Constructing families of highly similar but non-isomorphic tournaments

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Abstract

For any integer k, two tournaments T and T', on the same finite set V are *k-similar*, whenever they have the same score vector, and for every tournament H of size k the number of subtournaments of T (resp. T') isomorphic to H is the same. We study the 4-similarity. According to the decomposability, we contruct three infinite classes

of pairs of non-isomorphic 4-similar tournaments.

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1 Basic notions

A *tournament* T_n consists of n vertices $p_1, p_2, \cdots, p_n$ such that each pair of distinct vertices p_i and p_j is joined by one and only one of the oriented arcs (p_i, p_j) or (p_j, p_i). If the arc (p_i, p_j) is in T_n, then we say that p_i *dominates* p_j (symbolically, $p_i \rightarrow p_j$). We set $N^+(p_i) := \{p_j : p_i \rightarrow p_j\}$. Let A be a subset of $V := \{p_1, p_2, \cdots, p_n\}$ and let $x \in V$, if x dominates (resp. is dominated by) all elements of A, we write $x \rightarrow A$ (resp. $A \rightarrow x$). The *score* of p_i is $|N^+(p_i)|$ that is the number s_i of vertices that p_i dominates. The *score vector* of T_n, is the ordered n-tuple $(s_1, s_2, \cdots, s_n)$. We usually label the vertices so that $s_1 \leq s_2 \leq \cdots \leq s_n$. For more details see [7].
Let T be a tournament with V as set of vertices. With each non-empty subset X of V is associated the *subtournament* of T *induced by* X, that is the tournament $T(X)$ having X as set of vertices and the arcs (a, b) of T, where $a, b \in X$, as arcs. A sequence of arcs of the type $(a, b), (b, c), \cdots, (p, q)$ determines a *path* $P(a, q)$ from a to q. If the vertices $a, b, c, \cdots, q$ are all different and the arc (q, a) is in the tournament, then the arcs in $P(a, q)$ plus the arc (q, a) determine a *cycle* denoted $a \rightarrow b \rightarrow c \rightarrow \cdots \rightarrow q \rightarrow a$, or $(a, b, c, \cdots, p, q)$. The *length* of a path or a cycle is the number of arcs it contains. A *k-cycle* is a cycle of length k. It is well-known that, up to isomorphism, there is exactly one tournament with 4 vertices forming a 4-cycle, such a tournament is called a *4-cycle tournament*; hence in any tournament the number of 4-cycles and the number of 4-cycle subtournaments are the same (the 4-cycles are considered up to a circular permutation). The tournament T is *transitive* if, whenever u dominates v, and v dominates w, then u dominates w. A *chain* of T is a transitive subtournament of T. A *k-chain* is a chain with k vertices. A subset I of V is an *interval* of T, if for every $a, b \in I$, and every $x \in V \setminus I$, we have $a \rightarrow x$ iff $b \rightarrow x$. Clearly, the empty set, V and the singletons of V are intervals of T, called *trivial intervals*. The tournament T is said to be *indecomposable* whenever all its intervals are trivial, in the contrary T is said to be *decomposable*. Let $x \in V$ and A be a set disjoint from V. Let $T(A)$ be a tournament with A as set of vertices. We say that we *dilate* x by $T(A)$ if we replace x by $T(A)$, and for all $z \in V \setminus \{x\}$, if $z \rightarrow x$ then $z \rightarrow A$, and if $x \rightarrow z$ then $A \rightarrow z$; we obtain a new tournament T', and we say that T' is obtained from T *by*

dilatation, note that A is an interval of T'. A *diamond* D of T is a subtournament of T with 4 vertices, having only one 3-cycle, note that this 3-cycle dominates or is dominated by a vertex α of T called *the principal vertex* of D; if α dominates the 3-cycle, we say that D is a *positive diamond*, if α is dominated by the 3-cycle, we say that D is a *negative diamond*. Note that whenever T is a tournament with two vertices a, b, where $a \to b$, if we dilate b by a 3-cycle, we obtain a positive diamond and if we dilate a by a 3-cycle, we obtain a negative diamond. For vertices a, b, c of T, we say that a *separates* b and c if $b \to a \to c$ or $c \to a \to b$. Let T and T' be two tournaments with sets of vertices V and V', respectively. An *isomorphism* from T onto T' is a one-to-one correspondence f from V onto V' such that for all $x, y \in V$, $x \to y$ in T iff $f(x) \to f(y)$ in T'. The tournaments T and T' are then isomorphic, which is denoted by $T \simeq T'$, if there is an isomorphism from T onto T'.
Let k be a positive integer, and let T and T' be two tournaments on the same set V of n vertices. We say that T and T' are k-*similar* if they have the same score vector, and for every tournament H of size k the number of subtournaments of T isomorphic to H is equal to the number of subtournaments of T' isomorphic to H.

Let T_n be a tournament with score vector $(s_1, s_2, \cdots, s_n)$, and let $c_3(T_n)$ be the number of 3-cycles of T_n. It is known, see [7], that $c_3(T_n) = \binom{n}{3} - \sum_{i=1}^{n} \binom{s_i}{2}$. As a consequence we have:

Proposition 1.1 *If T and T' are two tournaments with the same score vector, then T and T' have the same number of 3-cycles and the same number of 3-chains.*

Note that there is no analogous result for 4-cycles and diamonds as shown by the following pair of tournaments $\{S, S'\}$.

Tournament S:

x:	vertices dominated by x
1:	2 3 4
2:	3
3:	4
4:	2 5
5:	1 2 3

Tournament S':

x:	vertices dominated by x
1:	2 3 4
2:	3
3:	5
4:	2 3
5:	1 2 4

The 4-cycles of S, (resp. S'), are $(1,3,4,5)$, $(2,3,4,5)$, (resp. $(1,2,3,5)$, $(1,4,3,5)$, $(2,3,5,4)$).
The diamonds of S are $1 \to (2,3,4)$ and $(1,4,5) \to 2$, but there is no diamond in S'.

From this pair, by dilatation, we can obtain an infinite class of pairs of decomposable non-isomorphic tournaments with the same score vector and satisfying the same conclusion as the initial pair.

It is well-known that every subtournament with 4 vertices, of a tournament is either a diamond, a 4-chain, or a 4-cycle subtournament.
For a tournament T, let $c_4(T)$, $\delta(T)$, $\delta^+(T)$, $\delta^-(T)$, $e_4(T)$ be the numbers of 4-cycles, diamonds, positive diamonds, negative diamonds, and 4-chains, respectively. In [8], the relationship between theses numbers is given by the following result.

Proposition 1.2 *([8]) Let T be a tournament with score vector $(s_1, s_2, \cdots, s_n)$ then:*
1) $c_4(T) = \binom{n}{4} - \delta(T) - e_4(T)$.
2) $e_4(T) = \sum_{i=1}^{n} \binom{s_i}{3} - \delta^+(T) = \sum_{i=1}^{n} \binom{n-1-s_i}{3} - \delta^-(T)$.
3) $c_4(T) = \binom{n}{4} - \delta^-(T) - \sum_{i=1}^{n} \binom{s_i}{3} = \binom{n}{4} - \delta^+(T) - \sum_{i=1}^{n} \binom{n-1-s_i}{3}$.

Proof. 1) Every subtournament with 4 vertices of T is either a diamond, or a 4-chain, or a 4-cycle subtournament. So if A is a 4-element set of vertices of T which forms neither a diamond nor a chain, then A forms a 4-cycle of T and only one.
2) From 1), $\binom{n}{4} = c_4(T) + e_4(T) + \delta^+(T) + \delta^-(T)$. We have $\sum_{i=1}^{n} \binom{s_i}{3} = \binom{n}{4} - \delta^-(T) - c_4(T)$. Then the first equality of 2) follows. To conclude it is sufficient to consider the tournament obtained from T by reversing all its arcs.
3) follows from 1) and 2). □

Proposition 1.3 *([8]) Let T and T' be two tournaments with the same score vector. If $f(T) = f(T')$ for some $f \in \{\delta^+, \delta^-, c_4, e_4\}$, then $g(T) = g(T')$ for all $g \in \{\delta^+, \delta^-, c_4, e_4\}$.*

Proof. It follows from 2) and 3) of Proposition 1.2. □

As a consequence of Propositions 1.1 and 1.3 we have:

Proposition 1.4 *Let T and T' be two tournaments with the same score vector. If $f(T) = f(T')$ for some $f \in \{\delta^+, \delta^-, c_4, e_4\}$, then T and T' have the same numbers of 3-cycles, 3-chains, 4-cycles, 4-chains, 4-cycles subtournaments, positive diamonds and negative diamonds.*

2 Introduction

W. Kocay [6] gave a list of all families of pairwise non-isomorphic tournaments of size 9, having the same number of 3-cycles and the same number

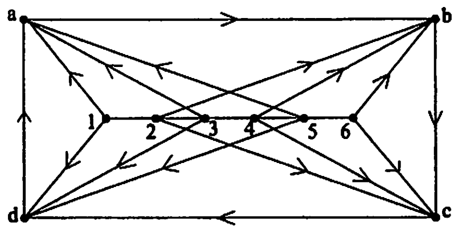
of positive (resp. negative) diamonds. In this list, each family has only two elements, except one with three elements. We remark that the tournaments of these families have the same score vector, and they are 4-similar. This list motivated our paper. Ulam's Reconstruction Conjecture [10] (see [1]), applied to tournaments, states that: "given two tournaments T and T' with the same set V of vertices, if for every $x \in V$, $T(V \setminus \{x\}) \simeq T'(V \setminus \{x\})$, then $T \simeq T'$". This hypothesis implies that T and T' have the same score vector (F. Harary and E. Palmer [3]); furthermore, for every tournament H, the number of subtournaments of T isomorphic to H is equal to the number of subtournaments of T' isomorphic to H (Kelly's Lemma [5]). Hence T and T' are k-similar for every k. P.K. Stockmeyer [9] gave an infinite family of pairs T, T' of non-isomorphic tournaments of size $2^p + 2^q$ satisfying Ulam's hypothesis. Note that T and T' are k-similar and non-isomorphic. A stronger condition than 4-similarity was studied by Y. Boudabbous [2]: let T and T' be two indecomposable tournaments on the same vertex set V satisfying $T_{\restriction X} \simeq T'_{\restriction X}$ for every k-element subset X of V, where $k \in \{2, 3, 4\}$; then $T \simeq T'$. From Kocay's list, one can easily obtain by dilatation an infinite family of pairs of 4-similar tournaments of arbitrary size $n \geq 9$, which are non-isomorphic.
We give another infinite family of pairs of 4-similar tournaments of arbitrary size $m \geq 8$, $m \not\equiv 3 \ (mod\ 4)$ which are indecomposable and non-isomorphic. We also construct an infinite class of pairs $\{T, T'\}$ of decomposable non-isomorphic 4-similar tournaments T and T', each one with a unique non trivial interval, in fact these non trivial intervals are of size 2. Finally we show that indecomposability is not preserved under the 4-similarity by constructing an infinite class of pairs $\{T, T'\}$ of 4-similar tournaments with T indecomposable, T' decomposable with a unique non trivial interval, in fact this interval is of size 2. Nevertheless, under Ulam's hypothesis, indecomposability is preserved, as has been shown by P. Ille [4].

3 Pairs of 4-similar, decomposable, non-isomorphic tournaments

Theorem 3.1 *For every integer $m \geq 7$, there are two decomposable, non-isomorphic, 4-similar tournaments of size m, such that each has a unique non trivial interval, where these intervals are of size 2; moreover they have the same diamonds.*

Proof.
Description of tournaments. Let $m \geq 6$ be an integer, we set $n := m-4$. Let T be the tournament with vertex set $V := \{a, b, c, d, 1, 2, \cdots, n\}$ and such that $a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$ is a cycle denoted Λ; $C := \{1, 2, \cdots, n\}$ is

a chain with $i \to j$ iff $i < j$. Moreover $c \to a$; $d \to b$; $N^+(i) = \{a, d, j \ : \ j \in C, \ i < j\}$ if i is odd; $N^+(i) = \{b, c, j \ : \ j \in C, \ i < j\}$ if i is even. The tournament T' is defined from T by reversing the directions of the arcs of the cycle Λ which becomes a cycle Λ' in T'. We denote by Γ the set of vertices of Λ. Note that T and T' have the same score vector.



T

The chain 1,2,...,6 is oriented from left to right.
Arcs (c,a), (d,b), (x,i) with x=a,b,c,d and i=1,2,...,6 are not represented.

Uniqueness and Interval size. Assume $m \geq 7$. The tournament T is decomposable since $\{a, d\}$ is a non trivial interval of T; and T' is decomposable since $\{b, c\}$ is a non trivial interval of T'. We shall prove that T and T' each have exactly one non trivial interval. Let I be an interval of T or T' having at least two elements x, y.

Case 1. $x, y \in C$.

If x, y do not have the same parity, then every element of Γ separates x and y, thus $\Gamma \subseteq I$. But every element of C separates a and b, hence $C \subseteq I$, thus $I = V$.

If x, y have the same parity, w.l.o.g. $x \to y$, so that $x \to x + 1 \to y$, hence $x + 1 \in I$, thus we conclude as above.

Case 2. $x, y \in \Gamma$.

Case 2.1. $x = a$ and $y \in \{b, c\}$. Every $z \in C$ separates a and y, so that $C \subseteq I$, thus we are in case 1.

Case 2.2. $x = a$ and $y = d$.

In T', $d \to b \to a$, so that $b \in I$. Then we are in case 2.1. In T we shall prove at the end that $I = \{a, d\}$ or $I = V$.

Case 2.3. $x = b$ and $y = c$.

In T, $c \to a \to b$, so that $a \in I$. Then we are in case 2.1. In T' we shall prove at the end that $I = \{b, c\}$ or $I = V$.

Case 2.4. $x = b$ and $y = d$. Then $a \in I$ because a separates b and d. Thus we are in case 2.1.

Case 2.5. $x = c$ and $y = d$. Every $z \in C$ separates c and d so that $C \subseteq I$. Thus we are in case 1.

Case 3. $x \in C$ and $y \in \Gamma$.

Case 3.1. x is odd.
If $y = a$, then if $x = 1$, 3 separates x and y, so that $3 \in I$. If $x \neq 1$, then 2 separates x and y, so that $2 \in I$. In both cases we are in case 1.
If $y = c$, then if $x = 1$, 2 separates x and y, then $2 \in I$. If $x \neq 1$, then 1 separates x and y, so that $1 \in I$. In both cases we are in case 1.
If $y = b$ (resp. $y = d$), then d (resp. b) separates x and y, so that $d \in I$ (resp. $b \in I$). Then we are in case 2.
Case 3.2. x is even.
If $y \in \{b, c\}$, then $x - 1$ separates x and y, so that $x - 1 \in I$. Then we are in case 1.
If $y \in \{a, d\}$, then for $x = 2$, 3 separates x and y, so that $3 \in I$; for $x > 2$, 2 separates x and y, so that $2 \in I$. In both cases we are in case 1.
Now we can complete Cases 2.2 and 2.3 using, in each one, Cases 2.1 and 3.

Non-isomorphism. Assume $m \geq 6$. We shall prove that T and T' are non-isomorphic by induction on n.
If $n = 2$, the score of a, b and 2 is 2 in both T and T', the score of c, d and 1 is 3 in both T and T'. By contradiction assume that T and T' are isomorphic then the set $\{a, b, 2\}$ is globaly invariant. But $T_{\restriction\{a,b,2\}}$ is a chain, however $T'_{\restriction\{a,b,2\}}$ is a cycle, that gives a contradiction. Let $n \geq 3$. By contradiction assume that f is an isomorphism from T onto T'. If $n = 3$, a and n are the unique vertices having the minimal score, and since $n \to a$ in both T and T' we have $f(a) = a$ and $f(n) = n$. If $n > 3$, since n is the unique vertex having the minimal score in T and T' then $f(n) = n$. Then in both cases $f(n) = n$. Let S and S' be respectively the tournaments obtained from T and T' by deleting the vertex n. Then S and S' are isomorphic. This contradicts the induction hypothesis.

Diamonds. Assume that $m \geq 6$. We claim that T and T' have the same diamonds. For it is sufficient to prove that every diamond of T or T' does not contain any arc of Λ in T, or any arc of Λ' in T'. In fact we shall prove that every diamond of T or T' has at most one vertex in common with Γ.
Let Δ be a diamond of T or T' and α be the principal vertex of Δ.
Fact 1. $\{a, b\}$ is not an arc of the 3-cycle of Δ. Assume the contrary and suppose that $\{a, b, \gamma\}$ is the 3-cycle of Δ, so $\Delta = \{a, b, \gamma, \alpha\}$. We have $b \to 2k+1 \to a$ and $a \to 2k \to b$, for every k, in both T and T'. Then $\alpha \notin C$, so $\alpha \in \{c, d\}$.
In T, since $b \to c \to a$, then $\alpha = d$. Since $d \to a$ then $d \to \gamma$, so $\gamma = 2p$ for some p, which contradicts $\{a, b, \gamma\}$ is a cycle.
In T', since $a \to d \to b$, then $\alpha = c$. From $c \to a$, we have $c \to \gamma$, so $\gamma = 2p + 1$ for some p, which contradicts $\{a, b, \gamma\}$ is a cycle.
Fact 1'. $\{c, d\}$ is not an arc of the 3-cycle of Δ. Assume the contrary and

suppose that $\{c, d, \gamma\}$ is the 3-cycle of Δ, so that $\Delta = \{c, d, \gamma, \alpha\}$. We have $c \rightarrow 2k + 1 \rightarrow d$ and $d \rightarrow 2k \rightarrow c$ for every k, in both T and T'. Then $\alpha \notin C$, so $\alpha \in \{a, b\}$. In T, since $d \rightarrow b \rightarrow c$, then $\alpha = a$. From $d \rightarrow a$, we get $\gamma \rightarrow a$, so $\gamma = 2p + 1$ for some p, which contradicts $\{c, d, \gamma\}$ is a cycle. In T', since $c \rightarrow a \rightarrow d$ then $\alpha = b$. From $c \rightarrow b$ we get $\gamma \rightarrow b$, so $\gamma = 2p$ for some p, which contradicts $\{c, d, \gamma\}$ is a 3-cycle.
Fact 2. $\{b, c\}$ is not an arc of the 3-cycle of Δ. Assume the contrary and suppose that $\{b, c, \gamma\}$ is the 3-cycle of Δ. Then $\gamma \notin C$ because all vertices of C dominates b and c or is dominated by b and c. In T' there is no solution for γ. In T, $\gamma \in \{a, d\}$, every vertex of C separates a and b, and separates d and c, so that $\alpha \notin C$, so $\alpha \in \Gamma$, thus $\Delta = \Gamma$: That is impossible.
Fact 2'. $\{a, d\}$ is not an arc of the 3-cycle of Δ. Assume the contrary and suppose that $\{a, d, \gamma\}$ is the 3-cycle of Δ, then $\gamma \notin C$ because all vertices of C dominates a and d or is dominated by a and d. Then no solution for γ in T. In T', $\gamma \in \{b, c\}$; every vertex of C separates b and d, and separates c and a, then $\alpha \notin C$, so $\alpha \in \Gamma$, thus $\Delta = \Gamma$: Again this is impossible.
Fact 3. $\{b, d\}$ is not an arc of the 3-cycle of Δ. Assume the contrary and suppose that $\{b, d, \gamma\}$ is the 3-cycle of Δ. Then $\alpha \notin \{a, c\}$ because a and c separate b and d in both T and T'. Then $\alpha \in C$, which contradicts the fact that α is the principal vertex of Δ.
Fact 3'. $\{a, c\}$ is not an arc of the 3-cycle of Δ. Assume the contrary and suppose that $\{a, c, \gamma\}$ is the 3-cycle of Δ. Then $\alpha \notin \{b, d\}$ because b and d separate a and c in both T and T'. Then $\alpha \in C$, which contradicts the fact that α is the principal vertex of Δ.

In conclusion, let $\{u, v, w\}$ be the 3-cycle of Δ in T or T'. Then $\{u, v, w\}$ has at most one vertex in Γ. Since $\{u, v, w\} \not\subseteq C$, then $\{u, v, w\}$ has exactly one vertex in Γ. Let u be this vertex, so $v, w \in C$. Then necessarily v and w have different parities. Since α does not separate v and w, we have $\alpha \notin \Gamma$. Consequently, Δ has only one vertex in common with Γ, and thus Δ is the same in T and T'.

Therefore the tournaments T and T' have the same diamonds and by Proposition 1.4, they are 4-similar. □

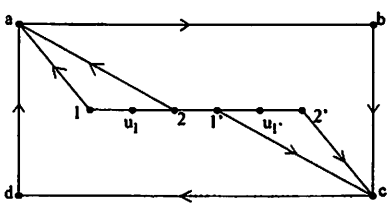
4 Pairs of 4-similar tournaments of another kind.

Theorem 4.1 *For every integer $n \geq 2$, there are two 4-similar tournaments of size $4n + 2$, one indecomposable, the other decomposable with a unique non trivial interval, where this interval is of size 2.*

Proof.
Description of tournaments. Given two integers $n \geq 2$ and $p \geq 2$, let $T_{n,p}$ be the tournament with vertex set:

$$V := \{a, b, c, d, 1, 2, \cdots, n, u_1, u_2, \cdots, u_{n-1}, 1', 2', \cdots, p', u_{1'}, u_{2'}, \cdots, u_{(p-1)'}\}$$

and such that $a \to b \to c \to d \to a$ is a cycle denoted Λ; $C := \{1, u_1, 2, u_2, \cdots, n-1, u_{n-1}, n\}$ is a chain with $1 \to u_1 \to 2 \to u_2 \to \cdots \to n-1 \to u_{n-1} \to n$ and the arcs obtained by transitivity; the other arcs are $c \to a$; $d \to b$; $i \to a$, $b \to i$, $c \to i$, $d \to i$ for $i \in \{1, 2, 3, \cdots, n-1, n\}$; $\{a, b, c, d\} \to u_i$ for $i \in \{1, 2, 3, \cdots, n-1\}$; $C' := \{1', u_{1'}, 2', u_{2'}, \cdots, (p-1)', u_{(p-1)'}, p'\}$ is a chain with $1' \to u_{1'} \to 2' \to u_{2'} \to \cdots \to (p-1)' \to u_{(p-1)'} \to p'$ and the arcs obtained by transitivity; the other arcs are $i' \to c$, $a \to i'$, $b \to i'$, $d \to i'$ for $i' \in \{1', 2', 3', \cdots, (p-1)', p'\}$; $\{a, b, c, d\} \to u_{i'}$ for $i' \in \{1', 2', 3', \cdots, (p-1)'\}$; and $x \to y$ for all $x \in C$ and $y \in C'$.
The tournament $T'_{n,p}$ is defined from $T_{n,p}$ by reversing the directions of the arcs of the cycle Λ which becomes a cycle Λ' in $T'_{n,p}$. We denote by Γ the set of vertices of Λ. Note that $T_{n,p}$ and $T'_{n,p}$ have the same score vector.



$T_{2,2}$
The chain 1, u_1, 2, 1', $u_{1'}$, 2' is oriented from left to right.
Arcs (c,a), (d,b), (x,y) with x=a,b,c,d and y in the chain above are not represented.

Intervals. The tournament $T'_{n,p}$ is decomposable since $\{1, b\}$ is a non trivial interval. We shall prove that $T_{n,p}$ is indecomposable and $\{1, b\}$ is the unique non trivial interval of $T'_{n,p}$. Let I be an interval of $T_{n,p}$ or $T'_{n,p}$ having at least two elements x, y.
Case 1. $x, y \in \Gamma$.
Case 1.1. $x = a$ and $y = b$.
In $T_{n,p}$, $c \in I$ because $b \to c \to a$. Since $c \to d \to a$, then $d \in I$, thus $\Gamma \subseteq I$.
In $T'_{n,p}$, $d \in I$ because $a \to d \to b$. Since $d \to c \to b$, then $c \in I$, thus $\Gamma \subseteq I$.
The elements of $\{1, 2, \cdots, n\}$ separate a and b, so that $\{1, 2, \cdots, n\} \subseteq I$.
The elements of $\{1', 2', \cdots, p'\}$ separate a and c, hence $\{1', 2', \cdots, p'\} \subseteq I$.
For every $x \in \{1, 2, \cdots, n-1\} \cup \{1', 2', \cdots, (p-1)'\}$, $x \to u_x \to x+1$, so that $u_x \in I$, and thus $C \cup C' \subseteq I$. Consequently $I = V$.

Case 1.2. $x = a$ and $y = c$. Then $b \in I$ because b separates a and c. Then we are in case 1.1.
Case 1.3. $x = a$ and $y = d$.
In $T_{n,p}$, $1 \in I$ because $d \to 1 \to a$. Now $b \in I$ because $d \to b \to 1$. In $T'_{n,p}$, $b \in I$ because b separates a and d. In both cases $b \in I$, then we are in case 1.1.
Case 1.4. $x = b$ and $y = c$.
In $T_{n,p}$, $a \in I$ because $c \to a \to b$. In $T'_{n,p}$, $b \to 1' \to c$, so that $1' \in I$. Now $c \to a \to 1'$, hence $a \in I$. In both cases $a \in I$, so that we are in case 1.1.
Case 1.5. $x = b$ and $y = d$. Then $a \in I$ because a separates b and d. Then we are in case 1.1.
Case 1.6. $x = c$ and $y = d$. In $T_{n,p}$, $d \to b \to c$, so that $b \in I$. Then we are in case 1.4.
In $T'_{n,p}$, $c \to a \to d$, so that $a \in I$. Then we are in case 1.2.
Case 2. $x, y \in C$.
Case 2.1. If $x \in \{1, 2, \cdots, n\}$ and $y = u_j$ for some j, then a separates x and y, so that $a \in I$. In $T_{n,p}$, b separates x and a, so that $b \in I$; in $T'_{n,p}$, d separates x and a, hence $d \in I$. In both cases we are in case 1.
Case 2.2. If $x, y \in \{1, 2, \cdots, n\}$, w.l.o.g. $x \to y$, then $x \to u_x \to y$, thus $u_x \in I$. Then we are in case 2.1.
Case 2.3. If $x = u_i$ and $y = u_j$ with $i < j$ then $x \to i+1 \to y$, so $i+1 \in I$. Then we are in case 2.1.
Case 3. $x, y \in C'$.
Case 3.1. If $x \in \{1', 2', \cdots, p'\}$ and $y = u_{j'}$.
$x \to c \to y$, so that $c \in I$. Also $c \to a \to x$, so that $a \in I$. Then we are in case 1.
Case 3.2. If $x, y \in \{1', 2', \cdots, p'\}$, w.l.o.g. $x \to y$, so that $x \to u_x \to y$, thus $u_x \in I$. Then we are in case 3.1.
Case 3.3. If $x = u_{i'}$ and $y = u_{j'}$ with $i' < j'$ then $x \to (i+1)' \to y$, hence $(i+1)' \in I$. Then we are in case 3.1.
Case 4. $x \in C$ and $y \in C'$.
Case 4.1. If $x \neq n$, then $x \to n \to y$, so that $n \in I$, thus we are in case 2.
Case 4.2. If $x = n$ and $y = k'$ with $k' \in \{1', 2', \cdots, p'\}$, then a, c separate x and y. Thus $a, c \in I$, so that we are in case 1.
Case 4.3. If $x = n$ and $y = u_{k'}$ with $k' \in \{1', 2', \cdots, (p-1)'\}$, then $x \to 1' \to y$, so that $1' \in I$. Then we are in case 3.
Case 5. $x \in C$ and $y \in \Gamma$.
Case 5.1. $x \in \{1, 2, \cdots, n\}$ and $y = a$.
In $T_{n,p}$, $a \to b \to x$, so that $b \in I$. In $T'_{n,p}$, $a \to d \to x$, so that $d \in I$. In both cases, we are in case 1.
Case 5.2. $x \in \{1, 2, \cdots, n\}$ and $y = b$.
In $T_{n,p}$, $x \to a \to b$, so that $a \in I$. Thus we are in case 1.
If $x \neq 1$ then 1 separates x and b, so that $1 \in I$. Thus, by case 2, $I = \{1, b\}$

or $I = V$.
Case 5.3. $x \in \{1, 2, \cdots, n\}$ and $y = c$.
In $T_{n,p}$, $c \to d \to x$, so that $d \in I$. In $T'_{n,p}$, $c \to b \to x$, so that $b \in I$. Then we are in case 1.
Case 5.4. $x \in \{1, 2, \cdots, n\}$ and $y = d$.
$d \to b \to x$, so that $b \in I$. Then we are in case 1.
Case 5.5. $x = u_t$ and $y = a$.
In $T_{n,p}$, $a \to b \to u_t$, so that $b \in I$. In $T'_{n,p}$, $a \to d \to u_t$, so that $d \in I$. In both cases, we are in case 1.
Case 5.6. $x = u_t$ and $y = b$.
In $T_{n,p}$, $b \to c \to u_t$, so that $c \in I$. In $T'_{n,p}$, $b \to a \to u_t$, so that $a \in I$. In both cases, we are in case 1.
Case 5.7. $x = u_t$ and $y = c$.
$c \to a \to u_t$, so that $a \in I$. Then we are in case 1.
Case 5.8. $x = u_t$ and $y = d$.
$d \to b \to u_t$, so that $b \in I$. Then we are in case 1.
Case 6. $x \in C'$ and $y \in \Gamma$.
$y \to u_1 \to x$, so that $u_1 \in I$. Thus we are in case 5.

Diamonds. We claim that $T_{n,n}$ and $T'_{n,n}$ have the same number of positive diamonds.
The tournaments $T_{n,n}$ and $T'_{n,n}$ have the same diamonds containing no arc of Λ in $T_{n,n}$, and no arc of Λ' in $T'_{n,n}$. We only list the positive diamonds of $T_{n,n}$ (resp. positive diamonds of $T'_{n,n}$) with at least one arc of Λ (resp. Λ'); such diamonds have at most two vertices i, j in $C \cup C'$. Note that, in $T_{n,n}$ (resp. $T'_{n,n}$), there is no diamond with at least one arc of Λ (resp. Λ') having a cycle of the form (x, y, z) with $\{x, y\} = \{a, c\}$ or $\{b, d\}$, and $z \in C \cup C'$.

The 3-cycles of $T_{n,n}$ having at least one vertex of Γ, except those of the form (x, y, z) with $\{x, y\} = \{a, c\}$ or {b,d} and $z \in C \cup C'$, are:
(a, b, c), (a, b, i) for every $i \in \{1, 2, \cdots, n\}$, (b, c, d), (c, d, i') for every $i' \in \{1', 2', \cdots, n'\}$, (a, u_i, j) for $i < j$ in $\{1, 2, \cdots, n\}$, (c, x, i') with $x \in C$ and $i' \in \{1', 2', \cdots, n'\}$, $(c, u_{i'}, j')$ for $i' < j'$ in $\{1', 2', \cdots, n'\}$.

The 3-cycles of $T'_{n,n}$ having at least one vertex of Γ, except those of the form (x, y, z) with $\{x, y\} = \{a, c\}$ or {b,d} and $z \in C \cup C'$, are:
(b, a, d), (c, b, i') for every $i' \in \{1', 2', \cdots, n'\}$, (d, c, a), (a, d, i) with $i \in \{1, 2, \cdots, n\}$, (a, u_i, j) for $i < j$ in $\{1, 2, \cdots, n\}$, (c, x, i') with $x \in C$ and $i' \in \{1', 2', \cdots, n'\}$, $(c, u_{i'}, j')$ with $i' < j'$ in $\{1', 2', \cdots, n'\}$.

Positive diamonds of $T_{n,n}$, with at least one arc of Λ, are:
$d(a, b, i)$ for $i \in \{1, 2, \cdots, n\}$,
$d(a, u_i, j)$ for $i < j$ in $\{1, 2, \cdots, n\}$,

$b(c, x, i')$ with $x \in C$ and $i' \in \{1', 2', \cdots, n'\}$,
$b(c, u_{i'}, j')$ for $i' < j'$ in $\{1', 2', \cdots, n'\}$.

Positive diamonds of $T'_{n,n}$, with at least one arc of Λ', are:
$d(c, b, i')$ for $i' \in \{1', 2', \cdots, n'\}$,
$b(a, u_i, j)$ for $i < j$ in $\{1, 2, \cdots, n\}$,
$d(c, x, i')$ with $x \in C$ and $i' \in \{1', 2', \cdots, n'\}$,
$d(c, u_{i'}, j')$ with $i' < j'$ in $\{1', 2', \cdots, n'\}$.

Clearly, $T_{n,n}$ and $T'_{n,n}$ have the same number of positive diamonds, and by Proposition 1.4, they are 4-similar. □

5 Pairs of 4-similar, indecomposable, non-isomorphic tournaments

Theorem 5.1 *For every integer $m \geq 8$, $m \not\equiv 3 \pmod 4$, there are two indecomposable, non-isomorphic, tournaments T and T' of size m which are 4-similar.*

Proof.
Description The following pair $\{T, T'\}$ of tournaments (see W. Kocay [6]) are indecomposable, have the same score vector, and the same number of positive (resp. negative) diamonds; the vertices are $1, 2, \cdots, 9$.

Tournament T:

x:	vertices dominated by x
1:	7 8
2:	1 3 6 9
3:	1 5 8 9
4:	1 2 3 8
5:	1 2 4 7
6:	1 3 4 5
7:	2 3 4 6
8:	2 5 6 7
9:	1 4 5 6 7 8

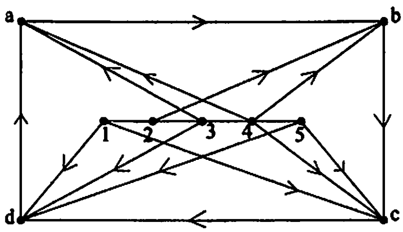
Tournament T':

x:	vertices dominated by x
1:	7 8
2:	1 6 7 9
3:	1 2 5 9
4:	1 2 3 8
5:	1 2 4 7
6:	1 3 4 5
7:	3 4 6 8
8:	2 3 5 6
9:	1 4 5 6 7 8

Tournaments T and T' are respectively isomorphic to the tournaments T_5 and T'_5 introduced below. In fact this pair $\{T, T'\}$ led us to construct the following class of tournaments T_n.

Given an integer $n \geq 4$, let T_n be the tournament with vertex set $V := \{a, b, c, d, 1, 2, \cdots, n\}$ such that $a \to b \to c \to d \to a$ is a cycle denoted Λ; $C := \{1, 2, \cdots, , n\}$ is a chain with $i \to j$ iff $i < j$. Moreover $c \to a$; $d \to b$; $N^+(i) = \{c, d, j \ : \ j \in C, \ i < j\}$ if $i \equiv 1 \ (mod\ 4)$; $N^+(i) = \{b, j \ : \ j \in C, \ i < j\}$ if $i \equiv 2 \ (mod\ 4)$; $N^+(i) = \{a, d, j \ : \ j \in C, \ i < j\}$ if $i \equiv 3 \ (mod\ 4)$; $N^+(i) = \{a, b, c, j \ : \ j \in C, \ i < j\}$ if $i \equiv 0 \ (mod\ 4)$.

The tournament T'_n is formed from T_n by reversing the directions of the arcs of the cycle Λ which becomes a cycle Λ' in T'_n. We denote by Γ the set of vertices of Λ. Note that T_n and T'_n have the same score vector.



T_5

The chain 1,2,...,5 is oriented from left to right.
Arcs (c,a), (d,b), (x,i) with x=a,b,c,d and i=1,2,...,5 are not represented.

Non-isomorphism. We show that T_n and T'_n are non-isomorphic by induction on $n \geq 4$.

Case 1. If $n = 4$, we have $s(a) = s(b) = s(2) = s(3) = s(4) = 3$, $s(c) = s(d) = 4$, $s(1) = 5$. Suppose that there is an isomorphism f from T_4 onto T'_4, then $f(1) = 1$, $f(c) = d$ and $f(d) = c$. Since $f(1) = 1$ and $\{a, b\} \to 1$, then $\{f(a), f(b)\} \to 1$, so $f(a), f(b) \in \{a, b\}$; thus $f(a) = b$ and $f(b) = a$. Now $f(\{2, 3, 4\}) = \{2, 3, 4\}$, so $f(2) = 2$, $f(3) = 3$, $f(4) = 4$. Thus, as $3 \to d$, then $3 \to c$; contradiction.

Now let $n > 4$, and suppose there is an isomorphism f from T_n onto T'_n. We have:

If $n \equiv 1 \ (mod\ 4)$ or $n \equiv 3 \ (mod\ 4)$, $s(n) = 2$, then $s(n) \neq s(x)$ for all $x \neq n$.

If $n \equiv 2 \ (mod\ 4)$, $s(n) = 1$, then $s(n) \neq s(x)$ for all $x \neq n$.

If $n \equiv 0 \ (mod\ 4)$, $s(n) = s(n-1) = s(n-2) = 3$, and $s(x) > 3$ for $x \notin \{n-2, n-1, n\}$. Then $f(i) = i$ for $i \in \{n-2, n-1, n\}$.

In all these cases, $f(n) = n$. Then there is an isomorphism between the tournaments obtained from T_n and T'_n by deleting the vertex n. That gives a contradiction since these tournaments are respectively T_{n-1} and T'_{n-1} which are non-isomorphic by the induction hypothesis.

Indecomposability. Given an integer $n \geq 4$, we shall prove that T_n and T'_n are indecomposable. Let I be an interval of T_n or $T'_{n,}$ having at least two elements x, y.

Case 1. $x, y \in \Gamma$.
Case 1.1. $x = a$ and $y = b$.
In T, $c \in I$ because $b \to c \to a$. Since $c \to d \to a$, we have $d \in I$, so that $\Gamma \subseteq I$.
In T', $d \in I$ because $a \to d \to b$. Since $d \to c \to b$, we have $c \in I$, so that $\Gamma \subseteq I$.
Each $x \in C$ separates at least two elements of Γ, so that $C \subseteq I$. Consequently $I = V$.
Case 1.2. $x = a$ and $y = c$. Then $b \in I$ because b separates a and c.
Case 1.3. $x = a$ and $y = d$.
$1 \in I$ because $a \to 1 \to d$. Now $b \in I$ because $d \to b \to 1$.
Case 1.4. $x = b$ and $y = c$.
In T, $a \in I$ because $c \to a \to b$. In T', $b \to 1 \to c$, so that $1 \in I$. Now $c \to a \to 1$, so that $a \in I$. Then in both cases $a \in I$.
Case 1.5. $x = b$ and $y = d$. Then $a \in I$ because a separates b and d.
In cases 1.2. to 1.5, we conclude using case 1.1.
Case 1.6. $x = c$ and $y = d$. In T, $d \to b \to c$, so that $b \in I$. Thus we are in case 1.4.
In T', $c \to a \to d$, so that $a \in I$. Thus we are in case 1.2.
Case 2. $x, y \in C$. We can assume $x < y$. Thus $x + 1 \in I$. There are at least two elements of Γ which separate x and $x + 1$. Then we are in case 1.
Case 3. $x \in C$ and $y \in \Gamma$.
Case 3.1. $x \equiv 1 \ (mod\ 4)$ and $y = a$.
$x \to c \to a$, so that $c \in I$. Then we are in case 1.
Case 3.2. $x \not\equiv 1 \ (mod\ 4)$ and $y = a$.
$a \to 1 \to x$, so that $1 \in I$. Then we are in case 2.
Case 3.3. $x \equiv 1 \ (mod\ 4)$ and $y = b$.
$x \to d \to b$, so that $d \in I$. Then we are in case 1.
Case 3.4. $x \not\equiv 1 \ (mod\ 4)$ and $y = b$.
$b \to 1 \to x$, so that $1 \in I$. Then we are in case 2.
Case 3.5. $x \equiv 1 \ (mod\ 4)$ or $x \equiv 2 \ (mod\ 4)$, and $y = c$.
$c \to a \to x$, so that $a \in I$. Then we are in case 1.
Case 3.6. $x \equiv 3 \ (mod\ 4)$ or $x \equiv 0 \ (mod\ 4)$, and $y = c$.
$c \to x - 1 \to x$, so that $x - 1 \in I$. Then we are in case 2.
Case 3.7. $x \equiv 1 \ (mod\ 4)$ or $x \equiv 3 \ (mod\ 4)$, and $y = d$.
$d \to b \to x$, so that $b \in I$. Then we are in case 1.
Case 3.8. $x \equiv 2 \ (mod\ 4)$ with $x \neq 2$ or $x \equiv 0 \ (mod\ 4)$, and $y = d$.
$d \to 2 \to x$, so that $2 \in I$. Then we are in case 2.
Case 3.9. $x = 2$ and $y = d$.
$2 \to 3 \to d$, so that $3 \in I$. Then we are in case 2.

Cycles. The tournaments T_n and T'_n have the same 4-cycles without any arc of Λ in T_n, or any arc of Λ' in T'_n. We have only to list the 4-cycles

of T_n (resp. 4-cycles of T'_n) with at least one arc of Λ (resp. Λ'). In what follows $0 \leq l \leq k$.
The 4-cycles of T_n, with at least one arc of Λ, are: (a,b,c,d), $(a,b,c,4k+3)$, $(a,b,4k+1,c)$, $(a,b,4k+1,d)$, $(a,b,4k+3,d)$, $(a,b,4l+1,4k+3)$, $(a,b,4l+1,4k+4)$, $(a,b,4l+3,4k+4)$, $(a,b,4l+3,4k+7)$, $(b,c,a,4k+2)$, $(b,c,d,4k+2)$, $(b,c,d,4k+4)$, $(b,c,4k+3,d)$, $(b,c,4l+2,4k+4)$, $(b,c,4l+2,4k+6)$, $(b,c,4l+3,4k+4)$, $(b,c,4l+3,4k+6)$, $(c,d,a,4k+1)$, $(c,d,b,4k+1)$, $(c,d,4l+2,4k+4)$, $(c,d,4l+2,4k+5)$, $(c,d,4l+4,4k+5)$, $(c,d,4l+4,4k+8)$, $(d,a,4l+1,4k+3)$, $(d,a,4l+1,4k+5)$, $(d,a,4l+2,4k+3)$, $(d,a,4l+2,4k+5)$.

The 4-cycles of T'_n, with at least one arc of Λ', are: (b,a,d,c), $(b,a,d,4k+2)$, $(b,a,d,4k+4)$, $(b,a,4k+1,c)$, $(b,a,4k+1,d)$, $(b,a,4l+1,4k+2)$, $(b,a,4l+1,4k+4)$, $(b,a,4l+2,4k+6)$, $(b,a,4l+2,4k+4)$, $(a,d,b,4k+3)$, $(a,d,c,4k+3)$, $(a,d,4l+2,4k+3)$, $(a,d,4l+2,4k+4)$, $(a,d,4k+4,c)$, $(a,d,4l+4,4k+7)$, $(a,d,4l+4,4k+8)$, $(d,c,a,4k+1)$, $(d,c,b,4k+1)$, $(d,c,b,4k+3)$, $(d,c,4l+2,4k+3)$, $(d,c,4l+2,4k+5)$, $(d,c,4l+3,4k+5)$, $(d,c,4l+3,4k+7)$, $(c,b,4l+1,4k+4)$, $(c,b,4l+1,4k+5)$, $(c,b,4l+3,4k+4)$, $(c,b,4l+3,4k+5)$.

Case 1. $m \equiv 0 \ (mod\ 4)$ i.e. $n = 4p$. We have $c_4(T_n) = c_4(T'_n) = 8p^2 + 10p + 1$.
Case 2. $m \equiv 1 \ (mod\ 4)$ i.e. $n = 4p+1$. We have $c_4(T_n) = c_4(T'_n) = 8p^2 + 14p + 5$.
Case 3. $m \equiv 2 \ (mod\ 4)$ i.e. $n = 4p+2$. We have $c_4(T_n) = c_4(T'_n) = 8p^2 + 16p + 7$.
Case 4. $m \equiv 3 \ (mod\ 4)$ i.e. $n = 4p+3$. We have $c_4(T_n) = 8p^2 + 20p + 13$ and $c_4(T'_n) = 8p^2 + 20p + 12$.
Remark. Thus, if $n \not\equiv 3 \ (mod\ 4)$, the tournaments T_n and T'_n have the same number of 4-cycles. Then by Proposition 1.4, they are 4-similar. Note that if $n \equiv 3 \ (mod\ 4)$, $c_4(T'_n) \neq c_4(T_n)$. □

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