Resonance of Plane Bipartite Graphs

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Abstract: A graph G is called resonant if the boundary of each face of G is an F-alternating closed trail with respect to some f-factor F of G. We show that a plane bipartite graph G is resonant if and only if it is connected and each edge of G is contained in an f-factor and not in another f-factor.

Keywords : Plane bipartite graph; Factor-covered graph; Factor-deleted graph; Resonance

In this paper, we consider the resonance of bipartite graphs. The concept for resonance originate from chemical significance of hexagonal systems and was extended to plane bipartite graphs. A *hexagonal system* [11] is a special connected plane graph without cut vertices, each interior face of which is surrounded by a regular hexagon of side length one. Some elegant characterizations for hexagonal system to be resonant were obtained in [12, 13] and there is a survey [15] about this topic. For plane elementary bipartite graphs, it also can be described by resonant faces [14].

We take the basic terminology from [7]. The graphs in this paper will be simple and finite. Let G be a plane bipartite graph with vertex set $V(G)$ and edge set $E(G)$. By a plane graph G we mean an embedding of a planar graph in the plane. This plane graph decomposes the plane into a number of open regions called *faces*. A bipartite graph means a graph for which the vertices are colored by white or black so that two adjacent vertices receive different colors.

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For a vertex x of graph G, the degree of x is denoted by $d_G(x)$. Let g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. A *(g, f)-factor* of G is a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. In particular, a (f, f)-factor is called an *f-factor*. If f is a constant function taking the value k, then an f-factor is said to be a *k-factor*. A *1-factor* of G is also a *perfect matching*. In 1952, Tutte [10] gave a criterion for the existence of an f-factor of a graph. A necessary and sufficient conditions for existence of a (g, f)-factor was obtained by Lovász [8] in 1970. Little [6] introduced the concept of factor-covered graph. Liu [2, 3] extended this concept to general situation. For a graph G with an f-factor is called *f-factor-covered* if each edge of G is contained in some f-factor. G is called *f-factor-deleted* if $G-e$ contains an f-factor for every edge e of G.

We now extend some concepts concerning 1-factor to f-factor, such as "allowed", "elementary" and "resonance" etc.. If $f \equiv 1$, then these concepts express the general meanings. Let G be a graph with an f-factor. An edge of G is called *f-allowed* if it lies in some f-factor of G and *f-forbidden* otherwise. An *f-fixed single edge* of G is the edge that there is no f-factor of G containing it. And an edge e of G is called *f-fixed double edge* if e belongs to all f-factors of G. G is said to be *f-elementary* if all its f-allowed edges form a connected subgraph of G. If G is 1-elementary, we always say that G is *elementary*. As early as in 1915, König [5] had employed this concept in studying the decomposition of a determinant. After nearly half a century, Hetyei [4] formally used the term "elementary" for this concept and obtained various properties of elementary bipartite graphs. Let F be an f-factor of G. A closed trail P of G is called *F-alternating* if the edges of P appear alternately in $E(F)$ and $E(G)\backslash E(F)$. A face φ of G is said to be *resonant* if G has an f-factor F such that the boundary of φ is an F-alternating closed trail. G is called *resonant* if each face of G is resonant.

From [14] we know the following result: Let G be a plane bipartite graph with more than two vertices, then each face of G is resonant if and only if G is elementary. But for a plane bipartite graph G with f-factors(where $f \not\equiv 1$), the conditions of f-elementary is not enough for G to be resonant. For example(see Fig. 1), there is no f-factor such that the faces φ_1 or φ_2 are resonant.

The symmetric difference of two finite sets A and B is defined by $A \bigoplus B := (A \bigcup B)\backslash(A \bigcap B)$. This binary operation is associative and commutative. Let M_1 and M_2 be two different perfect matchings of G. Then the symmetric difference $M_1 \bigoplus M_2$ consists of mutually disjoint (M_1, M_2)-alternating cycles. But this is different from f-factors.

Lemma 1[9]. *Let G be a connected graph with all its vertices in even*

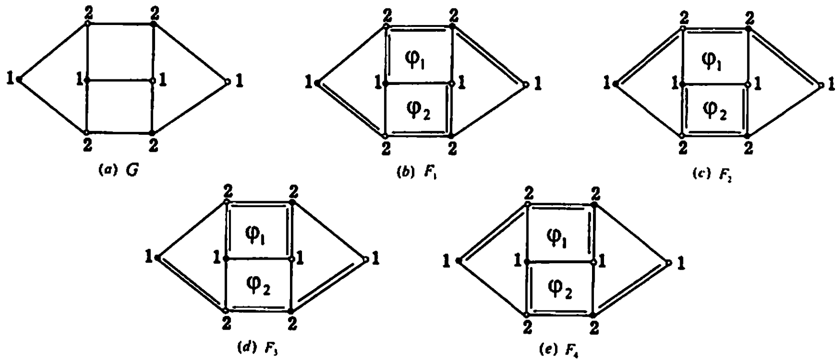

Fig. 1. Bipartite graph G and its f-factors

degree, and G has a 2-edge coloring (not necessary proper) with blue and red colors. If the number of red-color-edges equals to the number of blue-color-edges for every vertex of G, then G has an (red, blue)-alternating Euler tour.

Lemma 2. *If F_1 and F_2 are two different f-factors of G, then the connected components of $G' = (V, E(F_1) \bigoplus E(F_2))$ are (F_1, F_2)-alternating closed trails.*

Proof: Color the edges of F_1 and F_2 with red and blue respectively. For a vertex x of G, if there are $k(0 \leq k \leq f(x))$ edges incident with x which lie in both F_1 and F_2, then $d_{G'}(x) = 2(f(x) - k)$. Thus the number of red edges equals to the number of blue edges for every vertex of G'. By Lemma 1, the connected components of G' are (F_1, F_2)-alternating closed trails. □

Lemma 3[1]. *Let F be an f-factor of a bipartite graph G and P an alternating closed trail relative to F. Then P is the edge-disjoint union of alternating cycles relative to F.*

Corollary 4. *Let F be an f-factor of a bipartite graph G and P an alternating closed trail relative to F. Then the edges which do not belong to any cycle of P form an F-alternating path P'.*

Proof: By Lemma 3, the cycles of P are F-alternating. Assume that P' is not an F-alternating path. Then there are two adjacent edges e_1 and e_2 of P' such that $e_i \in F$ or $e_i \notin F (i = 1, 2)$. u is the same end of e_1 and e_2. Let P_1 be the F-alternating closed trail of P using u as its initial and terminal vertices, and $e_i \notin P_1$. Then $P_2 = e_1 P_1 e_2$ or $P_2 = e_2 P_1 e_1$ is not an F-alternating trail of P, which contradicts $P_2 \subseteq P$. □

Lemma 5[14]. *Let M be a perfect matching of a graph G and C an M-alternating cycle of G. Then $M \bigoplus E(C)$ is also a perfect matching of*

G and C is an $(M \bigoplus E(C))$*-alternating cycle of* G.

For f-factors, we have the similar result below.

Lemma 6. *Let* F *be an* f*-factor of a bipartite graph* G *and* P *an* F*-alternating closed trail of* G*. Then* $F \bigoplus E(P)$ *is also an* f*-factor of* G *and* P *is an* $(F \bigoplus E(P))$*-alternating closed trail of* G.

Proof: The result is obvious by using Lemmas 3 and 5. □

If a bipartite graph G is elementary, then G is 2-edge-connected. That is, G has no cut-edge. Conversely, for a bipartite graph G with perfect matchings, if G has a cut-edge, then the cut-edge is certainly forbidden, and G is certainly non-elementary. But for f-factors, this is different.

Lemma 7. *Assume that a bipartite graph* G *is* f*-elementary. If* G *has cut-edges, then all the cut-edges of* G *are* f*-fixed double edges.*

Proof: At first we show that all the cut-edges of G are allowed. Otherwise, suppose a cut-edge e is a fixed single edge. Then the union of all f-factors in G forms a disconnected subgraph of G, which contradicts that G is f-elementary.

Further we show that a cut edge ab is a fixed double edge. Otherwise ab is not a fixed double edge. Then there is an f-factor F of G such that $ab \notin F$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two connected components of $G - ab$. Then $F_1 := F \bigcap G_1$ is an f-factor of G_1. We have

$$\sum_{x \in V(G_1)} f(x) = \sum_{x \in V(G_1)} d_{F_1}(x) \equiv 0 (mod\ 2).$$

Since ab is an allowed edge, there is an f-factor F' of G such that $ab \in F'$. Then

$$\sum_{x \in V(G_1)} f(x) = \sum_{x \in V(G_1)} d_{F'}(x) = \sum_{x \in V(G_1)} d_{F' \cap G_1}(x) + 1.$$

So

$$\sum_{x \in V(G_1)} d_{F' \cap G_1}(x) \equiv 1 (mod\ 2),$$

contradicting the Handshaking Lemma. The theorem is proved. □

Corollary 8. *If a connected graph* G *is both* f*-factor-covered and* f*-factor-deleted, then* G *is 2-edge-connected.*

Proof: If G has a cut-edge e, by Lemma 7 e is a f-fixed double edge since G is f-elementary. Hence $G - e$ has no f-factor, which contradicts that G is f-factor-deleted. □

Lemma 9. *Suppose a graph* G *is both* f*-factor-covered and* f*-factor-deleted, and* e *is an edge of an* f*-factor* F *of* G*. Then there are an edge* e' *adjacent to* e *and an* f*-factor* F' *of* G *such that* $e' \notin F$, $e \notin F'$ *and* $e' \in F'$.

Proof: Let u be an endpoint of e. Assume that $e_1, e_2, \cdots, e_k$ are the edges incident with u but not belong to F. Because G is f-factor-deleted, we have $k = d_G(u) - f(u) \geqslant 1$ and $G - e$ has an f-factor F' such that F' contains some e_i. Otherwise, only $f(u) - 1$ edges incident with u belonging to F'. The result is proved. □

Theorem 10. *A connected plane bipartite graph G is both f-factor-covered and f-factor-deleted if and only if each face of G is resonant.*

Proof: Assume that each face of G is resonant. For any edge e of G, let P be a boundary of the faces containing e. Then there is an f-factor F such that P is an F-alternating closed trail, and by Lemma 6 $F \bigoplus E(P)$ is another f-factor. If $e \in F$, then $e \notin F \bigoplus E(P)$; if $e \notin F$, then $e \in F \bigoplus E(P)$. Hence G is both f-factor-covered and f-factor-deleted.

Conversely, assume that G is both f-factor-covered and f-factor-deleted. Let φ be a face of G. We will show that φ is resonant. For an edge e on the boundary of φ, there are two f-factors F_1 and F_2 such that $e \in F_1$ and $e \notin F_2$. By Lemmas 2 and 3, there is an (F_1, F_2)-alternating cycle C of $F_1 \bigoplus F_2$ such that $e \in E(C)$. Without loss of generality, assume that φ lies in the interior of C. Denote by $I[C]$ the subgraph of G consisting of C together with the interior. Obviously, $I[C]$ is connected. We now show that the every face of $I[C]$ is resonant by induction on the number m of edges contained in the interior of C.

If $m = 0$, the result is trivially true.

Now we suppose that the result is true for $0 \leq m \leq k$. That is: if the number of edges in the interior of an f-alternating cycle C is no more than k, then the every face of $I[C]$ is resonant. On the basis of this, we give the proof of $m = k + 1$ in the following.

In the interior of C, let e_1 be an edge with an endpoint on C. If the two ends of e_1 are both on C, then C and e_1 form two cycles C_1 and C_2. Whenever e_1 belongs to F_1(or F_2) or not, C_1 and C_2 are f-alternating cycles relative to some f-factors of G. And the number of edges in the interior of C_1 and C_2 are no more than k respectively. So by the induction hypothesis, each face of $I[C_1]$ and $I[C_2]$ is resonant.

If one of ends of e_1 is in the interior of C, then there are three cases to be considered.

Case 1: $e_1 \in F_1, e_1 \notin F_2$ (see Fig. 2 (a) and (b)).

By Lemmas 2 and 3, $F_1 \bigoplus F_2$ has a cycle C' containing e_1. Let P be an (F_1, F_2)-alternating path of C' such that only end vertices lie on C. If the two end vertices of P are different, then C and P form two cycles C_1 and C_2 such that $C_1 \bigcap C_2 = P$, see Fig.2(a). Without loss of generality, suppose C_1 is an F_1-alternating cycle and C_2 is thus an $F_1 \bigoplus E(C_1)$-alternating cycle. If the two end vertices of P are the same, then we can get that C_1 is an (F_1, F_2)-alternating cycle and C_2 is an (F_1, F_2)-alternating closed

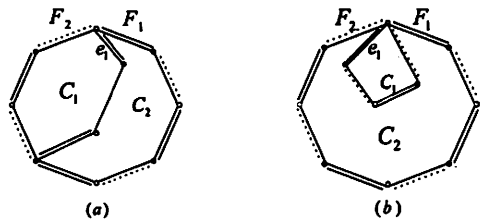

Fig. 2.

trail, see Fig.2(b). By the two narration above, the number of edges in the interior of the C_1 and C_2 are no more than k respectively. So by the induction hypothesis, each face of $I[C_1]$ and $I[C_2]$ is resonant. For $e_1 \notin F_1$ and $e_1 \in F_2$, use the same argument.

Case 2: $e_1 \in F_1 \bigcap F_2$ (see Fig. 3(a)).

By Lemma 9, there is an edge e_2 adjacent to e_1 and an f-factor F_3 such that $e_1 \notin F_3, e_2 \notin F_1, e_2 \in F_3$. Then by Lemma 2, an (F_1, F_3)-alternating closed trail C' of $F_1 \bigoplus F_3$ contains e_1 and e_2. Let P be a part(or the whole) of C' in $I[C]$ such that only end vertices lie on C. If the two end vertices of P are different, then by Corollary 4 we can get an (F_1, F_3)-alternating path by deleting the (F_1, F_3)-alternating cycles of P.

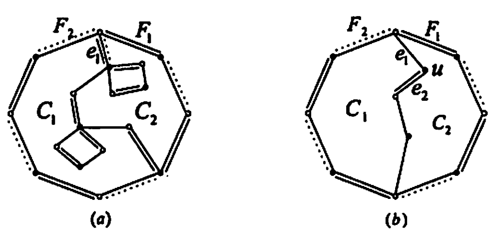

Fig. 3.

If the two end vertices of P are the same, keep an (F_1, F_3)-alternating cycle of P which has a vertex same to the ends of P and delete the other cycles. Then discussed as in Case 1, two cycles C_1 and C_2(or C_1 is a cycle and C_2 is a closed trail) is obtained, which alternate with respect to some f-factors of G. And the number of edges in the interior of the C_1 and C_2 are no more than k respectively. So by the induction hypothesis, each face of $I[C_1]$ and $I[C_2]$ is resonant.

Case 3: $e_1 \notin F_1$, $e_1 \notin F_2$ (see Fig. 3(b)).

Let u be an endpoint of e_1 in the interior of C and F_3 be an f-factor such that $e_1 \in F_3$. Among all the edges incident with u except e_1, there are $f(u)$ edges in F_1 and $f(u)-1$ edges in F_3. Then there are at least one edge e_2 incident with u, such that $e_2 \in F_1$, and $e_2 \notin F_3$. By Lemma 2, there is an alternating closed trail of $F_1 \bigoplus F_3$ containing e_1 and e_2. Similar to those methods used in Case 2, we can get an (F_3, F_1)-alternating path or cycle dividing C into two alternating cycles or closed trails C_1 and C_2. The number of edges in the interior of the C_1 and C_2 are no more than k respectively. So by the induction hypothesis, each face of $I[C_1]$ and $I[C_2]$ is resonant.

Therefore each face of $I[C]$ is resonant over the discussions above, and the theorem is proved. □

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