Super Edge-Magic Labelings of Generalized Petersen Graphs $P(n, 3)$ *

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Abstract

A graph G is called super edge-magic if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(u) + f(v) + f(uv) = C$ is a constant for any $uv \in E(G)$ and $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. Yasuhiro Fukuchi proved that the generalized Petersen graph $P(n, 2)$ is super edge-magic for odd $n \geq 3$. In this paper, we show that the generalized Petersen graph $P(n, 3)$ is super edge-magic for odd $n \geq 5$.
Keywords: *super edge-magic labeling, petersen graph, vertex labeling, edge labeling*

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1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $p = |V(G)|$, $q = |E(G)|$ be the number of vertices and edges of G, respectively. A bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p+q\}$ is called an edge-magic labeling of G, if there exists a constant C called the valence of f, such that $f(u)+f(v)+f(uv) = C$ for any edge $uv \in E(G)$. An edge-magic labeling f of G is called a super edge-magic labeling if $f(V(G)) = \{1, 2, \ldots, p\}$. We say that G is super edge-magic if there exists a super edge-magic labeling of G.

Kotzig and Rosa[4] introduced the notion of edge-magic labelings (in [4], edge-magic labelings are called magic valuations). They proved that complete bipartite graphs, cycles and caterpillars are edge-magic, and that the complete graph K_n is edge-magic if and only if $n = 1, 2, 3, 5$ or 6. They also conjectured that trees are edge-magic. Enomoto, Lladó, Nakamigawa and Ringel [1] introduced the notion of super edge-magic labelings. They proved that the cycle C_n is super edge-magic if and only if n is odd, and that the complete bipartite graph $K_{m,n}$ is super edge-magic if and only if $m = 1$ or $n = 1$, and that the complete graph K_n is super edge-magic if and only if $n = 1, 2$ or 3. They also conjectured that trees are super edge-magic. In addition, they proved that if $n \equiv 0(\mod 4)$, then the wheel graph W_n of order n is not edge-magic.

For the literature on super edge-magic graphs we refer to [3] and the relevant references given in it.

Let n, k be integers such that $n \geq 3$, $1 \leq k < n$ and $n \neq 2k$. For such n, k, the generalized Petersen graph $P(n, k)$ is defined by

$$V(P(n,k)) = \{ \ v_i \ \mid \ 1 \leq i \leq 2n \ \},$$
$$E(P(n,k)) = \{ \ v_i v_{1+(i \bmod n)} \ , \ v_i v_{i+n} \ , \ v_{n+i} v_{n+1+((i+k-1) \bmod n)} \ \mid \ 1 \leq \ i \leq n \ \}.$$

Now, we introduce some necessary conditions for a graph to be super edge-magic.

Enomoto et al. [1], Yasuhiro Fukuchi et al. [2] and F.M Figueroa-Centeno et al. [5] proved the following useful lemmas:

Lemma 1.1([1], *Lemma* 2.1) If G is super-edge-magic, then $|E(G)| \leq 2|V(G)| - 3$.

Lemma 1.2([2], *Lemma* 2) Let r be an odd integer. Let p be an integer, and let G be an r-regular graph such that $|V(G)| = p$.
(1)If $p \equiv 4(mod\ 8)$, then G is not edge-magic.
(2)If $p \equiv 0(mod\ 4)$, then G is not super edge-magic.

Lemma 1.3([5], *Lemma* 4) If G is an r-regular super edge-magic (p, q)-graph, where $r > 0$, then q is odd and the valence of any super edge-magic labeling of G is $(4p + q + 3)/2$.

It follows from *Lemma* 1.1 that if an r-regular graph is super edge-magic, then $r \leq 3$. Since generalized Petersen graphs $P(n, k)$ form an important class of 3-regular graphs, it is desirable to determine which of the $P(n, k)$ are super edge-magic.

$P(n, k)$ is a 3-regular graph with $2n$ vertices and $3n$ edges, as a corollary to *Lemma*1.1 – 1.3, we get the following result:

Corollary 1.4 If $P(n, k)$ is a super edge-magic graph, then n is odd and the valence of any super edge-magic labeling of $P(n, k)$ is $(11n + 3)/2$.

Yasuhiro Fukuchi [2] proved that $P(n, 2)$ is super edge-magic for odd $n \geq 3$. In this paper, we show that $P(n, 3)$ is super edge-magic for odd $n \geq 5$.

2 Statement of the Main Result

Theorem 2.1 $P(n, 3)$ is super edge-magic for odd $n \geq 5$.

Proof. Let $C = (11n + 3)/2$. We define a function :

$$f : V(P(n, 3)) \cup E(P(n, 3)) \to \{1, 2, \ldots, 5n\}$$

according to following three cases:

Case 1: $n \equiv 1\ mod\ 4$.

We label the vertices as follows:

$$f(v_i) = \begin{cases} 2n-(i-1)/2, & 1 \le i \le n \ \wedge \ i \bmod 2 = 1, \\ (3n+1)/2 - i/2, & 2 \le i \le n-1 \ \wedge \ i \bmod 2 = 0, \\ n-1-i/2, & n+1 \le i \le 2n-4 \ \wedge \ i \bmod 4 = 2, \\ (3n-i)/2+1, & n+2 \le i \le 2n-3 \ \wedge \ i \bmod 2 = 1, \\ n+3-i/2, & n+3 \le i \le 2n-2 \ \wedge \ i \bmod 4 = 0, \\ (n+1)/2, & i = 2n-1, \\ 2, & i = 2n. \end{cases}$$

And the edges as follows:

$$f(v_iv_j) = C - (f(v_i) + f(v_j)).$$

Firstly, we show that f is a bijective mapping from $V(G)$ onto $\{1, 2, \ldots, 2n\}$.

Denote by

$$S = \{ f(v_i) \mid 1 \le i \le 2n \}.$$

Then,

$$\begin{aligned} S_1 &= \{2n-(i-1)/2 \mid 1 \le i \le n \ \wedge \ i \bmod 2 = 1\} \\ &= \{2n, 2n-1, \ldots, (3n+1)/2\}, \\ S_2 &= \{(3n+1)/2 - i/2 \mid 2 \le i \le n-1 \ \wedge \ i \bmod 2 = 0\} \\ &= \{(3n+1)/2-1, (3n+1)/2-2, \ldots, n+1\}, \\ S_3 &= \{n-1-i/2 \mid n+1 \le i \le 2n-4 \ \wedge \ i \bmod 4 = 2\} \\ &= \{(n-3)/2, (n-7)/2, \ldots, 3, 1\}, \\ S_4 &= \{(3n-i)/2+1 \mid n+2 \le i \le 2n-3 \ \wedge \ i \bmod 2 = 1\} \\ &= \{n, n-1, \ldots, (n+3)/2+1\}, \\ S_5 &= \{n+3-i/2 \mid n+3 \le i \le 2n-2 \ \wedge \ i \bmod 4 = 0\} \\ &= \{(n+3)/2, (n-1)/2, \ldots, 4\}, \\ S_6 &= \{(n+1)/2 \mid i = 2n-1\} = \{(n+1)/2\}, \\ S_7 &= \{2 \mid i = 2n\} = \{2\}. \end{aligned}$$

Hence, $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7$ is the set of labels of all vertices, and

$$\begin{aligned} & S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6 \cup S_3 \cup S_7 \\ &= \{2n, 2n-1, 2n-2, \ldots, (3n+1)/2, (3n+1)/2-1, (3n+1)/2-2, \\ &\quad \ldots, n+1,\ n,\ n-1, \ldots, (n+3)/2+1,\ (n+3)/2,\ (n+1)/2, \\ &\quad (n-1)/2, (n-3)/2, \ldots, 4,\ 3,\ 2,\ 1\} \\ &= \{\ 2n,\ 2n-1, \ldots,\ 2,\ 1\}. \end{aligned}$$

It is clear that the labels of the vertices are different. So, f is a bijection from $V(G)$ onto $\{1, 2, \ldots, 2n\}$.

Secondly, we show that f is a bijective mapping from $E(G)$ onto $\{2n+1, 2n+2, \ldots, 5n\}$. Denote by

$$D \;=\; \{\, f(v_iv_j) \;\mid v_iv_j \in E(P(n,3))\}.$$

Let $D = D_1 \cup D_2 \cup D_3$, where

$$\begin{aligned}
D_1 &= \{f(v_iv_{1+(i \bmod n)}) \mid 1 \le i \le n\} = D_{11} \cup D_{12},\\
D_{11} &= \{f(v_iv_{1+(i \bmod n)}) \mid 1 \le i \le n-1\}\\
&= \{(11n+3)/2 - (7n+1)/2 + i \mid 1 \le i \le n-1\}\\
&= \{2n+2, 2n+3, \ldots, 3n\},\\
D_{12} &= \{f(v_iv_{1+(i \bmod n)}) \mid i = n\}\\
&= \{(11n+3)/2 - 4n + (n-1)/2\} = \{2n+1\},\\
D_2 &= \{f(v_iv_{i+n}) \mid 1 \le i \le n\}\\
&= D_{21} \cup D_{22} \cup D_{23} \cup D_{24} \cup D_{25},\\
D_{21} &= \{f(v_iv_{i+n}) \mid 1 \le i \le n-4 \;\wedge\; i \bmod 4 = 1\}\\
&= \{(11n+3)/2 - (5n-1)/2 + i \mid 1 \le i \le n-4 \wedge i \bmod 4 = 1\}\\
&= \{3n+3, 3n+7, \ldots, 4n-2\},\\
D_{22} &= \{f(v_iv_{i+n}) \mid 2 \le i \le n-3 \;\wedge\; i \bmod 2 = 0\}\\
&= \{(11n+3)/2 - (5n+3)/2 + i \mid 2 \le i \le n-3 \wedge i \bmod 2 = 0\}\\
&= \{3n+2, 3n+4, \ldots, 4n-3\},\\
D_{23} &= \{f(v_iv_{i+n}) \mid 3 \le i \le n-2 \;\wedge\; i \bmod 4 = 3\}\\
&= \{(11n+3)/2 - (5n+7)/2 + i \mid 3 \le i \le n-2 \wedge i \bmod 4 = 3\}\\
&= \{3n+1, 3n+5, \ldots, 4n-4\},\\
D_{24} &= \{f(v_iv_{i+n}) \mid i = n-1\}\\
&= \{(11n+3)/2 - (3n+3)/2 \mid i = n-1\} = \{4n\},\\
D_{25} &= \{f(v_iv_{i+n}) \mid i = n\}\\
&= \{(11n+3)/2 - 2n + (i-1)/2 - 2 \mid i = n\} = \{4n-1\},\\
D_3 &= \{f(v_iv_{n+1+((i+2) \bmod n)}) \mid n+1 \le i \le 2n\}\\
&= D_{31} \cup D_{32} \cup D_{33} \cup D_{34} \cup D_{35},\\
D_{31} &= \{f(v_iv_{n+1+((i+2) \bmod n)}) \mid n+1 \le i \le 2n-4 \wedge i \bmod 4 = 2\}\\
&= D_{311} \cup D_{312},
\end{aligned}$$

$$
\begin{aligned}
D_{311} &= \{f(v_i v_{n+1+((i+2) \bmod n)}) \mid n+1 \le i \le 2n-8 \wedge i \bmod 4 = 2\} \\
&= \{(11n+3)/2 - (5n-3)/2 + i \mid n+1 \le i \le 2n-8 \\
&\quad \wedge\ i \bmod 4 = 2\} \\
&= \{4n+4, 4n+8, \ldots, 5n-5\}, \\
D_{312} &= \{f(v_i v_{n+1+((i+2) \bmod n)}) \mid i = 2n-4\} \\
&= \{(11n+3)/2 - (n+3)/2 \mid i = 2n-4\} = \{5n\}, \\
D_{32} &= \{f(v_i v_{n+1+((i+2) \bmod n)}) \mid n+2 \le i \le 2n-3 \wedge i \bmod 4 = 3\} \\
&= D_{321} \cup D_{322}, \\
D_{321} &= \{f(v_i v_{n+1+((i+2) \bmod n)}) \mid n+2 \le i \le 2n-7 \wedge i \bmod 4 = 3\} \\
&= \{(11n+3)/2 - (5n-3)/2 + i \mid n+2 \le i \le 2n-7 \\
&\quad \wedge\ i \bmod 4 = 3\} \\
&= \{4n+5,\ 4n+9, \ldots,\ 5n-4\}, \\
D_{322} &= \{f(v_i v_{n+1+((i+2) \bmod n)}) \mid i = 2n-3\} \\
&= \{(11n+3)/2 - (n+9)/2 \mid i = 2n-3\} = \{5n-3\}, \\
D_{33} &= \{f(v_i v_{n+1+((i+2) \bmod n)}) \mid n+3 \le i \le 2n-6 \wedge i \bmod 4 = 0\} \\
&= \{(11n+3)/2 - (5n+5)/2 + i \mid n+3 \le i \le 2n-6 \\
&\quad \wedge\ i \bmod 4 = 0\} \\
&= \{4n+2,\ 4n+6,\ \ldots,\ 5n-7\}, \\
D_{34} &= \{f(v_i v_{n+1+((i+2) \bmod n)}) \mid n+4 \le i \le 2n-5 \wedge i \bmod 4 = 1\} \\
&= \{(11n+3)/2 - (5n+5)/2 + i \mid n+4 \le i \le 2n-5 \\
&\quad \wedge\ i \bmod 4 = 1\} \\
&= \{4n+3,\ 4n+7,\ \ldots,\ 5n-6\}, \\
D_{35} &= \{f(v_i v_{n+1+((i+2) \bmod n)}) \mid i = 2n-2\} \\
&= \{f(v_{2n-2} v_{n+1}) \mid i = 2n-2\} = \{(11n+3)/2 - (n+5)/2\} \\
&= \{5n-1\}, \\
D_{36} &= \{f(v_i v_{n+1+((i+2) \bmod n)}) \mid i = 2n-1\} \\
&= \{f(v_{2n-1} v_{n+2}) \mid i = 2n-1\} = \{(11n+3)/2 - (3n+1)/2\} \\
&= \{4n+1\}, \\
D_{37} &= \{f(v_i v_{n+1+((i+2) \bmod n)}) \mid i = 2n\} \\
&= \{f(v_{2n} v_{n+3}) \mid i = 2n\} = \{(11n+3)/2 - (n+7)/2\} \\
&= \{5n-2\}.
\end{aligned}
$$

Hence, $D = D_1 \cup D_2 \cup D_3$ is the set of labels of all edges, and

$$\begin{aligned}
D = \;& D_1 \cup D_2 \cup D_3 \\
= \;& D_{11} \cup D_{12} \cup D_{21} \cup D_{22} \cup D_{23} \cup D_{24} \cup D_{25} \cup D_{31} \cup D_{32} \cup D_{33} \\
& \cup D_{34} \cup D_{35} \cup D_{36} \cup D_{37} \\
= \;& D_{12} \cup D_{11} \cup D_{23} \cup D_{22} \cup D_{21} \cup D_{25} \cup D_{24} \cup D_{36} \cup D_{33} \cup D_{34} \\
& \cup D_{311} \cup D_{321} \cup D_{322} \cup D_{37} \cup D_{35} \cup D_{312} \\
= \;& \{2n+1,\ 2n+2, 2n+3, \ldots, 3n,\ 3n+1, 3n+5, \ldots, 4n-4, \\
& 3n+2, 3n+4, \ldots, 4n-3, 3n+3, 3n+7, \ldots, 4n-2,\ 4n-1, \\
& 4n,\ 4n+1,\ 4n+2, 4n+6, \ldots, 5n-7,\ 4n+3, 4n+7, \ldots, \\
& 5n-6, 4n+4, 4n+8, \ldots, 5n-5,\ 4n+5, 4n+9, \ldots, 5n-4, \\
& 5n-3,\ 5n-2,\ 5n-1,\ 5n\} \\
= \;& \{2n+1,\ 2n+2,\ \ldots,\ 5n\}.
\end{aligned}$$

It is clear that the labels of each edge are distinct, and the edge labels are $\{2n+1, 2n+2, \ldots, 5n\}$. According to the definition of super edge-magic labeling, we thus conclude that $P(n, 3)$ is super edge-magic for $n \equiv 1 \bmod 4$.

Case 2: $n \equiv 3 \bmod 8$.

We label the vertices as follows:

$$f(v_i) = \begin{cases}
2n - (i-1)/2, & 1 \le i \le n \wedge i \bmod 2 = 1, \\
(3n+1)/2 - i/2, & 2 \le i \le n-1 \wedge i \bmod 2 = 0, \\
(n-5)/2, & i = n+1, \\
4, & i = n+2, \\
n+2-i/2, & n+3 \le i \le 2n-8 \wedge i \bmod 4 = 2, \\
2 + (3n-i)/2, & n+4 \le i \le 2n-3 \wedge i \bmod 2 = 1, \\
(n+5)/2, & i = n+5, \\
n-2-i/2, & n+9 \le i \le 2n-10 \\
& \wedge\ i \bmod 8 = 4 \wedge n \ge 19, \\
n+6-i/2, & n+13 \le i \le 2n-6 \\
& \wedge\ i \bmod 8 = 0 \wedge n \ge 19, \\
2, & i = 2n-4, \\
5, & i = 2n-2, \\
(n+3)/2, & i = 2n-1, \\
1, & i = 2n.
\end{cases}$$

And the edges as follows:

$$f(v_i v_j) = C - (f(v_i) + f(v_j)).$$

By a proof similar to the one in Case 1, we have that this assignment provides a super edge-magic labeling for $P(n,3)$ with $n \equiv 3 \bmod 8$.

Case 3: $n \equiv 7 \bmod 8$.

For $n = 7$, we give a vertex and edge labeling of $P(7,3)$ shown in Figure 2.1.

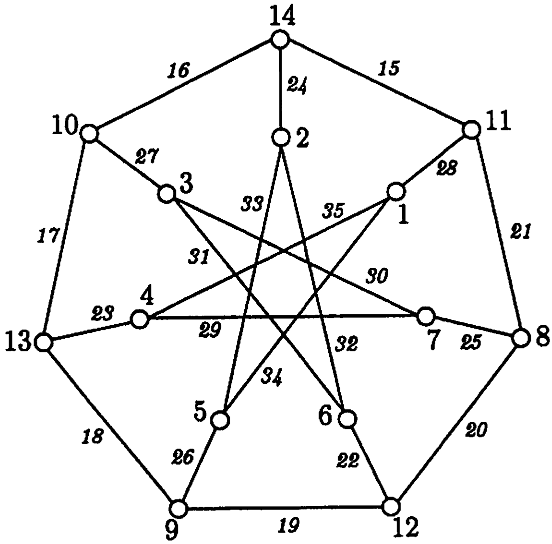

Figure 2.1 : The super edge-magic labeling of the graph $P(7,3)$.

According to the definition of super edge-magic labeling, it is clear that this assignment provides a super edge-magic labeling for $P(7,3)$.

For $n \geq 15$, we label the vertices as follows:

$$f(v_i) = \begin{cases} 2n-(i-1)/2, & 1 \leq i \leq n \wedge i \bmod 2 = 1, \\ (3n+1)/2 - i/2, & 2 \leq i \leq n-1 \wedge i \bmod 2 = 0, \\ (n-5)/2, & i = n+1, \\ 3, & i = n+2, \\ n+2-i/2, & n+3 \leq i \leq 2n-4 \wedge i \bmod 4 = 2, \\ 2+(3n-i)/2, & n+4 \leq i \leq 2n-5 \wedge i \bmod 2 = 1, \\ (n+5)/2, & i = n+5, \\ n-2-i/2, & n+9 \leq i \leq 2n-14 \\ & \wedge\ i \bmod 8 = 0 \wedge n \geq 23, \\ n+6-i/2, & n+13 \leq i \leq 2n-10 \\ & \wedge\ i \bmod 8 = 4 \wedge n \geq 23, \\ 2, & i = 2n-6, \\ (9-3n)/2+i, & 2n-3 \leq i \leq 2n-1 \wedge i \bmod 2 = 1, \\ 7, & i = 2n-2, \\ 1, & i = 2n. \end{cases}$$

And the edges as follows:

$$f(v_iv_j) = C - (f(v_i) + f(v_j)).$$

By a proof similar to the one in Case 1, we have that this assignment provides a super edge-magic labeling for $P(n, 3)$ with $n \equiv 7 \bmod 8$.

According to the proof of Case 1, Case 2 and Case 3, we thus conclude that $P(n, 3)$ is super edge-magic for odd $n \geq 5$.

In Figure 2.2, 2.3 and 2.4, we show our super edge-magic labelings for $P(17, 3)$, $P(19, 3)$ and $P(23, 3)$.

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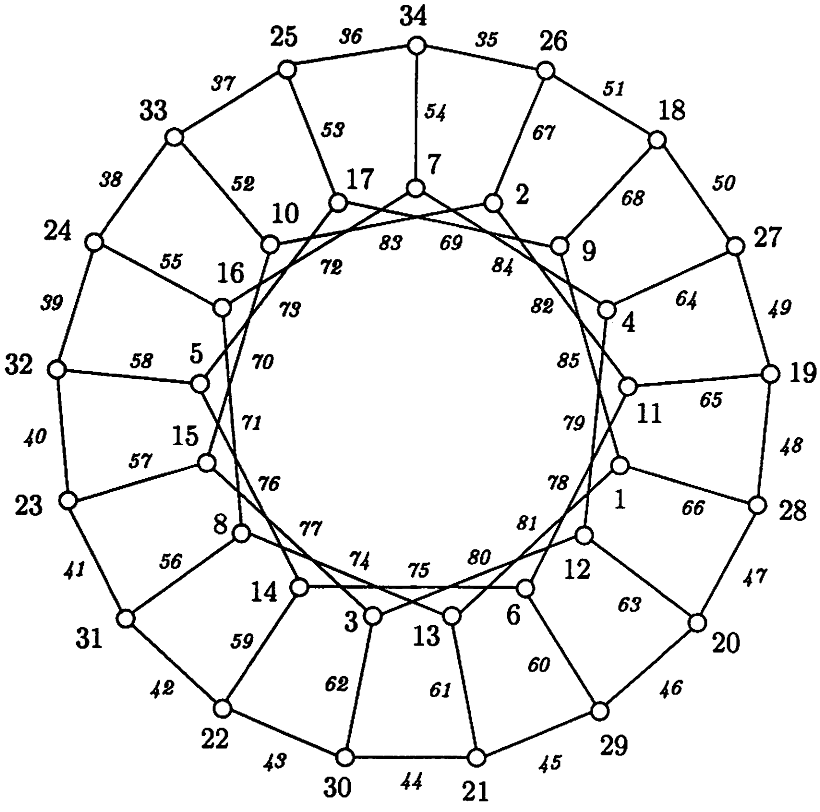

Figure 2.2 : The super edge-magic labeling of the graph $P(17, 3)$.

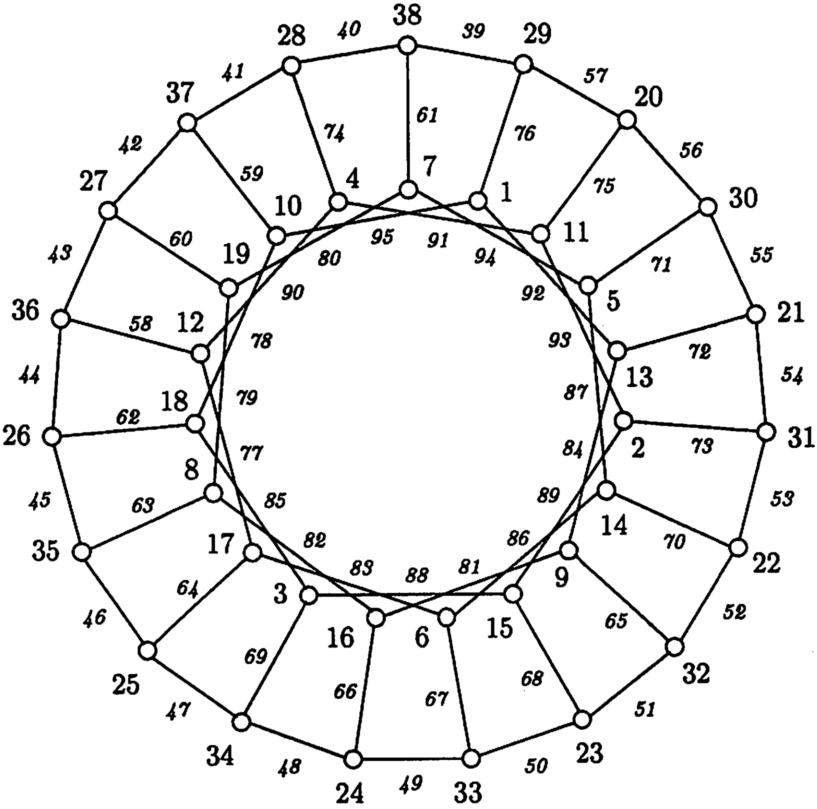

Figure 2.3 : The super edge-magic labeling of the graph $P(19,3)$.

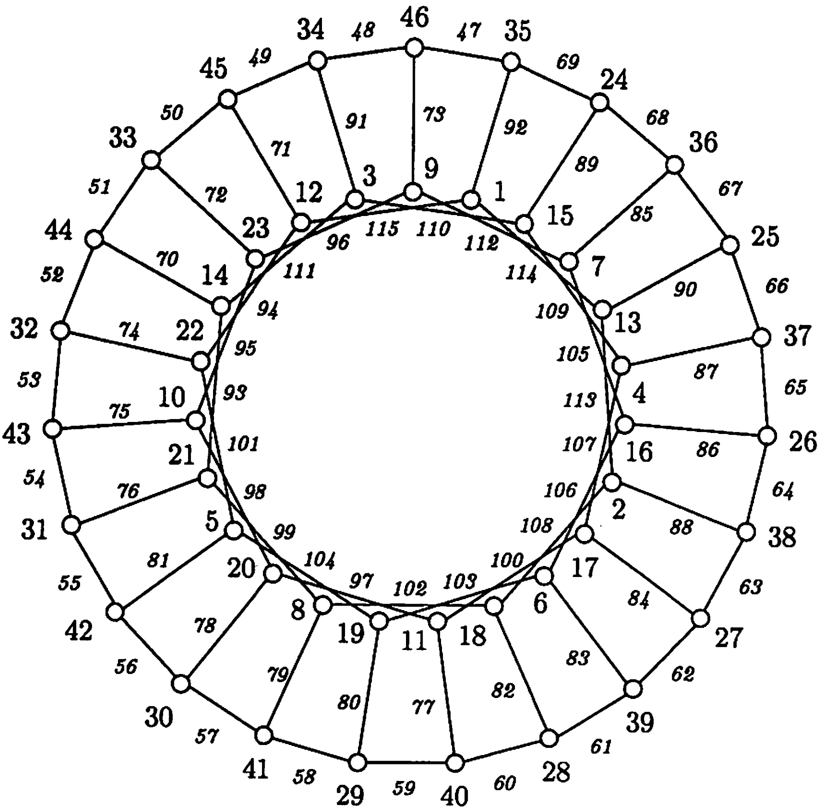

Figure 2.4 : The super edge-magic labeling of the graph $P(23,3)$.