# Edge-Neighbor-Scattering Number of Graphs

Zongtian Wei, Yang Li and Junmin Zhang
College of Science, Xi'an University of Architecture and Technology
Xi'an, Shaanxi 710055, P.R. China
Email: wzt6481@163.com

**Abstract.** The edge-neighbor-scattering number of a graph $G$ is defined to be $ENS(G) = \max_{S \subseteq E(G)} \{\omega(G/S) - |S|\}$, where $S$ is any edge-cut-strategy of $G$, $\omega(G/S)$ is the number of the components of $G/S$. In this paper, we give edge-neighbor-scattering number of some special classes of graphs, and then mainly discuss the general properties of the parameter.

# 1 Introduction

Gunther and Hartnell[1][2] introduced the idea of modelling a spy network by a graph whose vertices represent the stations and whose edges represent links of communication. If a station is destroyed, the adjacent stations will be betrayed so that the betrayed stations become useless to network as a whole. Therefore, instead of considering the stability of a communication network in standard sense, some new graphical parameters such as vertex-neighbor-integrity[3] and edge-neighbor-integrity[4] were introduced to measure the stability of communication networks in "neighbor" sense. The common ground of these parameters is that, when removing some vertices( or edges) from a graph, all of their adjacent vertices( or edges) are removed. So we call them the "neighbor stability" parameters. In[5] we introduced a graphical parameter called "vertex-neighbor-scattering number". Similarly, we can consider the edge analogue: edge-neighbor-scattering number of a graph.( in this paper, "graph" is equivalent to "communication network")

Let $G = (V, E)$ be a graph and $e$ be any edge in $G$. $N(e) = \{f \in E(G) | f \neq e, e \text{ and } f \text{ are adjacent}\}$ is the *open edge-neighborhood* of $e$, and $N[e] = N(e) \cup \{e\}$ is the *closed edge-neighborhood* of $e$. An edge $e$ in $G$ is said to be *subverted* when $N[e]$ is deleted from $G$. In other words, if $e = [u, v]$, $G - N[e] = G - \{u, v\}$. A set of edges $S \subseteq V(G)$ is called an *edge subversion strategy* of $G$ if each of the edges in $S$ has been subverted from $G$. The survival subgraph is denoted by $G/S$. An edge subversion strategy $S$ is called an *edge-cut-strategy* of $G$ if the survival subgraph $G/S$ is disconnected, or is a single vertex, or is $\phi$. The *edge- neighbor-connectivity* of $G$, $\Lambda(G)$, is the minimum size of all edge-cut-strategies of $G$. A graph $G$ is *$m$-edge-neighbor-connected* if $\Lambda(G) = m$.

Let $G$ be a connected graph. The *edge-neighbor-scattering number* of $G$, $ENS(G)$, is defined as $ENS(G) = \max_{S \subseteq E(G)} \{\omega(G/S) - |S|\}$, where $S$ is any edge-cut-strategy of $G$, and $\omega(G/S)$ is the number of the components of $G/S$. We call $S^* \subseteq E(G)$ a *$ENS-set$* of $G$ if $ENS(G) = \omega(G/S^*) - |S^*|$.

**Example 1:** Let $DS(n_1, n_2)$ be a double star with $\{n_1, n_2\}$ end-vertices, where $n_1 \geq 1$ and $n_2 \geq 1$, and a common edge $[u, v]$, as shown in Figure 1. $ENS(DS(n_1, n_2)) = n_1 + n_2 - 1$.

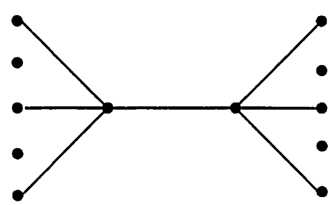


*Figure1* : $DS(n_1, n_2)$

In this paper, we give edge-neighbor-scattering number of several specific classes of graphs in section 2, and then discuss the bounds and other properties of the parameter in section 3 and section 4.

We use Bondy and Merty[6] for terminology and notation not defined here and consider only finite simple connected graphs. Throughout this paper, $\lceil x \rceil$ stands for the smallest integer greater than or equal to $x$, and $\lfloor x \rfloor$ stands for the greatest integer less than or equal to $x$.

# 2 Edge-neighbor-scattering number of Several Specific Classes of Graphs

**Theorem 1.** *Let $P_n$ be a path with order $n(\geq 2)$. Then*

$$ENS(P_n) = \begin{cases} -1, & if\ \ n = 2; \\ 0, & if\ \ n = 3\ ; \\ 1, & if\ \ n \geq 4. \end{cases}$$

*Proof.* The case $n = 2, 3$ is trivial. So we assume $n \geq 4$.

For any $S \subseteq E(P_n)$, $\omega(P_n/S) \leq |S| + 1$. Thus we have

$$ENS(P_n) = \max_{S \subseteq E(P_n)} \{\omega(P_n/S) - |S|\} \leq |S| + 1 - |S| = 1.$$

On the other hand, let $e = (x, y)$ be an edge in $P_n$ such that $d(x) = d(y) = 2$. Then $\omega(P_n/\{e\}) = 2$. So we have

$$ENS(P_n) = \max_{S \subseteq E(P_n)} \{\omega(P_n/S) - |S|\} \geq \omega(P_n/\{e\}) - |\{e\}| = 2 - 1 = 1.$$

Therefore $ENS(P_n) = 1$, and we complete the proof. □

**Theorem 2.** *Let $C_n$ be a cycle with order $n(\geq 3)$. Then*
$$ENS(C_n) = \begin{cases} -1, & \text{if } n = 4, 5; \\ 0, & \text{if } n = 3 \text{ or } n \geq 6. \end{cases}$$

*Proof.* $n = 3, 4, 5$ is trivial. So we assume $n \geq 6$.

For any $e \in E(C_n)$, $|N(e)| = 2$, $C_n/\{e\}$ is a path with order $n - 2(\geq 4)$. Thus for any $S \subseteq E(C_n)$, $\omega(C_n/S) \leq |S|$ and

$$ENS(C_n) = \max_{S \subseteq E(C_n)} \{\omega(C_n/S) - |S|\} \leq |S| - |S| = 0.$$

On the other hand, there must exist two edges $e, f$ in $C_n$ such that $\omega(C_n/\{e, f\}) = 2$, so we have

$$ENS(C_n) = \max_{S \subseteq E(C_n)} \{\omega(C_n/S) - |S|\} \geq \omega(C_n/\{e, f\}) - |\{e, f\}| = 2 - 2 = 0.$$

Therefore $ENS(C_n) = 0$, and the proof is completed. □

**Theorem 3.** *Let $K_n$ be a complete graph with order $n(\geq 3)$. Then $ENS(K_n) = 1 - \lfloor \frac{n}{2} \rfloor$.*

*Proof.* Observe that for any $e \in E(K_n)$, $K_n/\{e\} = K_{n-2}$. From $\Lambda(K_n) = \lfloor \frac{n}{2} \rfloor$( see[7]), if $S$ is an edge-cut-strategy of $K_n$, then $|S| \geq \lfloor \frac{n}{2} \rfloor$, and for any $S \subset E(K_n)$, $\omega(K_n/S) \leq 1$. So $ENS(K_n) = \max_{S \subseteq E(K_n)} \{\omega(K_n/S) - |S|\} \leq 1 - \lfloor \frac{n}{2} \rfloor$.

On the other hand, if $n$ is even, assume $M = \{(u_1, v_1), (u_2, v_2), \cdots, (u_{\frac{n}{2}}, v_{\frac{n}{2}})\}$ is a maximum matching in $K_n$. Replace $(u_1, v_1)$ by $(u_1, v_2)$, denote $M' = M - \{(u_1, v_1)\} \cup \{(u_1, v_2)\}$. Then $K_n/M' = v_1$. Thus we have $ENS(K_n) = \max_{S \subseteq E(K_n)} \{\omega(K_n/S) - |S|\} \geq \omega(K_n/M) - |M| \geq 1 - \frac{n}{2} = 1 - \lfloor \frac{n}{2} \rfloor$.

If $n$ is odd, then there is a maximum matching $M$ in $K_n$ such that $|M| = \frac{n-1}{2}$ and $K_n/M$ is an isolated vertex. So we have $ENS(K_n) = \max_{S \subseteq E(K_n)} \{\omega(K_n/S) - |S|\} \geq \omega(K_n/M) - |M| \geq 1 - \frac{n-1}{2} = 1 - \lfloor \frac{n}{2} \rfloor$ too.

Therefore $ENS(K_n) = 1 - \lfloor \frac{n}{2} \rfloor$. □

**Theorem 4.** *Let $K_{m,n}$ be a complete bipartite graph with a bipartition $(M, N)$, where $|M| = m, |N| = n$. Then $ENS(K_{m,n}) = |m - n| - 1$.*

*Proof.* Assume $m \geq n$. Let $S$ be any edge-cut strategy of $K_{m,n}$. By the structure of $K_{m,n}$, $|S| \geq n$ and if $V(K_{m,n}/S) \cap M \neq \phi$, then $(K_{m,n}/S) \cap N = \phi$, vice versa. This implies $\omega(K_{m,n}/S) \leq \max\{m, n\} - 1 = m - 1$. So $ENS(K_{m,n}) \leq m - n - 1$.

On the other hand, let $v$ be any vertex in $M$, $X$ be the edge set of all edges incident with $v$ in $K_{m,n}$. Then $|X| = n$ and $(K_{m,n}/X) = m - 1$. Thus $ENS(K_{m,n}) \geq m - n - 1$.

Therefore $ENS(K_{m,n}) = m - n - 1$ when $m \geq n$. By symmetry, when $n \geq m$, $ENS(K_{m,n}) = n - m - 1$. Finally, we have $ENS(K_{m,n}) = |m - n| - 1$. □

**Corollary** Let $S_{1,n}$, where $n > 1$, be a star graph. Then $ENS(S_{1,n}) = n - 2$.

A comet $C_{t,n}$ is a graph obtained by identifying one end of a path $P_t (t \geq 2)$ with the center of a star $S_{1,n} (n \geq 2)$. The center of $S_{1,n}$ is called the center of $C_{t,n}$.

**Theorem 5.** *Let $C_{t,n}$ be a comet, where $t \geq 3$ , $n \geq 2$. Then $ENS(C_{t,n}) = n$.*

*Proof.* Let $V(P_t) = \{v_1, v_2, \cdots, v_t\}$, and $v_1$ be the center in $C_{t,n}$. Obviously, if $S$ is an edge-cut strategy of $C_{t,n}$, then $|S| \geq 1$. Moreover, if $|S| = 1$ and $S = \{(v_1, v_2)\}$, then $\omega(C_{t,n}/S) = n + 1$; if $S \neq \{(v_1, v_2)\}$, then $\omega(C_{t,n}/S) < n + 1$.

If $|S| > 1$, clearly $\omega(C_{t,n}/S) \leq n + 1$. So $ENS(C_{t,n}) \leq n + 1 - 1 = n$.

On the other hand, since $\omega(C_{t,n}/\{(v_1, v_2)\}) = n+1$, we have $ENS(C_{t,n}) \geq n + 1 - 1 = n$.

Therefore $ENS(C_{t,n}) = n$, and we complete the proof. □

# 3 Bounds for Edge-Neighbor-Scattering Number

**Theorem 6.** *For any connected graph $G$ with order $n$, $ENS(G) \leq n - 3$.*

*Proof.* Let $S$ be any edge-cut strategy of $G$. Then $|S| \geq 1$. Notice that the subversion of an edge from $G$ is equivalent to removing it's two endpoints from $G$, so $\omega(G/S) \leq n - 2$. Thus we have $ENS(G) \leq \omega(G/S) - |S| = n - 2 - 1 = n - 3$. □

**Remark 1** The upper bound in Theorem 6 is the best possible. It can be shown by star or double star graphs.

**Theorem 7.** *For any connected graph $G$, $ENS(G) \leq \alpha(G) - \Lambda(G)$, where $\alpha(G)$ and $\Lambda(G)$ is the independent number and edge-neighbor-connectivity of $G$, respectively.*

*Proof.* For any $ENS$-set $S$ of $G$, $|S| \geq \Lambda(G)$, and $\omega(G/S) \leq \alpha(G)$, so $ENS(G) = \omega(G/S) - |S| \leq \alpha(G) - \Lambda(G)$. □

**Remark 2** The upper bound in Theorem 7 is the best possible. It can be shown by $C_{3,n}(n \geq 2)$ or double star graphs.

**Theorem 8.** *Let $G$ be a connected graph with order $n$. Then $ENS(G) \geq n - 3\alpha'(G)$, where $\alpha'(G)$ is the edge independence number of $G$.*

*Proof.* Let $M$ be a maximum matching in $G$. Then $|M| = \alpha'(G)$ and $G/M$ contains $n - 2\alpha'(G)$ isolated vertices. So $ENS(G) = \max_{S \subseteq E(G)} \{\omega(G/S) - |S|\} \geq \omega(G/M) - |M| = n - 3\alpha'(G)$. □

**Remark 3** The lower bound in Theorem 8 is the best possible. It can be shown by star graphs.

**Lemma 1** ([7]). *For any graph $G$ with order $n$, $\Lambda(G) \leq \lfloor \frac{n}{2} \rfloor$.*

**Theorem 9.** *Let $G$ be a connected graph with order $n(\geq 3)$. Then $ENS(G) \geq 1 - \lfloor \frac{n}{2} \rfloor$.*

*Proof.* Assume $S^*$ is a smallest edge-cut strategy of $G$, then $|S^*| = \Lambda(G)$. By Lemma 1, $|S^*| \leq \lfloor \frac{n}{2} \rfloor$. we distinguish two cases.
**Case 1.** $G/S^* \neq \phi$. Then $\omega(G/S^*) \geq 1$, so $ENS(G) = \max_{S \subseteq E(G)} \{\omega(G/S) - |S|\} \geq \omega(G/S^*) - |S^*| \geq 1 - \lfloor \frac{n}{2} \rfloor$.
**Case 2.** $G/S^* = \phi$. Notice that the subversion of an edge is equivalent to removing it's two endpoints from $G$. Since $n \geq 3$, then $|S^*| \geq 2$. Moreover, in $S^*$, any edge $e$ is not adjacent to other edges, otherwise, $S^* - \{e\}$ is also an edge-cut strategy of $G$, contradicted to the assumption. $G$ is connected, so for any edge $e = (u, v)$ in $S^*$, it's one endpoint, without loss of generality, assume $u$, is adjacent to at least one vertex in $S^*$ except $v$. Let $x$ be such one, set $S^{**} = (S^* - \{e\}) \cup \{(u, x)\}$, then $G/S^{**} = v$, an isolated vertex. Therefore $ENS(G) \geq \omega(G/S^{**}) - |S^{**}| \geq 1 - \Lambda(G) \geq 1 - \lfloor \frac{n}{2} \rfloor$.

The proof thus is completed. □

**Remark 4** The lower bound in Theorem 9 is the best possible. It can be shown by Theorem 3.

# 4 Neighbor Stability and Edge -Neighbor-Scattering Number of Graphs

The stability of a communication network is a prime important factor. As the network begins losing stations or links, eventually there is a loss in its effectiveness. Thus many graph theoretical parameters such as connectivity, edge-connectivity, scattering number, integrity, edge-integrity, tenacity and tenacity etc. have been introduced to describe the stability of communication networks. We call them the *standard stability parameters.*

When we consider the stability of a spy network, one failure station or link lead to all its neighbors to be useless for the network as a whole. Thus unlike standard stability parameters, some other parameters such as neighbor-connectivity, edge- neighbor-connectivity, vertex-neighbor-integrity, edge-neighbor-integrity, vertex-neighbor-scattering number and edge-neighbor-scattering number have been introduced to measure the stability of spy networks. As a generalization of the stability of a spy network, we call these parameters the *neighbor stability parameters.* Clearly, stability in standard and neighbor sense are different. For example, $K_n$, the complete graph with order $n$, $\kappa(K_n) = n - 1$, but $K(K_n) = 0$. That is, $K_n$ is the most stable in the standard sense, but is the most vulnerable in the neighbor sense. It is shown that measuring the neighbor stability of a network is more complex than measuring the standard stability of a network( see [5]). Therefore, only using one parameter to measure the neighbor stability of a network is not enough.

**Example 2:** $\Lambda(S_{1,n}) = \Lambda(DS(n_1, n_2)) = 1, ENI(S_{1,n}) = ENI(DS(n_1, n_2))$ $= 2$. However, $ENS(S_{1,n}) = n - 2$, $ENS(DS(n_1, n_2)) = n_1 + n_2 - 1$. This shown that $ENS$ does better than $\Lambda$ or $ENI$ in measuring the neighbor stability of networks in some cases.

From the definition of edge-neighbor-scattering number we know that, in edge-neighbor sense, the smaller the $ENS$ is, the more stable the network to be. For example, when $n \geq 6$, $ENS(C_n) = 0 < ENS(P_n) = 1$. In fact, Paths and cycles with same order($\geq$ 6), the former are more stable than the latter in the neighbor sense.

Obviously, $ENS$ has closed relation with $\Lambda$ Since $ENS$ considering not only the difficulty to break the network but also the damage has been caused, two networks with same $\Lambda$ maybe have different $ENS$( see Example 1). $ENS$ is also independent to $VNS$. Let $L(S_{1,n})$ be the line graph of star $S_{1,n}$. then $L(S_{1,n}) = K_n$ and $ENS(S_{1,n}) = n - 2 \neq VNS(L(S_{1,n})) = 1$.

Like other neighbor stability parameters, $ENS$ has its defects. For example, if $m = n + 1 (n \geq 4)$, then $ENS(K_{m,n}) = 0 = ENS(C_{m+n})$. But $\Lambda(K_{m,n}) = \min\{m, n\} = n > \Lambda(C_{m+n}) = 2$. In this case, $ENS$ can't measure the difference of neighbor stability between $K_{m,n}$ and $C_{m+n}$.

It seems that using several parameters at same time is a good method to measure the neighbor stability of a network. Unfortunately, it is not always true. For example, if $m = n + 1 (n \geq 4)$, then $\Lambda(K_{m,n}) = \min\{m, n\} = n > \Lambda(C_{m+n}) = 2$, but $VNS(K_{m,n}) = m - 2 > VNS(C_{m+n}) = 0$. Which is actually more stable in neighbor sense?

In order to avoid the difficulty, we divide neighbor stability parameters into two classes: ""vertex neighbor" such as neighbor-connectivity, vertex-neighbor-integrity and vertex-neighbor-scattering number and ""edge neighbor" such as edge-neighbor-connectivity, edge-neighbor-integrity and edge-neighbor-scattering number. This division is necessary and reasonable, since the role of vertices or edges in a network are different, we often need to distinguish them. For example, when measuring the neighbor stability of a network, we can select parameters from one of the two classes according to the importance of vertices and edges in the network.

**Acknowledgements.** This work was supported by NSFC(No. 60642002) , SRF for ROCS of SEM, and BSF(No. AJ12046) of XAUAT. The authors are grateful to an anonymous referee for helpful comments on an earlier version of this article.

# References

[1] G. Gunther and B. L. Hartnell, On Minimizing the Effects of Betrayals in a Resistance Movement, *Proc. English Manitoba Conference on Numerical Mathematics and Computing*, (1978), 285-306.

[2] G. Gunther, On the Neighbor-Connectivity in Regular Graphs, *Discrete Applied Mathematics* **11** (1985), 233-243.

[3] M. B. Cozzens and S.-S. Y. Wu, Vertex-Neighbour-Integrity of Trees, *Ars Combinatoria* **43**, (1996), 169-180.

[4] M. B. Cozzens and S.-S. Y. Wu, Edge-Neighbour-Integrity of Trees, *Australasian Journal of Combinatorics* **10**, (1994), 163-174.

[5] Z. wei and Y. Li, Vertex-Neighbour-Scattering Number of Graphs, submitted.

[6] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London; Elsevier, New York, 1976.

[7] M. B. Cozzens and S.-S. Y. Wu, Extreme Values of the Edge-Neighbor-Connectivity, *Ars Combinatoria* **39**, (1995), 199-210.