Path Decompositions of $\lambda K_{n,n}$

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Abstract. Let P_k denote a path with k vertices and $k-1$ edges. And let $\lambda K_{n,n}$ denote the λ-fold complete bipartite graph with both parts of size n. A P_k-decomposition $\mathscr{D}$ of $\lambda K_{n,n}$ is a family of subgraphs of $\lambda K_{n,n}$ whose edge sets form a partition of the edge set of $\lambda K_{n,n}$ such that each member of $\mathscr{D}$ is isomorphic to P_k. Necessary conditions for the existence of a P_k-decomposition of $\lambda K_{n,n}$ are (i) $\lambda n^2 \equiv 0 \pmod{k-1}$ and (ii) $k \leq n+1$ if $\lambda = 1$ and n is odd, or $k \leq 2n$ if $\lambda \geq 2$ or n is even. In this paper, we show these necessary conditions are sufficient except for the possibility of the case that $\lambda = 3$, $n = 15$, and $k = 28$.

1 Introduction

For a positive integer k, let P_k denote a path with k vertices and $k-1$ edges. Suppose that G is a multigraph. A P_k *– decomposition* or a P_k *– design* $\mathscr{D}$ of G is a family of subgraphs of G such that the edge sets of these subgraphs form a partition of the edge set of G and each member of $\mathscr{D}$ is isomorphic to P_k.

For a graph G and a positive integer λ, we use λG to denote the multigraph obtained from G by replacing each edge e of G by λ edges with the same end vertices as e. Let λK_n denote the λ-fold complete graph on n vertices, and let $\lambda K_{m,n}$ denote the λ-fold complete bipartite graph

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with parts of size m and n. A complete r-partite graph with r-partition $(V_1, V_2, \ldots, V_r)$ and $|V_i| = m_i$ is denoted by $K_{m_1, m_2, \ldots, m_r}$.

Decompositions of various graphs into paths have been studied extensively. In 1983, the question of P_k-decompositions of λK_n was completely settled by Tarsi [10]. We next consider the question of P_k-decompositions of G where G is other than a complete graph. Kotzig [5] had much earlier shown that a connected graph has a P_3-decomposition if and only if it has an even number of edges. When $k \geq 4$, the question is unsolved. When the problem is restricted to regular graphs, Bouchet and Fouquet [1] and Kotzig [5] independently showed that a 3-regular graph G has a P_4-decomposition if and only if G has a 1-factor. Heinrich, Liu, and Yu [2] proved that a connected 4-regular graph G admits a P_4-decomposition if and only if $|E(G)|$ is divisible by 3. Jacobson, Truszczyński, and Tuza [4] proved that every 4-regular bipartite graph has a P_5-decomposition.

As to the P_k-decomposition of complete bipartite graph $K_{m,n}$, there is one obvious necessary condition. The number of edges in the path must divide the number of edges in the complete bipartite graph. Parker [8] divided the remaining necessary conditions into 8 cases which based on the parity of the path length $k-1$ and the size of the parts m and n. And Parker showed that these necessary conditions are also sufficient. The parity conditions are summarized in Table 1.

Table 1 Necessary Conditions

Case	$k-1$	m	n	Necessary Conditions
1.	even	even	even	$k \leq 2m+1, k \leq 2n+1$, not both equalities
2.	even	even	odd	$k \leq 2m-1, k \leq 2n+1$
3.	even	odd	even	$k \leq 2m+1, k \leq 2n-1$
4.	even	odd	odd	not possible
5.	odd	even	even	$k \leq 2m, k \leq 2n$
6.	odd	even	odd	$k \leq 2m, k \leq n+1$
7.	odd	odd	even	$k \leq m+1, k \leq 2n$
8.	odd	odd	odd	$k \leq m+1, k \leq n+1$

Truszczyński [11] considered the question of decomposing $\lambda K_{m,n}$ into P_k and gave some necessary and/or sufficient conditions for the decomposition to exist. Shyu and Lin [9] provided a necessary and sufficient condition for the existence of P_k-decompositions of crown graphs. The question of P_4-decompositions of $K_{m_1, m_2, \ldots, m_r}$ was solved by Kumar [6]. As for the other relative problems in path decompositions, there was an excellent survey giving by Heinrich [3].

In [11], the following result concerning the path decomposition of λ-fold complete bipartite graphs $\lambda K_{n,n}$ was proved.

Theorem 1.1 *Let λ, n, and k be positive integers such that λ or n are even. The graph $\lambda K_{n,n}$ has a P_k-decomposition if and only if $\lambda n^2 \equiv 0 \pmod{k-1}$ and $k \leq 2n$* □

In this paper, the P_k-decompositions of $\lambda K_{n,n}$ are investigated, and we obtain the following results.

Theorem 1.2 *Let λ, n and k be positive integers where λ and n are odd. The graph $\lambda K_{n,n}$ has a P_k-decomposition if and only if the following conditions,* (i) *and* (ii), *are satisfied; except for the possibility of the case that $\lambda = 3$, $n = 15$, and $k = 28$.*

(i) $\lambda n^2 \equiv 0 \pmod{k-1}$,

(ii) $k \leq \begin{cases} n+1 & \text{if } \lambda = 1, \\ 2n & \text{if } \lambda \geq 2. \end{cases}$

The following theorem follows immediately from Theorem 1.1 and Theorem 1.2.

Theorem 1.3 *Let λ, n and k be positive integers. The graph $\lambda K_{n,n}$ has a P_k-decomposition if and only if the following conditions ,* (i) *and* (ii), *are satisfied; except for the possibility of the case that $\lambda = 3$, $n = 15$, and $k = 28$.*

(i) $\lambda n^2 \equiv 0 \pmod{k-1}$,

(ii) $k \leq \begin{cases} n+1 & \text{if } \lambda = 1 \text{ and } n \text{ is odd}, \\ 2n & \text{if } \lambda \geq 2 \text{ or } n \text{ is even}. \end{cases}$ □

The remaining open problem is to show if the P_k-decomposition of $\lambda K_{n,n}$ exists when $\lambda = 3$, $n = 15$, and $k = 28$.

2 Main Result

Before we prove the main result, we need a couple of lemmas.

Lemma 2.1 *Suppose that a multigraph G can be decomposed into t nontrivial paths and that G contains v vertices with odd degrees. Then $v \leq 2t$.*

Proof. Suppose that G is decomposed into nontrivial paths $Q_1, Q_2, \ldots, Q_t$. Since each vertex of G with odd degree must be an end vertex of at least one Q_i $(1 \leq i \leq t)$, the result follows. □

For our discussion we need the following definitions and notations. A $v_0 - v_r$ *walk* of length r in a multigraph G is a sequence of vertices of the form $v_0, v_1, \ldots, v_r$ where $v_{i-1}v_i \in E(G)$ for $i = 1, 2, \ldots, r$; this walk is denoted by $v_0v_1 \ldots v_r$. A trail is a walk in which all edges are distinct. Suppose Q: $x_1x_2x_3 \ldots x_n$ is a trail in a multigraph. Then the trail T: $x_kx_{k+1}x_{k+2} \ldots x_s$ $(1 \leq k \leq s \leq n)$ is called a *subtrail* of Q. A trail is *closed* if its starting vertex and ending vertex are the same. A closed trail of a multigraph G containing all the edges of G is called an *Euler tour* of G. A *path* in a multigraph is a trail in which no vertex is repeated. A closed trail in which no vertex is repeated is called a *cycle*. A cycle of a multigraph G containing every vertex of G is called a *Hamiltonian cycle* of G. Suppose that (A, B) is the bipartition of $\lambda K_{n,n}$ such that $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$. In the sequel of the paper, the subscripts of a_i and b_j are taken modulo n. For any edge a_ib_j in $\lambda K_{n,n}$, the *label* of a_ib_j is $(j - i)$ mod n. For vertices $a_{i_1}, b_{j_1}, a_{i_2}, b_{j_2}, \ldots, a_{i_k}, b_{j_k}$ in $\lambda K_{n,n}$, we use $< a_{i_l}b_{j_l} >_{l=1}^{k}$ to mean the walk $a_{i_1}b_{j_1}a_{i_2}b_{j_2} \ldots a_{i_k}b_{j_k}$, and $< a_{i_1} >$ the trivial walk a_{i_1}. Suppose W_1: $x_1x_2 \ldots x_t$ and W_2: $y_1y_2 \ldots y_s$ are walks. If x_ty_1 is an edge, we use W_1W_2 to denote the walk $x_1x_2 \ldots x_ty_1y_2 \ldots y_s$. If $x_t = y_1$, we use $W_1 + W_2$ to denote the walk $x_1x_2 \ldots x_ty_2y_3 \ldots y_s$. For walks $W_1, W_2, \ldots, W_v$ in a multigraph, $W_1 + W_2 + \cdots + W_v$ is similarly defined. For an integer t and a walk W: $x_1x_2 \ldots x_s$ in a multigraph, we use $W + t$ to denote the walk $x_{1+t}x_{2+t} \ldots x_{s+t}$.

Given a closed trail $T = v_{i_1}v_{i_2} \ldots v_{i_q}$ of a multigraph G, let $g(T)$ denote the minimum k such that a segment of $k + 1$ consecutive vertices $v_{i_j}, v_{i_{j+1}} \ldots v_{i_{j+k}}$ in T where $1 \leq j \leq j + k \leq q$, is a cycle of length k in G, that is, when $v_{i_j} = v_{i_{j+k}}$ and the vertices $v_{i_j}, v_{i_{j+1}} \ldots v_{i_{j+k-1}}$ are distinct. For example, let W be the closed trail $a_0b_2a_4b_3a_0b_5a_4b_1a_5b_2a_3b_0a_0$ in $K_{6,6}$. Then $g(W) = 4$.

For $i = 0, 1, 2, \ldots, n - 1$, let C_i denote the Hamiltonian cycle $< a_lb_{i+l} >_{l=0}^{n-1} < a_0 >$ of $\lambda K_{n,n}$, i.e., C_i is the cycle $a_0b_ia_1b_{i+1}a_2b_{i+2} \ldots a_nb_{i+(n-1)}a_0$. For positive integer t, let $t * C_i = \underbrace{C_i + C_i + \cdots + C_i}_{\text{t copies of } C_i}$.

Lemma 2.2 *Let i, j, s, t and n be positive integers where $i + j \leq n - 1$. Then $g(s * C_i + t * C_{i+j}) = 2n - 2j$.*

Proof. Firstly, it is easy to see that $g(s * C_i) = g(t * C_{i+j}) = 2n$. Secondly, $C_i + C_{i+j}$ is the closed trail $< a_lb_{i+l} >_{l=0}^{n-1} < a_lb_{i+j+l} >_{l=0}^{n-1} < a_0 >$, i.e., $C_i + C_{i+j}$ is the trail $a_0b_ia_1b_{i+1}a_2b_{i+2} \ldots a_nb_{i+(n-1)}a_0b_{i+j}a_1b_{i+j+1}a_2b_{i+j+2} \ldots a_nb_{i+j+(n-1)}a_0$. There are $2n$ edges between two appearances of a_l $(l =$

$0, 1, \ldots, n-1$), $2n-2j$ edges between two appearances of b_l for $l = i + j, i+j+1, \ldots, i+n-1$, and $4n-2j$ edges between two appearances of b_l for $l = i, i+1, \ldots, i+j-1$. It implies that $g(C_i + C_{i+j}) = 2n - 2j$. Thus $g(s * C_i + t * C_{i+j}) = 2n - 2j$. □

The following Lemma was introduced by Tarsi [10], and was henceforth used in many papers, e.g. [2, 7, 8, 9, 11].

Lemma 2.3 *Let $k_1, k_2, \ldots, k_m$ be positive integers, and let E be an Euler tour of a multigraph G with $k_1 + k_2 + \cdots + k_m = |E(G)|$ and $k_i < g(E)$, for $i = 1, 2, \ldots, m$. Then G can be decomposed into m paths of lengths $k_1, k_2, \ldots, k_m$.*

Proof. From the starting vertex we cut the Euler tour E into subtrails with k_i edges, $i = 1, 2, \ldots, m$. These subtrails are all paths since $k_i < g(E)$. Thus G can be decomposed into m paths of lengths $k_1, k_2, \ldots, k_m$. □

Suppose that G is a multigraph. For $x, y \in V(G)$ with $x \neq y$, we use $m_G(x, y)$ to denote the number of edges joining x and y in G. Suppose that $G_1, G_2, \ldots, G_t$ are spanning subgraphs of a multigraph G. We use $G_1 \oplus G_2 \oplus \cdots \oplus G_t$ to denote the spanning subgraph S such that for $x, y \in V(S)$ with $x \neq y$, $m_S(x, y) = \sum_{i=1}^{t} m_{G_i}(x, y)$. The graph $S = G_1 \oplus G_2 \oplus \cdots \oplus G_t$ is called the *edge sum* of $G_1, G_2, \cdots, G_t$.

Suppose that G is a subgraph of $\lambda K_{n,n}$ and H is a subgraph of G such that the edges of G can be decomposed into subgraphs $H, H+1, H+2, \ldots, H+j$ for some integer j. Then H is called a *base graph* of this decomposition. For $i = 0, 1, \ldots, n-1$, let F_i denote the spanning subgraph of $\lambda K_{n,n}$ with edge set $\{a_0 b_i, a_1 b_{i+1}, \ldots, a_{n-1} b_{i+(n-1)}\}$, i.e., F_i is a 1-factor. In the sequel of the paper, the subscripts of F_i are taken modulo n.

Lemma 2.4 *Let n and k be positive integers with $2 \leq k \leq 2n$, and let $G_i = F_{(2n-1)+i} \oplus F_{(2n-2)+i} \oplus \cdots \oplus F_{(2n-k+1)+i}$, for $i = 0, 1, \ldots, n-1$. Then G_i has a P_k-decomposition.*

Proof. For $i = 0, 1, 2, \ldots, n-1$, let Q_i be the base path $< a_l b_{(n-1)-l+i} >_{l=0}^{\lfloor \frac{k}{2} \rfloor - 1}$, i.e., Q_i is the path

$$a_0 b_{n-1+i} a_1 b_{n-2+i} a_2 b_{n-3+i} \cdots a_{\lfloor \frac{k}{2} \rfloor - 1} b_{n-\lfloor \frac{k}{2} \rfloor + i}$$

, if k is even and let Q_i be the base path $< a_l b_{(n-1)-l+i} >_{l=0}^{\lfloor \frac{k}{2} \rfloor - 1} < a_{\lfloor \frac{k}{2} \rfloor} >$, i.e., Q_i is the path

$$a_0 b_{n-1+i} a_1 b_{n-2+i} a_2 b_{n-3+i} \cdots a_{\lfloor \frac{k}{2} \rfloor - 1} b_{n-\lfloor \frac{k}{2} \rfloor + i} a_{\lfloor \frac{k}{2} \rfloor}$$

, if k is odd.

We can see that Q_i is a path of length $k-1$, and consists of edges with labels in order of $2n-1+i, 2n-2+i, \ldots, 2n-k+1+i \pmod{n}$. It is easy to see that G_i can be decomposed into paths $Q_i, Q_i+1, Q_i+2, \ldots, Q_i+(n-1)$. That implies G_i has a P_k–decomposition. □

Now we prove the main theorem.

Proof of Theorem 1.2(Necessity) Condition (i) is trivial. Now we prove Condition (ii). Since λ and n are odd, all of the $2n$ vertices in $\lambda K_{n,n}$ have odd degrees. And there are $\frac{\lambda n^2}{k-1}$ paths in the decomposition. By Lemma 2.1, we have $2n \le 2\frac{\lambda n^2}{k-1}$, i.e., $k \le \lambda n + 1$. On the other hand, P_k has k vertices, and $\lambda K_{n,n}$ has $2n$ vertices, so we have $k \le 2n$. This completes Condition (ii).
(Sufficiency) To avoid trivialities we assume $n \ge 3$ and $k \ge 2$. Since λ and n are odd, and $\lambda n^2 \equiv 0 \pmod{k-1}$, we have that k is even.

Case 1. $k \le n+1$.

Suppose that $\lambda = 2p+1$, for some nonnegative integer p. Let $G_1 = F_{n-1} \oplus F_{n-2} \oplus \cdots \oplus F_{n-k+1}$, and $G_2 = \lambda K_{n,n} - E(G_1)$. Assume that $E = p*C_0 + (p+1)*C_1 + p*C_2 + (p+1)*C_3 + \cdots + p*C_{n-k-1} + (p+1)*C_{n-k}$ $+ p*C_{n-k+1} + p*C_{n-k+2} + \cdots + p*C_{n-1}$. Since $p*C_0 + p*C_1 + \cdots + p*C_{n-1}$ is an Euler tour of $(\lambda-1)K_{n,n}$ and $C_1 + C_3 + \cdots + C_{n-k-2} + C_{n-k}$ is an Euler tour of $K_{n,n} - E(G_1)$, E is an Euler tour of G_2. By Lemma 2.2, we have $g(E) \ge 2n-4$ ($g(E) = 2n-4$ if $p=0$), by Lemma 2.3, G_2 has a P_k–decomposition. As for the decomposition of G_1, by Lemma 2.4, G_1 has a P_k–decomposition. From the P_k–decomposition of G_1 and of G_2, we obtain the P_k–decomposition of $\lambda K_{n,n}$.

Case 2. $n+2 \le k \le 2n-4$.

Since $k \ge n+2$, by Condition (ii), we have $\lambda \ge 3$. Suppose that $\lambda = 2p+3$, for some nonnegative integer p and $k-1 = 2n-m$, with $5 \le m \le n-1$. Let $G_1 = F_{(2n-1)} \oplus F_{(2n-2)} \oplus \cdots \oplus F_{(2n-k+1)} = K_{n,n} \oplus F_{n-1} \oplus F_{n-2} \oplus \cdots \oplus F_m$, and $G_2 = \lambda K_{n,n} - E(G_1)$. Assume that $E = (p+1)*C_0 + (p+1)*C_1 + \cdots + (p+1)*C_m + p*C_{m+1} + (p+1)*C_{m+2} + p*C_{m+3} + (p+1)*C_{m+4} + \cdots + p*C_{n-3} + (p+1)*C_{n-2} + p*C_{n-1}$. Since $p*C_0 + p*C_1 + \cdots + p*C_{n-1}$ is an Euler tour of $(\lambda-3)K_{n,n}$ and $C_0 + C_1 + C_2 + \cdots + C_m + C_{m+2} + C_{m+4} \cdots + C_{n-4} + C_{n-2}$ is an Euler tour of $3K_{n,n} - E(G_1)$, E is an Euler tour of G_2. By Lemma 2.2, $g(E) \ge 2n-4$ ($g(E) = 2n-4$ if $p=0$), we by Lemma 2.3, that G_2 has a P_k–decomposition. As to the decomposition of G_1, by Lemma 2.4, G_1 has a P_k–decomposition. This completes case 2.

Case 3. $k = 2n-2$.

By the assumption, n is odd and $n = 3$ belong to case 1, we assume $n \geq 5$. By condition (ii), we have $\lambda \geq 3$. We distinguish two cases as follows.

Case 3.1. $\lambda \geq 5$.

Suppose that $\lambda = 2p + 5$, for some nonnegative integer p. Let $G_1 = F_{(2n-1)} \oplus F_{(2n-2)} \oplus \cdots \oplus F_{(2n-k+1)} = K_{n,n} \oplus F_{n-1} \oplus F_{n-2} \cdots \oplus F_3$ and $G_2 = \lambda K_{n,n} - E(G_1)$. Since $(p+1)*C_0+(p+1)*C_1+\cdots+(p+1)*C_{n-1}$ is an Euler tour of $(\lambda-3)K_{n,n}$ and $C_0+C_1+C_2+C_3+C_5+C_7+\cdots+C_{n-4}+C_{n-2}$ is an Euler tour of $3K_{n,n} - E(G_1)$, $E = (p+2)*C_0 + (p+2)*C_1 + (p+2)*C_2 + (p+2)*C_3 + (p+1)*C_4 + (p+2)*C_5 + \cdots + (p+2)*C_{n-4} + (p+1)*C_{n-3} + (p+2)*C_{n-2} + (p+1)*C_{n-1}$ is an Euler tour of G_2. By Lemma 2.2, $g(E) = 2n-2$, and hence by Lemma 2.3, has a P_{2n-2}–decomposition. As to the decomposition of G_1, by Lemma 2.4, G_1 has a P_{2n-2}–decomposition. Thus $\lambda K_{n,n}$ has a P_{2n-2}–decomposition.

Case 3.2. $\lambda = 3$.

Since $gcd(2n-3, n) = 1$ or 3, we consider two cases as follows.

Case 3.2.1. $gcd(2n-3, n) = 1$.

By Condition (i) $\lambda n^2 \equiv 0 \pmod{2n-3}$, we have $\lambda \equiv 0 \pmod{2n-3}$, i.e., $3 \equiv 0 \pmod{2n-3}$. Thus $2n-3 = 1$ or 3. By assumption, we complete the case 3.2.1.

Case 3.2.2. $gcd(2n-3, n) = 3$.

By condition (i) $\lambda n^2 \equiv 0 \pmod{2n-3}$, we have $9\lambda \equiv 0 \pmod{2n-3}$, i.e., $27 \equiv 0 \pmod{2n-3}$. Thus the value of $2n-3$ will be either 1, 3, 9, or 27. By the assumption, $n \geq 5$ and odd, we leave the case for $n = 15$. The case for $\lambda = 3$, $n = 15$, and $k = 28$ is still open.

Case 4. $k = 2n$.

Since $\lambda n^2 \equiv 0 \pmod{2n-1}$ and $gcd(2n-1, n) = 1$, we have $\lambda \equiv 0 \pmod{2n-1}$. Suppose that $\lambda = s(2n-1)$, for some positive integer s. Let G_1 be the following edge sum

$$\underbrace{[2K_{n,n} - E(F_1)] \oplus [2K_{n,n} - E(F_1)] \oplus \cdots \oplus [2K_{n,n} - E(F_1)]}_{\text{s copies of } [2K_{n,n} - E(F_1)]} \oplus$$

$$\underbrace{[2K_{n,n} - E(F_2)] \oplus [2K_{n,n} - E(F_2)] \oplus \cdots \oplus [2K_{n,n} - E(F_2)]}_{\text{s copies of } [2K_{n,n} - E(F_2)]} \oplus$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \oplus$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \oplus$$

$$\underbrace{[2K_{n,n}-E(F_{n-1})]\oplus[2K_{n,n}-E(F_{n-1})]\oplus\cdots\oplus[2K_{n,n}-E(F_{n-1})]}_{\text{s copies of } [2K_{n,n}-E(F_{n-1})]}.$$

And let $G_2 = s(2n-1)K_{n,n} - E(G_1)$. For $i = 1, 2, \ldots, n-1$, $2K_{n,n} - E(F_i) = F_{(2n-1)+i} \oplus F_{(2n-2)+i} \oplus \cdots \oplus F_{1+i}$. By Lemma 2.4, $2K_{n,n} - E(F_i)$ has a P_{2n}–decomposition. That implies G_1 has a P_{2n}–decomposition. As for the decomposition of G_2, let $H = [2K_{n,n} - E(F_1)] \oplus [2K_{n,n} - E(F_2)] \oplus \cdots \oplus [2K_{n,n} - E(F_{n-1})]$. Then G_2 is the edge sum of s copies of $(2n-1)K_{n,n} - E(H)$. Since $(2n-1)K_{n,n} - E(H) = K_{n,n} \oplus F_{n-1} \oplus F_{n-2} \oplus \cdots \oplus F_1 = F_{2n-1} \oplus F_{2n-2} \oplus \cdots \oplus F_1$, by Lemma 2.4 again, G_2 has a P_{2n}–decomposition. This completes case 4. □

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