COLORING THE LINE

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ABSTRACT. The distance graph $G(S, D)$ has vertex set $V(G(S, D)) = S \subseteq \mathbb{R}^n$ and two vertices u and v are adjacent if and only if their distance $d(u, v)$ is an element of the distance set $D \subseteq \mathbb{R}_+$.

We determine the chromatic index, the choice index, the total chromatic number and the total choice number of all distance graphs $G(\mathbb{R}, D)$, $G(\mathbb{Q}, D)$ and $G(\mathbb{Z}, D)$ transferring a theorem of de Bruijn and Erdős on infinite graphs. Moreover, we prove that $|D| + 1$ is an upper bound for the chromatic number and the choice number of $G(S, D)$, $S \subseteq \mathbb{R}$.

1. INTRODUCTION

If S is a subset of the n-dimensional Euclidean space, $S \subseteq \mathbb{R}^n$, and D a set of positive real numbers, $D \subseteq \mathbb{R}_+$, then the *distance graph* $G(S, D)$ is defined to be the graph G with vertex set $V(G) = S$ and two vertices u and v are adjacent if and only if their distance $d(u, v)$ is an element of the so-called *distance set* D. The graphs $G(\mathbb{Z}^n, D)$ with $D \subseteq \mathbb{N}$ are called *integer distance graphs* and the graphs $G(\mathbb{Q}^n, D)$ with $D \subseteq \mathbb{Q}_+$ *rational distance graphs.*

A *(vertex) coloring* of a graph $G = (V(G), E(G))$ is an assignment of colors to the vertices of G such that adjacent vertices are colored differently. The minimum number of colors necessary to color the vertices of G is the *chromatic number* $\chi(G)$ of G.

If $L = \{L(v) : v \in V(G)\}$ is a set of lists of colors then an *L-list (vertex) coloring* of a graph G is a coloring of the vertices of G such that each vertex obtains a color from its own list and adjacent vertices are colored differently. A graph G is called *k-choosable* if such a coloring exists for each choice of lists $L(v)$ of cardinality k. The minimum k such that G is k-choosable is the *choice number* $ch(G)$ of G.

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Coloring problems on distance graphs are motivated by the famous Hadwiger-Nelson unit distance plane coloring problem which asks for the minimum number of colors necessary to color the points of the Euclidean plane (i. e., $V(G) = S = \mathbb{R}^2$) such that pairs of points of unit distance (i. e., $D = \{1\}$) are colored differently. In [9] Hadwiger gives a tiling of the plane in seven sets of congruent hexagons such that no set contains two points of distance 1. On the other hand, there exist 4-chromatic unit distance graphs in the plane (see [16], e. g.). Therefore, we have $4 \leq \chi\big(G(\mathbb{R}^2, \{1\})\big) \leq 7$. No substantial progress on the problem has been made till now [18].

The restriction of the problem to rational or integer points of the plane is solved. It is proved in [26] that $\chi\big(G(\mathbb{Q}^2,\{1\})\big) = 2$. Obviously, $\chi\big(G(\mathbb{Z}^2,\{1\})\big) = 2$ since the lattice graph $G(\mathbb{Z}^2, \{1\})$ is bipartite.

The *coloring number* $col(G)$ of a graph G is defined to be the minimum cardinal number α for which a well-ordering of the vertex set exists such that every vertex $v \in V(G)$ is adjacent to less than α vertices which are smaller than v [7].

If G is finite then $col(G) = \max_{H \subseteq G} \delta(H) + 1$ (see [20]) where $\delta(H)$ is the *minimum degree* of the graph H.

By definition of $\chi(G)$, $ch(G)$ and $col(G)$ it is obvious (see, e.g., [10]) that

(1) $\quad \chi(G) \leq ch(G) \leq col(G).$

An important statement on colorings of infinite graphs is the following result of de Bruijn and Erdős which gives a relationship between the chromatic number of an infinite graph and the chromatic numbers of its finite subgraphs.

Theorem (de Bruijn, Erdős [3]). *If $\chi(H) \leq k$ for all finite subgraphs H of an infinite graph G then $\chi(G) \leq k$.*

This theorem implies that in case of finite chromaticity of an infinite graph its chromatic number is attained by a finite subgraph.

Johnson [11] transferred the result of de Bruijn and Erdős to an analogous statement for the choice number of infinite graphs which can also be obtained by some minor modifications of the original proof.

Theorem (Johnson [11]). *If $ch(H) \leq k$ for all finite subgraphs H of an infinite graph G then $ch(G) \leq k$.*

On the other hand, the result cannot be transferred to an analogous statement for the coloring number of infinite graphs (see [7]).

An *edge coloring* of a graph G is an assignment of colors to the edges of G such that adjacent edges are colored differently. A *total coloring* is an assignment of colors to the vertices and edges such that neighbored elements – that are two adjacent vertices or two adjacent edges or a vertex and an incident edge – are colored differently, respectively. The minimum number of colors necessary to color the edges of G is the *chromatic index* $\chi'(G)$ and to color the vertices and edges the *total chromatic number* $\chi''(G)$.

Again, if the colors belong to specific lists assigned to the edges or to vertices and edges of G, respectively, and the cardinality of the lists is k then G is called *k-edge choosable* or *k-total choosable*, respectively, if such colorings exist for each choice of lists. The minimum k such that G is k-edge choosable is the *choice index* $ch'(G)$ of G, and the minimum k such that G is k-total choosable is the *total choice number* $ch''(G)$.

In this note we consider distance graphs $G(S, D)$ with vertex set $V(G(S, D)) = S \subseteq \mathbb{R}$ which we will call *1-dimensional distance graphs* or *line distance graphs*. Among others, we prove that $|D| + 1$ is an upper bound for the chromatic number, the choice number and the coloring number (for finite D in case of coloring number) of line distance graphs. Moreover, we determine the chromatic index, the choice index, the total chromatic number and the total choice number of the line distance graphs $G(S, D)$, $S = \mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Z}$.

2. Vertex colorings

Line distance graphs $G(S, D)$ were introduced by Eggleton, Erdős, and Skilton [5]. They proved that $\chi(G(\mathbb{Z}, D)) = \chi(G(\mathbb{R}, D))$ if D is a subset of the set $\mathbb{N}$ of positive integers.

There exist several papers in which the chromatic number of certain line integer distance graphs are determined (see, e.g., [4, 5, 6, 13, 14, 15, 22, 28]). For example, $\chi(G(\mathbb{Z}, \mathbb{P})) = 4$ is proved in [6] where $\mathbb{P}$ is the set of primes. If $D \subseteq \mathbb{N}$ also contains nonprimes it turns out that the determination of $\chi(G(\mathbb{Z}, D))$ is in general difficult if $|D| \geq 3$ and D contains elements of distinct parity.

If d is an arbitrary divisor of the elements $d_1, d_2, \dots$ of $D \subseteq \mathbb{N}$ then the integer distance graph $G(\mathbb{Z}, \{d_1, d_2, \dots\})$ is isomorphic to d copies of $G(\mathbb{Z}, \{\frac{d_1}{d}, \frac{d_2}{d}, \dots\})$. Therefore, we can restrict ourselves to distance sets whose elements have greatest common divisor 1, that is, $G(\mathbb{Z}, D)$ is connected.

If $D \subseteq \mathbb{N}$ is finite then $\chi(G(\mathbb{Z}, D)) \leq |D| + 1$ [24]. Obviously, $\chi(G(\mathbb{Z}, D)) = 2$ for 1-element distance sets D. If D contains only odd integers then $\chi(G(\mathbb{Z}, D)) = 2$ (color all vertices alternately with two colors). Therefore, it holds $\chi(G(\mathbb{Z}, D)) = 2$ for 2-element distance sets if D contains two odd elements and $\chi(G(\mathbb{Z}, D)) = 3$ if D consists of two coprime elements of distinct parity.

If $|D| = 3$ and the greatest common divisor of $D = \{x, y, z\}$ is 1 then $\chi(G(\mathbb{Z}, D)) = 4$ if and only if $D = \{1, 2, 3n\}$ or $D = \{x, y, x+y\}$ and $x \not\equiv y \pmod 3$. If x, y, z are odd then $\chi(G(\mathbb{Z}, D)) = 2$. For all other 3-element distance sets D, $\chi(G(\mathbb{Z}, D)) = 3$ [23, 28].

On the other hand, if $|D| = 3$ and $D \not\subseteq r \cdot \mathbb{N}$ where r is a positive real number then $\chi(G(\mathbb{R}, D)) \leq 3$ [27].

If $|D| \geq 4$ then a complete characterization of line integer distance graphs with respect to chromatic number is not known till now.

If $|D| = 4$ and the greatest common divisor of D is 1 then $\chi(G(\mathbb{Z}, D)) = 5$ if $D_1 = \{1, 2, 3, 4n\}$ or $D_2 = \{x, y, x+y, |y-x|\}$ and $x \equiv y \equiv 1 \pmod 2$ [13]. We conjecture that there are no other 4-element distance sets D such that $\chi(G(\mathbb{Z}, D)) = 5$ (see [14]). The conjecture is supported by [19] where it is proved that $\chi(G(\mathbb{Z}, D)) \leq 4$ if $D = \{a, b, c, d\}$ and $a < b < c < d \leq 2000$ and $D \neq D_1$ and $D \neq D_2$.

If $D = \{1, 2, \ldots, k-1, kn\}$ then $\chi(G(\mathbb{Z}, D)) = |D| + 1$ [12]. In [19] it is proved that $\chi(G(\mathbb{Z}, D)) = |D|+1$ if $D = \{1, 4, 5, 6, 7\}$ or $D = \{1, 2, \ldots, 2k-1, 2k+1, 4k\}$.

We generalize the above mentioned result of [24] by showing that $|D| + 1$ is an upper bound for the choice number of an arbitrary line distance graph $G(S, D)$, $S \subseteq \mathbb{R}$.

A graph G is called *k-degenerate* if each subgraph of G contains a vertex of degree at most k, that is, if $col(G) \leq k + 1$ for finite graphs.

Theorem 1. *If $S \subseteq \mathbb{R}$, $S \neq \emptyset$ and $D \subseteq \mathbb{R}_+$, then $\chi(G(S, D)) \leq ch(G(S, D)) \leq |D| + 1$.*

Proof: Let H be a finite subgraph of $G(S, D)$ and H' be an arbitrary subgraph of H, $H' \subseteq H$. The vertex $v = \min V(H')$ is adjacent to at most $|D|$ vertices which implies that H is $|D|$-degenerate. Therefore, $\chi(H) \leq ch(H) \leq col(H) = \max_{H' \subseteq H} \delta(H') + 1 \leq |D| + 1$ which gives $\chi(G(S, D)) \leq ch(G(S, D)) \leq |D| + 1$ by the Theorem of Johnson. □

For example, $\chi(G(\mathbb{R}, D)) = ch(G(\mathbb{R}, D)) = |D| + 1 = 3$ if $D = \{1, 2\}$.

Since the result of de Bruijn and Erdős cannot be transferred to an analogous statement for the coloring number of infinite graphs (see Introduction) we cannot use the method of Theorem 1 to prove that $col(G(S, D)) \leq |D| + 1$ for $S \subseteq \mathbb{R}$.

Theorem 2. *If $S \subseteq \mathbb{R}$, $S \neq \emptyset$ and $D \subseteq \mathbb{R}_+$ finite then $col(G(S,D)) \leq |D|+1$.*

Proof: Let $0 < x < \min D$. We partition $\mathbb{R}$ into semi-open intervals $I_i = [ix, (i+1)x)$, $i \in \mathbb{Z}$, of length x.

Let $\prec_0$ be a well-ordering of I_0. We extend this to a well-ordering $\prec$ of $\mathbb{R}$ as follows:

If $r_1, r_2 \in I_i$ then $r_1 \prec r_2$ if and only if $r_1 - ix \prec_0 r_2 - ix$. If $r_1 \in I_{i_1}$, $r_2 \in I_{i_2}$, $i_1 \neq i_2$, then $r_1 \prec r_2$ if and only if $|i_1| < |i_2|$ or $i_1 = -i_2 > 0$.

This is a well-ordering of $\mathbb{R}$ since each non-empty subset $T \subseteq \mathbb{R}$ has a smallest element which is determined as follows: Let $r \in I_{i(r)}$. We choose the smallest i_0 in the well-ordering $0, 1, -1, 2, -2, \ldots$ of $\mathbb{Z}$ which is in the set $J = \{i(r) : r \in T\}$. Since I_{i_0} is well-ordered the elements of $T \cap I_{i_0}$ have a smallest element which obviously is the smallest element of T.

Each vertex r of $G(S,D)$ has at most $|D|$ neighbors s with $s \prec r$ since $x < \min D$ and $r \prec r + d$ if $r \geq 0$ and $r \prec r - d$ if $r < 0$ for all $d \in D$. Therefore, $col(G(S,D)) \leq col(G(\mathbb{R},D)) \leq |D|+1$. □

Let $D = \{d_1, \ldots, d_s\}$ and H be a subgraph of $G(S,D)$ with vertex set $\{0, d_1, \ldots, d_s, d_1+d_2, d_1+d_3, \ldots, d_{s-1}+d_s, d_1+d_2+d_3, \ldots, d_1+d_2+\cdots+d_s\}$ (V_H consists of 0 and all sums of elements of D). Then $\delta(H) = |D|$ and therefore $col(G(S,D)) \geq col(H) = |D|+1$. This gives

Corollary 1. *If $D = \{d_1, \ldots, d_s\}$ and $\{0, d_1, \ldots, d_s, d_1+d_2, d_1+d_3, \ldots, d_{s-1} + d_s, d_1+d_2+d_3, \ldots, d_1+d_2+\cdots+d_s\} \subseteq S$ then $col(G(S,D)) = |D|+1$.*

This implies in particular that $col(G(\mathbb{Z},D)) = col(G(\mathbb{Q},D)) = col(G(\mathbb{R},D)) = |D|+1$ if D is finite and $D \subseteq \mathbb{N}$, $D \subseteq \mathbb{Q}_+$, $D \subseteq \mathbb{R}_+$, respectively.

3. Edge colorings

The Theorem of de Bruijn and Erdős can be transferred to edge colorings by considering the *line graph* $L(G) = (V(L(G)), E(L(G)))$ of $G = (V(G), E(G))$ which is defined by $V(L(G)) = E(G)$ and two vertices of $V(L(G))$ are adjacent if and only if the corresponding edges of $E(G)$ are adjacent.

Theorem 3. *If $\chi'(H) \leq k$ (or $ch'(H) \leq k$, respectively) for all finite subgraphs H of an infinite graph G then $\chi'(G) \leq k$ (or $ch'(G) \leq k$, respectively).*

Proof: Let $G = (V(G), E(G))$ be an infinite graph. Let H' be a finite subgraph of $L(G)$ with $V(H') = \{e_1, \ldots, e_m\} \subseteq E(G)$ and $H = G[\{e_1, \ldots, e_m\}]$ the subgraph of G induced by these edges. Then the vertex sets of H' and $L(H)$ coincide which implies $H' \subseteq L(H)$. Therefore, $\chi(H') \leq \chi(L(H)) = \chi'(H) \leq k$ by assumption. Since H' is an arbitrary subgraph of $L(G)$ we obtain $\chi(L(G)) = \chi'(G) \leq k$ by the Theorem of de Bruijn and Erdős.

If we apply the Theorem of Johnson instead we get the analogous result for the choice index. □

Theorem 3 can also be proved by modifying the original proof of the Theorem of de Bruijn and Erdős [3]; see also [21].

We use this result to determine the chromatic index and the choice index of $G(S, D)$ for $S = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$.

Theorem 4. *If $S \subseteq \mathbb{R}$, $S \neq \emptyset$, then $\chi'(G(S,D)) \leq ch'(G(S,D)) \leq 2|D|$.*

Proof: Let $K \subseteq G(S, D)$ be a finite subgraph. If $E(K) = \emptyset$ then obviously $\chi'(K) = ch'(K) \leq 2|D|$. Otherwise let H' be an arbitrary subgraph of the line graph $L(K)$ with $V(H') = \{e_1, \ldots, e_m\}$ and $H = K[\{e_1, \ldots, e_m\}]$. We consider the vertex $v = \min V(H)$ and the edge $e = vw$ with $w = \min N_H(v)$ where $N_H(v)$ is the *neighborhood* of v in the graph H.

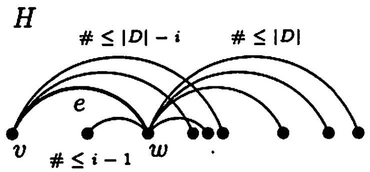

FIGURE 1. Edge $e = vw$ with its adjacent edges in H.

Let $d = w - v \in D$ and $i = |\{x : x \in D, x \leq d\}|$. Then $e = vw$ is adjacent to at most $2|D| - 1$ edges since e is adjacent to at most $|D| - i$ edges that are incident to v and to at most $|D| + i - 1$ edges that are incident to w (see Figure 1). Since $e \in V(H') = V(L(H))$ we obtain $\delta(H') \leq 2|D| - 1$. This implies that $L(K)$ is $(2|D| - 1)$-degenerate, that is, $\chi(L(K)) = \chi'(K) \leq ch(L(K)) = ch'(K) \leq col(L(K)) \leq 2|D|$. Therefore, $\chi'(G(S,D)) \leq ch'(G(S,D)) \leq 2|D|$ by Theorem 3. □

If $\Delta(G(S,D)) = 2|D|$ then equality holds in Theorem 4.

Corollary 2. *If $S \subseteq \mathbb{R}$, $S \neq \emptyset$ and $\Delta(G(S,D)) = 2\,|D|$ then $\chi'(G(S,D)) = ch'(G(S,D)) = 2\,|D|$.*

For example, let $I_N^D \subset N$ for $N \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ be a closed interval of length $2 \max D$. If $I_{\mathbb{R}}^D \subseteq S$ or ($I_{\mathbb{Q}}^D \subseteq S$ and $D \subseteq \mathbb{Q}_+$) or ($I_{\mathbb{Z}}^D \subseteq S$ and $D \subseteq \mathbb{N}$) then $\chi'(G(S,D)) = ch'(G(S,D)) = 2\,|D|$ since in these cases we have $\Delta(G(S,D)) = 2\,|D|$. (Note that $d \in \mathbb{R}_+ \setminus \mathbb{Q}_+$ ($d \in \mathbb{R}_+ \setminus \mathbb{N}$) yields no edges in $G(S,D)$ if $S \subseteq \mathbb{Q}$ ($S \subseteq \mathbb{Z}$).)

Corollary 3. *If $S = \mathbb{R}$ or ($S = \mathbb{Q}$ and $D \subseteq \mathbb{Q}_+$) or ($S = \mathbb{Z}$ and $D \subseteq \mathbb{N}$) then $\chi'(G(S,D)) = ch'(G(S,D)) = 2\,|D|$.*

4. Total colorings

The Theorem of de Bruijn and Erdős can also be transferred to total colorings by considering the *total graph* $T(G) = (V(T(G)), E(T(G)))$ of $G = (V(G), E(G))$ which has vertex set $V(T(G)) = V(G) \cup E(G)$ and two elements of $V(T(G))$ are adjacent if and only if the elements are adjacent or incident in G.

Theorem 5. *If $\chi''(H) \leq k$ (or $ch''(H) \leq k$, respectively) for all finite subgraphs H of an infinite graph G then $\chi''(G) \leq k$ (or $ch''(G) \leq k$, respectively).*

Proof: Let $G = (V(G), E(G))$ be an infinite graph. Let H'' be a finite subgraph of the total graph $T(G)$ with vertices $V(H'') = \{v_1, \ldots, v_n, e_1, \ldots, e_m\} \subseteq V(T(G))$, $v_1, \ldots, v_n \in V(G)$, $e_1, \ldots, e_m \in E(G)$. If $V = \{v_1, \ldots, v_n\} \cup \{u_i, w_i : e_i = u_i w_i, i = 1, \ldots, m\}$ then let $H = G[V]$ be the subgraph of G induced by V. Then $H'' \subseteq T(H)$ which implies $\chi(H'') \leq \chi(T(H)) = \chi''(H) \leq k$ by assumption. Since H'' is an arbitrary finite subgraph of $T(G)$ we get $\chi(T(G)) = \chi''(G) \leq k$ applying the Theorem of de Bruijn and Erdős.

Using the Theorem of Johnson instead we obtain the analogous result for the total choice number. $\square$

We apply Theorem 5 to determine the total chromatic number and the total choice number for line distance graphs.

Theorem 6. *If $S \subseteq \mathbb{R}$, $S \neq \emptyset$, then $\chi''(G(S,D)) \leq ch''(G(S,D)) \leq 2\,|D| + 1$.*

Proof: Let K be a finite subgraph of $G(S,D)$. If $E(K) = \emptyset$ then $\chi''(K) = ch''(K) = 1 \leq 2\,|D| + 1$. Otherwise let H'' be an arbitrary subgraph of the total graph $T(K)$ with $V(H'') = \{e_1, \dots, e_m, v_1, \dots, v_n\}$, $e_i \in E(K)$, $i = 1, \dots, m$, and $v_j \in V(K)$, $j = 1, \dots, n$.

Let $V = \{v_1, \dots, v_n\} \cup \{u_i, w_i : e_i = u_i w_i, i = 1, \dots, m\}$ and $H = K[V]$ be the subgraph of K induced by V. This implies $H'' \subseteq T(H)$. Consider $v = \min V(H)$. If $v \in V(H'')$ then v is adjacent to at most $|D|$ vertices and is incident to at most $|D|$ edges in H i.e. $d(v) \leq 2|D|$ in H''.

If $v \notin V(H'')$ then v is incident to at least one edge e_i, $1 \leq i \leq m$. We consider the induced subgraph $H[\{e_1, \dots, e_m\}]$ and the edge $e_j = vw$, $1 \leq j \leq m$, where $w = \min N_{H[\{e_1,\dots,e_m\}]}(v)$. Then edge e_j is adjacent to at most $2|D| - 1$ edges in $H[\{e_1, \dots, e_m\}]$ (see proof of Theorem 4). Therefore, the degree $d(e_j)$ in H'' is at most $2|D| - 1 + 1$ since $v \notin V(H'')$. In both cases we have $\delta(H'') \leq 2|D|$, that is, $T(K)$ is $2|D|$-degenerate.

This implies that $\chi(T(K)) = \chi''(K) \leq ch(T(K)) = ch''(K) \leq col(T(K)) \leq 2|D| + 1$ which gives $\chi''(G(S,D)) \leq ch''(G(S,D)) \leq 2|D| + 1$ by Theorem 5. □

In case of $\Delta(G(S,D)) = 2|D|$ we obtain equality in Theorem 6.

Corollary 4. *If $S \subseteq \mathbb{R}$, $S \neq \emptyset$ and $\Delta(G(S,D)) = 2|D|$ then $\chi''(G(S,D)) = ch''(G(S,D)) = 2|D| + 1$.*

Corollary 5. *If $S = \mathbb{R}$ or ($S = \mathbb{Q}$ and $D \subseteq \mathbb{Q}_+$) or ($S = \mathbb{Z}$ and $D \subseteq \mathbb{N}$) then $\chi''(G(S,D)) = ch''(G(S,D)) = 2|D| + 1$.*

Proof: In all three cases we have $\Delta(G(S,D)) = 2|D|$. □

Corollaries 3 and 5 generalize results of [15] where the chromatic index and the choice index such as the total chromatic number and the total choice number of line integer distance graphs have been determined by constructions. Moreover, open questions of [15] are answered in the affirmative.

5. Higher dimensional distance graphs

In this note we consider colorings of 1-dimensional distance graphs $G(S,D)$ with $V(G(S,D)) = S \subseteq \mathbb{R}$ and $D \subseteq \mathbb{R}_+$. For higher dimensional distance graphs $G(S,D)$, $S \subseteq \mathbb{R}^n$, $D \subseteq \mathbb{R}_+$, it is obvious that $\chi'(G(S,D))$, $ch'(G(S,D))$, $\chi''(G(S,D))$ and $ch''(G(S,D))$ are infinite if $S = \mathbb{R}^n$ or $S = \mathbb{Q}^n$ and $D \subseteq \mathbb{Q}_+$, and $n \geq 2$ and $D \neq \emptyset$ (see [15]).

The integer case of Corollary 3 can be generalized as follows:

Theorem 7. $\chi'(G(\mathbb{Z}^n, D)) = \Delta(G(\mathbb{Z}^n, D))$
where $\Delta(G(\mathbb{Z}^n, D)) = \sum_{d \in D} |\mathbb{Z}^n \cap \partial S_d^n(0)|$.

Proof: Let $\Delta = \Delta(G(\mathbb{Z}^n, D))$. The vertices $v \in \mathbb{Z}^n$ of the boundary ∂ of the n-dimensional sphere $S_d^n(0)$ with radius $d \in D$ and center 0 are all the neighbors of vertex 0 with distance d which implies for the degree $d(0)$ of vertex 0 that $d(0) = \sum_{d \in D} |\mathbb{Z}^n \cap \partial S_d^n(0)| = \Delta$.

Since $-x \in \mathbb{Z}^n \cap \partial S_d^n(0)$ whenever $x \in \mathbb{Z}^n \cap \partial S_d^n(0)$ and every pair $x, -x$ determines a direction in $\mathbb{R}^n$, all edges $\{v, v+x\}$, $v \in \mathbb{Z}^n$, with $||x|| = d$ are on infinite paths which can be 2-edge colored. Therefore, all edges of $G(\mathbb{Z}^n, D)$ can be colored with $\frac{\Delta}{2} \cdot 2 = \Delta$ colors. □

It is proved in [8] that every r-degenerate graph G with $\Delta(G) \geq 2r$ is class 1. Theorem 7 can also be proved by using this result.

It is proved in [10] that $ch(G(\mathbb{R}^2, \{1\}))$ is infinite by embedding the hypercubes Q_n in the distance graph $G(\mathbb{R}^2, \{1\})$. Since this class contains subgraphs of arbitrarily high coloring number the choice number of this class is not bounded (see [1]). Obviously, this implies that $ch(G(\mathbb{R}^n, D))$ is infinite for $n \geq 2$. By an analogous argument we obtain $ch(G(\mathbb{Q}^n, D)) = \infty$ if $n \geq 2$.

The determination of $ch(G(\mathbb{Z}^n, D))$, $ch'(G(\mathbb{Z}^n, D))$, $\chi''(G(\mathbb{Z}^n, D))$ and $ch''(G(\mathbb{Z}^n, D))$ is an unsolved problem for $n \geq 2$.

Only few results are known for the chromatic number $\chi(G(S, D))$ for higher dimensional distance graphs. For example, even for a unit distance space coloring (i.e., $V = S = \mathbb{R}^n$, $n \geq 2$, $D = \{1\}$, generalizing the Hadwiger-Nelson unit distance plane coloring problem) none of the chromatic numbers is known. There only exist upper and lower bounds for $\chi(G(\mathbb{R}^n, \{1\}))$ (see, e.g., [2, 17, 25]).

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