partially balanced incomplete block (PBIB) designs have been obtained and new group divisible and nested group divisible designs have been identified in Section 2. In Section 3, we give a method of construction of extended group divisible (EGD) designs with three factors using the self-complementary rectangular designs and their particular cases. Rectangular designs have also been obtained using self-complementary balanced incomplete block (BIB) designs. Catalogues of EGD designs and rectangular designs obtainable from this method of construction with number of replications $r\leq 10$ and block size $k\leq 10$ have been given.

2. Use of Complementary Property in PBIB Designs

In this section we exploit the complementary property of binary block designs in obtaining necessary conditions for non-existence of PBIB designs. The results have also been used for identifying non-existent group divisible designs. Some new group divisible and nested group divisible designs have also been identified using the complementary property. In some cases, the new designs require less number of experimental units than those catalogued earlier with same number of treatments, blocks and same association scheme. In some cases, however, new designs identified require more experimental units than their counterparts, yet they are important and helpful in completing the catalogue of PBIB designs with $r,k\leq 10$.

Consider a PBIB design D based on m-associate class association scheme with parameters $v, b, r, k, \lambda_i, n_i$; $i = 1, \ldots, m$ where parameters have their usual meanings, see e.g., Dey (1986) and Raghavarao (1971). Since an association scheme is invariant with respect to complementation of the design, Using Lemma 1.1, we can easily see that the complementary design of D is also an m-associate class PBIB design with same association scheme as that of D and with parameters $v^* = v, b^* = b, r^* = b - r$, $k^* = v - k, \lambda_i^* = b - 2r + \lambda_i, n_i$; $i = 1, \ldots, m$. Using this, some new group divisible and nested group divisible designs have been obtained.

Group Divisible Designs

A group divisible design (2-associate PBIB design based on a group divisible association scheme of mn treatments) with parameters $v = 16, b = 16, r = 9, k = 9$, $\lambda_1 = 2, \lambda_2 = 5$, $m = 8, n = 2$ is new, as it is not listed in Clatworthy (1973), Sinha (1991) and Ghosh and Divecha (1995). This design can be obtained as a complementary solution of a regular group divisible design with parameters $v = 16$, $b = 16$, $r = 7, k = 7, \lambda_1 = 0, \lambda_2 = 3$, $m = 8, n = 2$ obtained by Dey (1977) and listed at serial number 28 of Sinha (1991).

Nested Group Divisible Designs

Duan and Kageyama (1993), and Miao, Kageyama and Duan (1996) gave catalogues of nested group divisible designs (3-associate PBIB designs based on a nested group divisible association scheme of pmn treatments). Duan and Kageyama (1993) presented the catalogue for $r, k \leq 15$, whereas Miao, Kageyama and Duan (1996) presented the catalogue with $r, k \leq 10$. Taking the complements of the nested group divisible designs given in the above catalogues several new nested group divisible designs are obtained. These designs with $r, k \leq 10$ are presented in Table 2.1 in the appendix. It may be noted that the designs at serial numbers 3, 4, 5, 6 and 7 in Table 2.1 require fewer replications and less number of experimental units than their counterparts (designs whose comlementaty solution yields the designs listed in Table 2.1). Furthermore, the nested group divisible designs of Duan and Kageyama (1993) at serial numbers 42, 57, 61 and 67 can be obtained as complementary design of designs at serial numbers 31, 45, 53 and 58 respectively.

It has also been observed that two more designs with fewer replications than those given by Duan and Kageyama (1993) and $k > 10$ can be obtained. The parameters of these designs obtainable from DK64 and DK65 respectively are:

D1: $v = 27, b = 9, r = 4, k = 12, p = 3, m = 3, n = 3, \lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 1$.

D2: $v = 27, b = 18, r = 8, k = 12, p = 3, m = 3, n = 3, \lambda_1 = 8, \lambda_2 = 6, \lambda_3 = 2$.

It can easily be observed that the design **D2** can be obtained by taking 2-copies of design **D1**. This is due to the fact that the design DK65 can be obtained by taking 2-copies of DK64.

Non-existence of PBIB designs

We know that a necessary condition for the existence of a PBIB design based on m-class association scheme with parameters $v, b, r, k, \lambda_i, n_i; i = 1,\ldots,m$ is that the parameters $\lambda_i, i = 1,\ldots,m$, are non-negative integers. Using the fact that the complementary design of a m-associate PBIB design is also a m-associate class PBIB design, $b - 2r + \lambda_i, i = 1,\ldots,m$ should also be non-negative integers. We, therefore, have the following result:

Result 2.1: A necessary condition for the existence of an m-associate class PBIB design with parameters $v, b, r, k, \lambda_i, n_i, i = 1,\ldots,m$, is that $b - 2r + \lambda_i, i = 1,\ldots,m$ are non-negative integers.

Example 2.1: The parameters $v = 4, b = 12, r = 9, k = 3$, $\lambda_1 = 8, \lambda_2 = 5$, $m = 2$, $n = 2$ satisfy the parametric relationships of a regular group divisible design. For these parameters $b - 2r + \lambda_1 = 2$ and $b - 2r + \lambda_2 = -1$. Therefore, the design with the above parameters is non-existent. Similarly, there does not exist a regular group divisible design with parameters $v = 6, b = 15, r = 10, k = 4$, $\lambda_1 = 9, \lambda_2 = 4, m = 2, n = 3$.

Disconnected group divisible designs have been used in construction of binary balanced block designs. These designs are also quite useful in factorial experiments. A group divisible design with parameters $v, b \geq mn, r, k, \lambda_1, \lambda_2 = 0, m, n$ can always be obtained by writing a BIB design with parameters $v^* = n, b^* = b/m, r^*, k^*, \lambda_1$ for each row of the association scheme and then taking the union of all blocks. Using this fact, we have the Result 2.2 on the existence of group divisible designs.

Result 2.2: A necessary condition for the existence of a group divisible design with $v = mn$ (m groups of n treatments each) and $\lambda_2 = 0$ is that b should be multiple of m.

Example 2.2: A group divisible design with parameters $v = 12, b = 15, r = 5, k = 4, \lambda_1 = 3$, $\lambda_2 = 0, m = 2, n = 6$ is non-existent, as $b = 15$ is not a multiple of $m (= 2)$. 105 such designs with $r, k \leq 10$ have been identified and a catalogue of such designs is available with the first author.

We can also say that if a group divisible design with parameters $v, b, r, k, \lambda_1, \lambda_2, m, n$ is non-existent, then its complementary design is also non-existent. Here using Example 2.2 we can say the following:

Example 2.3: A group divisible design with parameters $v = 12, b = 15$, $r = 10, k = 8$, $\lambda_1 = 8, \lambda_2 = 5, m = 2, n = 6$ is non-existent.

The parameters of complementary designs of all the 105 non-existent disconnected group divisible designs as per Result 2.2 have been obtained. Furthermore, it can be said that these complementary designs are also non-existent. It was observed that for all these designs r and/or $k \geq 10$ except the design given in Example 2.3.

3. PBIB Designs Using Self-Complementary Designs

In this section, we give a method of construction of PBIB designs using the incidence matrices of the original binary, equireplicate, proper block designs and their complementary designs. The designs obtained in this section are essentially EGD designs. An EGD design always possesses the property of orthogonal factorial structure with balance, see e.g., Gupta and Mukerjee (1989). Therefore, these designs are quite useful for 2 and 3-factor factorial experiments.

A rectangular design is a 3-associate PBIB design based on a rectangular association scheme of m rows and n columns, introduced by Vartak (1955), see also Raghavarao (1971).

Method 3.1: Suppose that there exists a self-complementary rectangular design D with parameters $v = mn = 2k, b = 2r, r, k, \lambda_1, \lambda_2, \lambda_3, m, n$. Let **N** be the incidence matrix of D and $\overline{\mathbf{N}}$ be the incidence matrix of the complementary design of D. Then the design D* with incidence matrix

$$\mathbf{N}^* = \begin{bmatrix} \mathbf{N} \\ \overline{\mathbf{N}} \end{bmatrix}$$

is a self-complementary EGD design for 3-factors, introduced by Hinkelmann and Kempthorne (1963) and Hinkelmann (1964). The parameters of design D* are $v^* = 2v$ *or* $2mn$, $b^* = b$, $r^* = r$, $k^* = 2k = v$, $\lambda_1^* = \lambda_1, \lambda_2^* = \lambda_2$, $\lambda_3^* = \lambda_3$, $\lambda_4^* = 0$, $\lambda_5^* = r - \lambda_1$, $\lambda_6^* = r - \lambda_2, \lambda_7^* = r - \lambda_3, p^* = 2, m^* = m, n^* = n$.

The EGD association scheme with three factors may be simplified as follows: Let $v = pmn$ be the number of treatment combinations of a 3-factor experiment with levels as p, m and n, respectively. Let the factors be represented by F_1, F_2, F_3, respectively. Write the pmn treatment combinations in the lexicographic order given by $\mathbf{a}_1 \times \mathbf{a}_2 \times \mathbf{a}_3$, where $\times$ denotes the symbolic direct product and $\mathbf{a}_1 = (0,1,\ldots,p-1)'$; $\mathbf{a}_2 = (0,1,\ldots,m-1)'$; $\mathbf{a}_3 = (0,1,\ldots,n-1)'$. Here, the elements of $\mathbf{a}_i's$ are the levels of i^{th} factor $i = 1,2,3$. Arrange the pmn treatment combinations in p sets of mn treatment combinations each such that the treatment combinations within each set have same level of factor F_1. Divide the mn treatment combinations within sets into an $m \times n$ array such that the levels of factor F_2 are fixed in a row. If we denote the seven associate classes 001, 010, 011, 100, 101, 110, 111 of the EGD association scheme as 1,2,3,4,5,6 and 7 respectively and treatment combinations in the lexicographic order as 1,2, ..., v, then the association scheme is defined as: (i) any two treatments from the same set and the same row are 1st (001) associates; (ii) any two treatments from the same set and the same column are 2nd (010) associates; (iii) any two treatments from the same set and different row and column are 3rd (011) associates; (iv) any two treatments from different sets and the same position are 4th (100) associates; (v) any two treatments from different sets, the same row number and not the same column number are 5th (101) associates; (vi) any two treatments from different sets, same column number and not the same row number are 6th (110) associates and (vii) any two treatments from different sets, not the same row number and not the same column number are 7th (111) associates.

Remark 3.1: A rectangular design with $\lambda_2 = \lambda_3$ becomes a group divisible design. Therefore, if the design D in Method 3.1 is a self-complementary group divisible design, then we get an EGD design with $\lambda_2^* = \lambda_3^*$ and $\lambda_6^* = \lambda_7^*$.

Remark 3.2: It can easily be seen that for $m = n$ and $\lambda_1 = \lambda_2$, a rectangular design is same as a PBIB design based on an L_2-association scheme. Hence, if the design D in Method 3.1 is a self-complementary PBIB design based on L_2-association scheme, then we get an EGD design with $\lambda_1^* = \lambda_2^*$ and $\lambda_5^* = \lambda_6^*$.

Remark 3.3: If we take the design D in Method 3.1 as a self-complementary BIB design D with parameters $v = 2k, b, r, k, \lambda$, then we get a self-complementary rectangular design. The parameters of the design D* are $v^* = 2v, b^* = b, r^* = r, \ k^* = v, \ \lambda_1 = \lambda, \lambda_2 = 0, \lambda_3 = r - \lambda$, $m = 2, n = v$.

A catalogue of rectangular designs obtainable through this method using self-complementary BIB design with $r, k \le 10$ has been prepared and is given in Table 3.1.

Remark 3.4: It is interesting to note here that the parameters v, b, r, k of the rectangular designs obtained through the above method are same as that of a singular group divisible design obtained by replacing each treatment of the corresponding BIB design with two treatments. The difference, however, is that for a singular group divisible design the roles of m and n get interchanged. Further, the eigenvalues of the information matrix for reduced normal equations for the rectangular designs obtained through Method 3.1, Remark 3.3 and corresponding singular group divisible designs are the same including multiplicities. As a consequence the average variance of all possible pairwise treatment comparisons through the designs are also same.

Using the self-complementary group divisible designs, PBIB designs based on an L_2-association scheme and the self-complementary rectangular designs given in Table 3.1, a catalogue of EGD designs with $r, k \leq 10$ has been prepared and is given in Table 3.2.

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Table 2.1: Nested Group Divisible Designs Obtained as Complementary of Designs given in Duan and Kageyama (1993) and Miao, Kageyama and Duan (1996)

SNo.	v	b	r	k	p	m	n	λ_1	λ_2	λ_3	Source
1	12	12	8	8	3	2	2	8	4	5	MKD18
2	16	16	10	10	4	2	2	4	7	6	DK22
3	18	9	4	8	3	3	2	4	3	1	DK38
4	18	18	8	8	3	3	2	8	6	2	DK39
5	18	18	8	8	3	2	3	7	6	2	DK40
6	24	12	5	10	3	4	2	5	4	1	DK59
7	24	24	10	10	3	4	2	10	8	2	DK60

DK#: denotes the nested group divisible design listed at serial number # in Duan and Kageyama (1993).
MKD#: denotes the nested group divisible design listed at serial number # in Miao, Kageyama and Duan (1996).

Table 3.1: Rectangular designs obtainable from Method 3.1.

SNo.	v	b	r	k	λ_1	λ_2	λ_3	m	n	Source
1	8	6	3	4	1	0	2	2	4	BIB(4,6,3,2,1)
2	12	10	5	6	2	0	3	2	6	BIB(6,10,5,3,2)
3	16	14	7	8	3	0	4	2	8	BIB(8,14,7,4,3)
4	20	18	9	10	4	0	5	2	10	BIB(10,18,9,5,4)

Table 3.2: Extended Group Divisible designs for three factors for $r, k \leq 10$ obtainable from Method 3.1.

SNo.	v	b	r	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	p	m	n	Source
1	8	4	2	4	0	1	1	0	2	1	1	2	2	2	SR1
2	8	8	4	4	0	2	2	0	4	2	2	2	2	2	SR2
3	8	8	4	4	2	1	1	0	2	3	3	2	2	2	R1
4	8	10	5	4	3	1	1	0	2	4	4	2	2	2	R2
5	8	10	5	4	1	2	2	0	4	3	3	2	2	2	R3
6	8	12	6	4	4	1	1	0	2	5	5	2	2	2	R4
7	8	12	6	4	0	3	3	0	6	3	3	2	2	2	SR3
8	8	14	7	4	5	1	1	0	2	6	6	2	2	2	R5
9	8	14	7	4	3	2	2	0	4	5	5	2	2	2	R6
10	8	14	7	4	1	3	3	0	6	4	4	2	2	2	R7
11	8	16	8	4	6	1	1	0	2	7	7	2	2	2	R8

SNo.	v	b	r	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	p	m	n	Source
12	8	16	8	4	4	2	2	0	4	6	6	2	2	2	R9
13	8	16	8	4	2	3	3	0	6	5	5	2	2	2	R10
14	8	16	8	4	0	4	4	0	8	4	4	2	2	2	SR4
15	8	18	9	4	7	1	1	0	2	8	8	2	2	2	R11
16	8	18	9	4	5	2	2	0	4	7	7	2	2	2	R12
17	8	18	9	4	1	4	4	0	8	5	5	2	2	2	R13
18	8	20	10	4	8	1	1	0	2	9	9	2	2	2	R14
19	8	20	10	4	6	2	2	0	4	8	8	2	2	2	R15
20	8	20	10	4	4	3	3	0	6	7	7	2	2	2	R16
21	8	20	10	4	2	4	4	0	8	6	6	2	2	2	R17
22	8	20	10	4	0	5	5	0	10	5	5	2	2	2	SR5
23	12	4	2	6	0	1	1	0	2	1	1	2	3	2	SR18
24	12	6	3	6	2	1	1	0	1	2	2	2	3	2	R42
25	12	8	4	6	0	2	2	0	4	2	2	2	3	2	SR19
26	12	12	6	6	0	3	3	0	6	3	3	2	3	2	SR20
27	12	12	6	6	3	2	2	0	3	4	4	2	2	3	R43
28	12	12	6	6	4	2	2	0	2	4	4	2	3	2	R44
29	12	14	7	6	4	2	2	0	3	5	5	2	2	3	R45
30	12	14	7	6	2	3	3	0	5	4	4	2	3	2	R46
31	12	16	8	6	0	4	4	0	8	4	4	2	3	2	SR21
32	12	16	8	6	5	2	2	0	3	6	6	2	2	3	R47
33	12	16	8	6	4	3	3	0	4	5	5	2	3	2	R48
34	12	18	9	6	6	2	2	0	3	7	7	2	2	3	R49
35	12	18	9	6	6	3	3	0	3	6	6	2	3	2	R50
36	12	18	9	6	2	4	4	0	7	5	5	2	3	2	R51
37	12	18	9	6	3	4	4	0	6	5	5	2	2	3	R52
38	12	20	10	6	7	2	2	0	3	8	8	2	2	3	R53
39	12	20	10	6	0	5	5	0	10	5	5	2	3	2	SR22
40	16	6	3	8	3	1	1	0	0	2	2	2	4	2	S6
41	16	6	3	8	1	0	2	0	2	3	1	2	2	4	D1:Table 3.1
42	16	8	4	8	0	2	2	0	4	2	2	2	4	2	SR36
43	16	10	5	8	3	2	2	0	2	3	3	2	4	2	R97
44	16	12	6	8	0	3	3	0	6	3	3	2	4	2	SR37
45	16	12	6	8	2	3	3	0	4	3	3	2	2	4	SR38
46	16	12	6	8	6	2	2	0	0	4	4	2	4	2	S7
47	16	16	8	8	0	4	4	0	8	4	4	2	4	2	SR39
48	16	16	8	8	4	3	3	0	4	5	5	2	2	4	R98
49	16	16	8	8	6	3	3	0	2	5	5	2	4	2	R99

SNo.	v	b	r	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	p	m	n	Source
50	16	18	9	8	5	3	3	0	4	6	6	2	2	4	R100
51	16	18	9	8	3	4	4	0	6	5	5	2	4	2	R101
52	16	18	9	8	9	3	3	0	0	6	6	2	4	2	S8
53	16	20	10	8	6	3	3	0	4	7	7	2	2	4	R102
54	16	20	10	8	6	4	4	0	4	6	6	2	4	2	R103
55	16	20	10	8	0	5	5	0	10	5	5	2	4	2	SR40
56	20	8	4	10	0	2	2	0	4	2	2	2	5	2	SR52
57	20	10	5	10	4	2	2	0	1	3	3	2	5	2	R139
58	20	12	6	10	0	3	3	0	6	3	3	2	5	2	SR53
59	20	14	7	10	4	3	3	0	3	4	4	2	5	2	R140
60	20	16	8	10	0	4	4	0	8	4	4	2	5	2	SR54
61	20	20	10	10	0	5	5	0	10	5	5	2	5	2	SR55
62	20	20	10	10	5	4	4	0	5	6	6	2	2	5	R141
63	20	20	10	10	8	4	4	0	2	6	6	2	5	2	R142

X# denotes the design catalogued in Clatworthy (1973).