Remarks on MacMahon's identity for sums of cubes of binomial coefficients[1]

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The question concerning a possible generalization of the identity of MacMahon [4] (Cf. Gould [3, formula (6.7)])

$$\sum_{k=0}^{n}\binom{n}{k}^3 u^k v^{n-k} = \sum_{0\le k\le n/2}\binom{n}{2k}\binom{2k}{k}\binom{n+k}{k} u^k v^k (u+v)^{n-2k} \qquad (1)$$

is implicitly answered in [2]. In formula (12) of that paper take $c = d = 1$, $e = -a$, $x = -u$ and we get

$$\sum_{k=0}^{\infty}\binom{a}{k}^3 u^k = (1+u)^a \sum_{r=0}^{\infty} \frac{(-\frac{a}{2})_r(-\frac{a}{2}+\frac{1}{2})_r(a+1)_r}{r!\,r!\,r!}\,\frac{4^r u^r}{(1+u)^{2r}}$$

$$= \sum_{r=0}^{\infty}\binom{a}{2r}\binom{2r}{r}\binom{a+r}{r} u^r (1+u)^{a-2r}, \qquad (2)$$

which is the desired result. (Taking $v = 1$ does not involve any loss in generality.)

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[1] This paper was begun in 1965 and laid aside to be developed later.
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It may be of interest to note that

$$\sum_{r=0}^{\infty} \binom{a}{2r}\binom{2r}{r}\binom{a+r}{r} u^{r}(1+u)^{a-2k}$$

$$= \sum_{r=0}^{\infty} \binom{a}{2r}\binom{2r}{r}\binom{a+r}{r} u^{r} \sum_{j=0}^{\infty} (-1)^{j} \binom{-a+2r+j-1}{j} u^{j}$$

$$= \sum_{n=0}^{\infty} u^{n} \sum_{r+j=n} \binom{a}{2r}\binom{2r}{r}\binom{a+r}{r}\binom{a-2r}{j}.$$

The inner sum is equal to

$$\sum_{r+j=n} \binom{2r}{r}\binom{a+r}{r}\binom{a}{2r+j}\binom{2r+j}{2r} = \sum_{r=0}^{n} \binom{2r}{r}\binom{a+r}{r}\binom{a}{n+r}\binom{n+r}{2r}.$$

Comparing coefficients in (2) we get

$$\sum_{r=0}^{n} \binom{2r}{r}\binom{a+r}{r}\binom{a}{n+r}\binom{n+r}{2r} = \binom{a}{n}^{3}. \qquad (3)$$

Now

$$\binom{2r}{r}\binom{a+r}{r}\binom{a}{n+r}\binom{n+r}{2r} = \binom{a+r}{r}\binom{a}{n+r} \frac{(n+r)!}{(n-r)!\, r!\, r!}$$

$$= \frac{(a+1)_r}{r!} \binom{a}{n} \frac{(-a+n)_r (n+1)_r (-n)_r}{(n+1)_n\, r!\, r!},$$

so that (3) becomes

$$\sum_{r=0}^{\infty} \frac{(-n)_r (a+1)_r (-a+n)_r}{r!\, r!\, r!} = \binom{a}{n}^{2}, \qquad (4)$$

but this is implied by Saalschütz's theorem, so that (3) is really nothing new. However this does show that (2) is a consequence of Saalschütz's theorem, which in turn is a special case of the remark following formula (13) in [2]. This remark had to do with a certain ${}_3F_2$ transformation which Bailey had noted was a consequence of Saalschütz's theorem.

In [2] it was shown that

$$
{}_4F_3\left[\begin{matrix} e\,, & 1+a-c-d\,, & 1+a-b-d\,, & 1+a-b-c\,; & x \\ & 1+a-b\,, & 1+a-c\,, & 1+a-d & \end{matrix}\right]
$$

$$
= \sum_{r=0}^{\infty} \frac{(a)_r\,(b)_r\,(c)_r\,(d)_r\,(e)_{2r}}{r!(1+a-b)_r\,(1+a-c)_r\,(1+a-d)_r\,(a)_{2r}}\, F(e+2r,\, e-a;\, 1+a+2r;\, x), \tag{5}
$$

where it is assumed that $1 + 2a = b + c + d + e$.

It would be nice to find something more elegant than this that is equivalent to Dougall's theorem.

When $x = -1$, we note that formula (5) reduces to

$$
{}_4F_3\left[\begin{matrix} e\,, & 1+a-c-d\,, & 1+a-b-d\,, & 1+a-b-c\,; & -1 \\ & 1+a-b\,, & 1+a-c\,, & 1+a-d & \end{matrix}\right]
$$

$$
= 2^{-e}\, \frac{\Gamma(1+a)\,\Gamma(\frac{1}{2})}{\Gamma(1+a-\frac{e}{2})\Gamma(\frac{1}{2}+\frac{e}{2})} \cdot
$$

$$
\cdot\, {}_6F_5\left[\begin{matrix} a\,, & 1+a/2\,, & b\,, & c\,, & d\,, & e/2\,; & -1 \\ & a/2\,, & 1+a-b\,, & 1+a-c\,, & 1+a-d\,, & 1+a-e/2 & \end{matrix}\right] \tag{6}
$$

This formula is not given explicitly in Bailey [1]; however formula (2), page 28, of the tract gives us

$$
{}_6F_5\left[\begin{matrix} a\,, & 1+a/2\,, & b\,, & c\,, & d\,, & e\,; & -1 \\ & a/2\,, & 1+a-b\,, & 1+a-c\,, & 1+a-d\,, & 1+a-e & \end{matrix}\right]
$$

$$= \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}\ {}_3F_2\left[\begin{matrix}1+a-b-c, & d, & e; & 1\\ & 1+a-b, & 1+a-c & \end{matrix}\right] \qquad (7)$$

so there are ways to further modify our formula (6).

Saalschütz's formula says that

$${}_3F_2\left[\begin{matrix}a, & b, & -n & ; & 1\\ & c, & 1+a+b-c-n & & \end{matrix}\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}. \qquad (8)$$

When $n \to \infty$ it reduces to Gauss's theorem

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad (9)$$

valid for $R(c - a - b) > 0$.

When $a = -n$, then (9) is nothing other than the standard Vandermonde convolution. In fact, many binomial coefficient summation identities, such as are listed by Gould in [3] are consequences of the Saalschütz formula. Saalschütz's formula is given in binomial coefficient form as formula (11.1) in [3] and the formula of Gauss is formula (7.1) there.

References

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