Classification of Optimal Linear $\mathbb{Z}_4$ Rate 1/2 Codes of Length ≤ 8 [1]

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Abstract

In this paper, we classify all optimal linear $[n, n/2]$ codes over $\mathbb{Z}_4$ up to length $n = 8$, and determine the number of optimal codes which are self-dual and formally self-dual. Optimal codes with linear binary images are identified. In particular, we show that for length 8, there are nine optimal codes for the Hamming distance, one optimal code for the Lee distance, and two optimal codes for the Euclidean distance.

Keywords–Optimal linear codes, Codes over $\mathbb{F}_4$.

1 Introduction

Linear codes over $\mathbb{Z}_4$ have received a great deal of interest since the pioneering work of Nechaev [6, 7] and Hammons *et al.* [5]. To date, there are few results on optimal codes due to the focus on self-dual codes [3, 4, 8]. This is addressed here where we complete the classification of all optimal rate 1/2 code over $\mathbb{Z}_4$ up to length 8.

A quaternary linear $[n, k]$ code C is a submodule of $\mathbb{Z}_4^n$, where the elements of $\mathbb{Z}_4$ are taken to be $\{0, 1, 2, 3\}$. The elements of C are called codewords.

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The Gray map is defined as $\psi : \mathbb{Z}_4 \to \mathbb{F}_2^2$ by $\psi(0) = 00$, $\psi(1) = 01, \psi(2) = 11, \psi(3) = 10$. This is a non-linear weight preserving map.

There are three weights attached to vectors over $\mathbb{Z}_4$. The Hamming weight of a vector is the number of non-zero coordinates in the vector. The Lee weight of the vector is the Hamming weight of its image under the Gray map ψ. The Euclidean weight of a vector $v = (v_1, v_2, \ldots, v_n)$ is $\sum_{i=1}^{n} \min\{v_i^2, (-v_i)^2\}$. We denote the minimum Hamming weight of a code C by $d_H(C)$, the minimum Lee weight by $d_L(C)$ and the minimum Euclidean weight by $d_E(C)$.

The weight enumerator of C is $W(X) = \sum_{i=0}^{n} A_i X^i$, where A_i is the number of codewords of weight i in C. We use W_H to denote Hamming weight, W_L to denote Lee weight, and W_E to denote Euclidean weight.

The dual code $C^\perp$ of C is defined as $C^\perp = \{x \in \mathbb{Z}_4^n \mid x \cdot y = 0 \text{ for all } y \in C\}$. A code C is *self-dual* if $C = C^\perp$. This corresponds to the family 4^Z in [9]. A code C is *formally self-dual* (FSD) if C and $C^\perp$ have identical weight enumerators. A code is isodual if it is equivalent to its dual, and so is also formally self-dual. Self-dual codes are by definition also isodual.

A linear $[n, k]$ code C is *optimal* if C has the highest minimum weight among all linear $[n, k]$ codes. It is trivial that there is a unique optimal [1,0.5] code with generator matrix $G_{1,1} = (2)$. The Lee, Hamming and Euclidean weight enumerators for this code are

$$\begin{aligned} W_L(G_{1,1}) &= 1 + X^2, \\ W_H(G_{1,1}) &= 1 + X, \\ W_E(G_{1,1}) &= 1 + X^4, \end{aligned}$$

respectively. This is a self-dual code and has as its binary image (via the Gray map) the linear [2,1] repetition code. A natural question that arises is how many inequivalent optimal linear rate 1/2 codes are there for small lengths? In the remainder of this paper, we answer this question by completing the classification of these codes up to length 8. As a result, we determine for what lengths the self-dual codes previously given in [3] are optimal.

2 Classification of Optimal $\mathbb{Z}_4$ Linear Codes

Any linear code over $\mathbb{Z}_4$ is permutation equivalent to a code with generator matrix

$$G = \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2D \end{pmatrix}, \tag{1}$$

where I_k is the k by k identity matrix, A and D are $\mathbb{Z}_2$ matrices, and B is a $\mathbb{Z}_4$ matrix. The *rank* of a code C generated by G is $k = k_1 + \frac{k_2}{2}$. A code of this form is said to be type $\{k_1, k_2\}$, and contains $4^{k_1}2^{k_2}$ codewords. The rate of the code is $(k_1 + \frac{k_2}{2})/n$.

In order to obtain all inequivalent $[n, n/2]$ $\mathbb{Z}_4$ linear codes, A, B and D matrices were considered with the required combinations of k_1 and k_2. Equivalent matrices were eliminated to minimize the computational complexity.

In the classification of codes, we are only interested in one representative from each code equivalence class. Two codes are said to be *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates [3]. For a more theoretical justification of this definition, see [12].

From the inequivalent optimal codes, self-dual codes, isodual codes and FSD codes were identified using programs in the C language, and Magma[1] was used to identify codes that have linear binary images.

The subscript notation n, i is used for the codes and their generator matrices, with a third subscript j used to denote multiple codes with the same weight enumerator. For example, the first optimal $[4, 2]$ $\mathbb{Z}_4$ linear code is denoted as $C_{4,1,1}$ (since there are two codes with this weight enumerator), and its generator matrix is written as $G_{4,1,1}$. Subsequent appearances of the same matrix are denoted the same. To save space, we do not list all generator matrices here, but a complete list of codes up to length 7 can be found in [13] (all optimal rate 1/2 codes of length 8 are given in this paper).

2.1 Classification by Lee Distance

This section presents the classification of rate 1/2 Lee distance optimal $\mathbb{Z}_4$ codes up to length 8.

[1] For more information on Magma, see http://magma.maths.usyd.edu.au/magma.

Length 2

All optimal $\mathbb{Z}_4$ codes of length two are equivalent to one of the three codes generated by

$$G_{2,1,1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \; G_{2,1,2} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \; G_{2,2} = \begin{pmatrix} 1 & 2 \end{pmatrix}.$$

The corresponding Lee weight enumerators are

$$\begin{aligned} W_L(G_{2,1}) &= 1 + 2X^2 + X^4, \\ W_L(G_{2,2}) &= 1 + X^2 + 2X^3, \end{aligned}$$

respectively. The code generated by $G_{2,1,1}$ is self-dual. The other two codes are isodual. All three codes have linear binary images.

Length 3

There are eight inequivalent optimal $\mathbb{Z}_4$ codes of length three generated by

$$G_{3,1,1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \; G_{3,1,2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \; G_{3,2,1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{pmatrix},$$

$$G_{3,3} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \; G_{3,4,1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix}, \; G_{3,5} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

$$G_{3,4,2} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}, G_{3,2,2} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix},$$

with Lee weight enumerators

$$\begin{aligned} W_L(G_{3,1}) &= 1 + 3X^2 + 3X^4 + X^6, \\ W_L(G_{3,2}) &= 1 + 2X^2 + 5X^4, \\ W_L(G_{3,3}) &= 1 + 2X^2 + 2X^3 + X^4 + 2X^5, \\ W_L(G_{3,4}) &= 1 + X^2 + 4X^3 + X^4 + X^6, \\ W_L(G_{3,5}) &= 1 + 6X^2 + X^4, \end{aligned}$$

respectively. There is one self-dual code generated by $G_{3,1,1}$. In addition, there are two more FSD codes generated by $G_{3,1,2}$ and $G_{3,3}$, both of which are isodual. All eight codes have a linear binary image.

Length 4
There are two inequivalent length 4 codes with the following generator matrices

$$G_{4,1,1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}, \; G_{4,1,2} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

The Lee weight enumerator for these codes is

$$W_L(G_{4,1}) = 1 + 14X^4 + X^8.$$

The code generated by $G_{4,1,1}$ is self-dual, and the code generated by $G_{4,1,2}$ is isodual. Both codes have linear binary images.
Length 5
There are five optimal codes of length 5 generated by

$$G_{5,1,1} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix}, \; G_{5,1,2} = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 0 & 2 \end{pmatrix},$$

$$G_{5,1,3} = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix}, \; G_{5,1,4} = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 & 0 \end{pmatrix},$$

$$G_{5,2} = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 \end{pmatrix}.$$

The Lee weight enumerators for these codes are

$$\begin{aligned} W_L(G_{5,1}) &= 1 + 18X^4 + 8X^6 + 5^8, \\ W_L(G_{5,2}) &= 1 + 16X^4 + 12X^6 + 3X^8, \end{aligned}$$

respectively. None of these is self-dual, isodual or FSD. The first four codes have a linear binary image.
Length 6
the number of inequivalent optimal codes is 46. The completed list of generator matrices is given in [13]. There is one optimal self-dual code

generated by

$$G_{6,1,1} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2 \end{pmatrix}.$$

In addition to this self-dual code, there are 15 additional FSD codes, all of which are isodual. These codes are generated by

$$G_{6,1,2} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{pmatrix}, \; G_{6,1,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 & 0 \end{pmatrix},$$

$$G_{6,2,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 \end{pmatrix}, \; G_{6,1,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2 \end{pmatrix},$$

$$G_{6,1,5} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2 \end{pmatrix}, \; G_{6,1,6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{pmatrix},$$

$$G_{6,1,7} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}, \; G_{6,3,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 3 \end{pmatrix},$$

$$G_{6,1,8} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{pmatrix}, \; G_{6,2,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix},$$

$$G_{6,1,9} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{pmatrix}, \; G_{6,3,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix},$$

$$G_{6,1,10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{pmatrix}, \; G_{6,4} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 3 & 2 \end{pmatrix},$$

$$G_{6,1,11} = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{pmatrix}.$$

The corresponding Lee weight enumerators are

$$\begin{aligned} W_L(G_{6,1}) &= 1 + 15X^4 + 32X^6 + 15X^8 + X^{12}, \\ W_L(G_{6,2}) &= 1 + 10X^4 + 16X^5 + 12X^6 + 16X^7 + 5X^8 + 4X^{10}, \\ W_L(G_{6,3}) &= 1 + 9X^4 + 18X^5 + 13X^6 + 12X^7 + 6X^8 + 2X^9 + 3X^{10}, \\ W_L(G_{6,4}) &= 1 + 6X^4 + 24X^5 + 16X^6 + 9X^8 + 8X^9, \end{aligned}$$

respectively. Six of these codes do not have a linear binary image, namely $G_{6,2,1}, G_{6,1,7}$, $G_{6,2,2}, G_{6,3,1}, G_{6,3,2}$ and $G_{6,4}$. In total, 22 of the optimal codes do not have linear binary images.

Length 7

The number of optimal length seven $\mathbb{Z}_4$ linear codes is 257, however, only seven of these are FSD. The complete list of generator matrices is given at [13]. The only self-dual code is generated by

$$G_{7,1,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 2 & 0 & 2 \end{pmatrix}.$$

The other six FSD codes are generated by

$$G_{7,2,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 \end{pmatrix}, \quad G_{7,2,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \end{pmatrix},$$

$$G_{7,2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 \end{pmatrix}, \quad G_{7,1,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 0 \end{pmatrix},$$

$$G_{7,3,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 & 2 \end{pmatrix}, \quad G_{7,3,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 \end{pmatrix}.$$

The Lee weight enumerators for these codes are

$$\begin{aligned} W_L(G_{7,1}) &= 1+14X^4+49X^6+49X^8+14X^{10}+X^{14}, \\ W_L(G_{7,2}) &= 1+9X^4+16X^5+24X^6+32X^7+19X^8+16X^9+8X^{10} \\ &\quad +3X^{12}, \\ W_L(G_{7,3}) &= 1+7X^4+20X^5+24X^6+28X^7+23X^8+12X^9+8X^{10} \\ &\quad +4X^{11}+1X^{12}, \end{aligned}$$

respectively. It is interesting that the first 4 of these 6 codes are isodual, while the last two are duals of each other. These are the first examples of optimal FSD codes which are not isodual. None of these codes has a linear binary image, but 118 of the 357 optimal codes do have a linear binary image.

Length 8

There is one optimal length eight code generated by

$$G_{8,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 & 3 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 1 & 3 & 3 \end{pmatrix},$$

with weight enumerator

$$W_L(G_{8,1}) = 1+112x^6+30x^8+112x^{10}+x^{16}.$$

This is code O_8 (octacode) in [9, 3], and it does not have a linear binary image.

2.2 Classification by Hamming Distance

This section presents the classification of rate 1/2 Hamming distance optimal linear $\mathbb{Z}_4$ codes up to length 8.

Length 2

There is one optimal length 2 code generated by $G_{2,1,2}$ given in the previous section. It has Hamming weight enumerator

$$W_H(G_{2,1,2}) = 1+3X^2.$$

Length 3

There is one optimal code of length 3 and it is generated by $G_{3,2,1}$ given in the previous section. The Hamming weight enumerator for this code is

$$W_H(G_{3,2,1}) = 1 + 5X^2 + 2X^3.$$

Length 4

The number of optimal codes of length 4 is nine. The inequivalent codes are generated by $G_{4,1,1}$ and $G_{4,1,2}$ in the previous section and

$$G_{4,2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}, G_{4,3} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$G_{4,4} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, G_{4,5} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

$$G_{4,6} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}, G_{4,7} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix},$$

$$G_{4,8} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix},$$

The corresponding Hamming weight enumerator for these codes are

$$\begin{aligned}
W_H(G_{4,1,1}) &= 1 + 6X^2 + 9X^4, \\
W_H(G_{4,1,2}) &= 1 + 2X^2 + 8X^3 + 5X^4, \\
W_H(G_{4,2}) &= 1 + 8X^2 + 4X^3 + 3X^4, \\
W_H(G_{4,3}) &= 1 + 9X^2 + 6X^3, \\
W_H(G_{4,4}) &= 1 + 6X^2 + 9X^4, \\
W_H(G_{4,5}) &= 1 + 3X^2 + 6X^3 + 6X^4, \\
W_H(G_{4,6}) &= 1 + 4X^2 + 4X^3 + 7X^4, \\
W_H(G_{4,7}) &= 1 + 5X^2 + 6X^3 + 4X^4, \\
W_H(G_{4,8}) &= 1 + X^2 + 10X^3 + 4X^4,
\end{aligned}$$

respectively.

As noted in the previous section, $G_{4,1,1}$ generates a self-dual code. In addition, there are 5 other FSD codes generated by $G_{4,4}$, $G_{4,5}$, $G_{4,6}$, $G_{4,8}$, and $G_{4,1,2}$, which are all isodual. The codes generated by $G_{4,3}$, $G_{4,5}$, $G_{4,7}$ and $G_{4,8}$ do not have linear binary images.

Length 5

There are 27 inequivalent codes of length 5, and these are listed in [13]. There is one FSD (isodual) code generated by

$$G_{5,3} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 & 0 \end{pmatrix},$$

and the Hamming weight enumerator of this code is

$$W_H(G_{5,3}) = 1 + 2X^2 + 8X^3 + 13X^4 + 8X^5.$$

Thirteen of the 27 codes do not have a linear binary image, including the code given above.

Length 6

There are five optimal Hamming distance length 6 codes with generator matrices $G_{6,4}$ given previously and

$$G_{6,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, G_{6,6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$G_{6,7} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{pmatrix}, G_{6,8} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix},$$

with Hamming weight enumerators

$$\begin{aligned} W_H(G_{6,4}) &= 1 + 4X^3 + 33X^4 + 12X^5 + 14X^6, \\ W_H(G_{6,5}) &= 1 + 10X^3 + 15X^4 + 30X^5 + 8X^6, \\ W_H(G_{6,6}) &= 1 + 12X^3 + 9X^4 + 36X^5 + 6X^6, \\ W_H(G_{6,7}) &= 1 + 8X^3 + 21X^4 + 24X^5 + 10X^6, \\ W_H(G_{6,8}) &= 1 + 6X^3 + 27X^4 + 18X^5 + 12X^6, \end{aligned}$$

respectively. All five codes are isodual, but none have linear binary images.

Length 7

There are 11 inequivalent length 7 Hamming distance $\mathbb{Z}_4^7$ linear codes. The self-dual code is generated by $G_{7,1,1}$ and has Hamming weight enumerator

$$W_H(G_{7,1}) = 1 + 7X^3 + 21X^4 + 42X^5 + 42X^6 + 15X^7.$$

In addition, there is another isodual code generated by $G_{7,1,2}$ with Hamming weight enumerator $W_H(G_{7,1})$. The remaining nine codes are generated by

$$G_{7,4} = \begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&1&0&0&1&0&1 \\ 0&0&1&0&1&1&0 \\ 0&0&0&2&2&2&2 \end{pmatrix}, \quad G_{7,5} = \begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&1&0&0&1&0&1 \\ 0&0&1&0&1&1&1 \\ 0&0&0&2&2&2&0 \end{pmatrix},$$

$$G_{7,6} = \begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&1&0&0&1&0&1 \\ 0&0&1&0&1&1&3 \\ 0&0&0&2&2&2&0 \end{pmatrix}, \quad G_{7,7} = \begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&1&0&0&1&1&2 \\ 0&0&1&0&1&2&1 \\ 0&0&0&2&2&2&2 \end{pmatrix},$$

$$G_{7,8} = \begin{pmatrix} 1&0&0&0&1&1&1 \\ 0&1&0&0&1&2&3 \\ 0&0&1&0&1&3&2 \\ 0&0&0&2&0&2&2 \end{pmatrix}, \quad G_{7,9} = \begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&1&0&0&1&0&1 \\ 0&0&1&1&0&0&1 \\ 0&0&0&2&2&2&0 \end{pmatrix},$$

$$G_{7,10} = \begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&1&0&0&1&0&1 \\ 0&0&1&1&0&2&1 \\ 0&0&0&2&2&2&0 \end{pmatrix}, \quad G_{7,11} = \begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&1&0&0&1&1&2 \\ 0&0&1&1&0&1&2 \\ 0&0&0&2&2&0&2 \end{pmatrix},$$

$$G_{7,12} = \begin{pmatrix} 1&0&0&0&1&1&1 \\ 0&1&0&0&1&2&3 \\ 0&0&1&1&1&0&3 \\ 0&0&0&2&0&2&2 \end{pmatrix},$$

with Hamming weight enumerators

$$\begin{aligned}
W_H(G_{7,4}) &= 1 + 13X^3 + 21X^4 + 54X^5 + 26X^6 + 13X^7, \\
W_H(G_{7,5}) &= 1 + 15X^3 + 13X^4 + 66X^5 + 18X^6 + 15X^7, \\
W_H(G_{7,6}) &= 1 + 11X^3 + 29X^4 + 42X^5 + 34X^6 + 11X^7, \\
W_H(G_{7,7}) &= 1 + 9X^3 + 37X^4 + 30X^5 + 42X^6 + 9X^7, \\
W_H(G_{7,8}) &= 1 + 7X^3 + 45X^4 + 18X^5 + 50X^6 + 7X^7, \\
W_H(G_{7,9}) &= 1 + 13X^3 + 13X^4 + 54X^5 + 18X^6 + 29X^7,
\end{aligned}$$

$$
\begin{aligned}
W_H(G_{7,10}) &= 1 + 11X^3 + 21X^4 + 42X^5 + 26X^6 + 27X^7, \\
W_H(G_{7,11}) &= 1 + 9X^3 + 29X^4 + 30X^5 + 34X^6 + 25X^7, \\
W_H(G_{7,12}) &= 1 + 7X^3 + 37X^4 + 18X^5 + 42X^6 + 23X^7,
\end{aligned}
$$

respectively. None of these codes has a linear binary image.

Length 8

There are nine optimal length eight codes generated by $G_{8,1}$ and

$$
G_{8,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \ G_{8,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \end{pmatrix},
$$

$$
G_{8,4,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 & 2 \end{pmatrix}, \ G_{8,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 & 0 \end{pmatrix},
$$

$$
G_{8,4,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 & 2 \end{pmatrix}, \ G_{8,6,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 3 & 2 & 1 \end{pmatrix},
$$

$$
G_{8,4,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 3 & 2 & 3 \end{pmatrix}, \ G_{8,6,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 & 3 & 2 & 1 \end{pmatrix},
$$

with Hamming weight enumerators

$$
\begin{aligned}
W_H(G_{8,1}) &= 1 + 14X^4 + 112X^5 + 112X^7 + 17X^8, \\
W_H(G_{8,2}) &= 1 + 34X^4 + 32X^5 + 120X^6 + 32X^7 + 37X^8, \\
W_H(G_{8,3}) &= 1 + 38X^4 + 16X^5 + 144X^6 + 16X^7 + 41X^8, \\
W_H(G_{8,4}) &= 1 + 30X^4 + 48X^5 + 96X^6 + 48X^7 + 33X^8, \\
W_H(G_{8,5}) &= 1 + 26X^4 + 64X^5 + 72X^6 + 64X^7 + 29X^8, \\
W_H(G_{8,6}) &= 1 + 22X^4 + 80X^5 + 48X^6 + 80X^7 + 25X^8,
\end{aligned}
$$

respectively. None of these codes has a linear binary image. $C_{8,1}$ and $C_{8,4,3}$ are the self-dual codes codes O_8 and E_8, respectively, in [3]. The remaining 7 codes are isodual.

2.3 Classification by Euclidean Distance

This section presents the classification of rate 1/2 Euclidean distance optimal $\mathbb{Z}_4$ codes up to length 8.

Length 2

The are two Euclidean distance optimal [2,1] linear codes generated by $G_{2,1}$ and $G_{2,2}$. The corresponding Euclidean weight enumerators are

$$\begin{aligned} W_E(G_{2,1,1}) &= 1 + 2X^4 + X^8, \\ W_E(G_{2,2}) &= 1 + X^4 + 2X^5, \end{aligned}$$

respectively.

Length 3

There are four optimal length 3 linear codes with generator matrices $G_{3,1,1}$, $G_{3,1,2}$, $G_{3,4,1}$ and $G_{3,2,2}$. The Euclidean weight enumerators of these codes are

$$\begin{aligned} W_E(G_{3,1,1}) &= 1 + 3X^4 + 3X^8 + X^{12}, \\ W_E(G_{3,1,2}) &= 1 + 2X^4 + 2X^5 + X^8 + 2X^9, \\ W_E(G_{3,4,1}) &= 1 + X^4 + 4X^5 + X^8 + X^{12}, \\ W_E(G_{3,2,2}) &= 1 + 2X^4 + 4X^6 + X^8, \end{aligned}$$

respectively.

Length 4

There are thirteen optimal Euclidean distance length 4 codes generated by $G_{4,1,1}, G_{4,1,2}$, and the following

$$G_{4,9} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad G_{4,10} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

$$G_{4,11} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}, \quad G_{4,12} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix},$$

$$G_{4,13} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad G_{4,14} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

$$G_{4,15} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, G_{4,16} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

$$G_{4,17} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}, G_{4,18} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix},$$

$$G_{4,19} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix},$$

The Euclidean weight enumerators for these codes are

$$\begin{aligned}
W_E(G_{4,1,1}) &= 1 + 8X^4 + 6X^8 + X^{16}, \\
W_E(G_{4,1,2}) &= 1 + 4X^4 + 8X^6 + 2X^8 + X^{16}, \\
W_E(G_{4,9}) &= 1 + 4X^4 + 6X^8 + 4X^{12} + X^{16}, \\
W_E(G_{4,10}) &= 1 + 3X^4 + 2X^5 + 3X^8 + 4X^9 + X^{12} + 2X^{13}, \\
W_E(G_{4,11}) &= 1 + 2X^4 + 4X^5 + 2X^8 + 4X^9 + 2X^{12} + X^{16}, \\
W_E(G_{4,12}) &= 1 + X^4 + 6X^5 + 3X^8 + 3X^{12} + 2X^{13}, \\
W_E(G_{4,13}) &= 1 + 3X^4 + 4X^6 + 3X^8 + 4X^{10} + X^{12}, \\
W_E(G_{4,14}) &= 1 + 2X^4 + 8X^6 + 2X^8 + 2X^{12} + X^{16}, \\
W_E(G_{4,15}) &= 1 + 10X^4 + 2X^8 + 2X^{12} + X^{16}, \\
W_E(G_{4,16}) &= 1 + 3X^4 + 8X^7 + 3X^8 + X^{12}, \\
W_E(G_{4,17}) &= 1 + 2X^4 + 4X^5 + X^8 + 4X^9 + 4X^{10}, \\
W_E(G_{4,18}) &= 1 + X^4 + 4X^5 + 2X^6 + 4X^7 + X^8 + 2X^{10} + X^{12}, \\
W_E(G_{4,19}) &= 1 + 2X^4 + 2X^5 + 4X^6 + X^8 + 4X^9 + 2X^{13},
\end{aligned}$$

respectively.

There are two self-dual codes, generated by $G_{4,1,1}$ and $G_{4,9}$. In addition to the two self-dual codes, there are another six isodual codes generated by $G_{4,1,2}$, $G_{4,10}$, $G_{4,14}$, $G_{4,17}$, $G_{4,18}$, and $G_{4,19}$. All 13 codes have linear binary images.

Length 5

The number of Euclidean distance optimal length 5 codes is 51. The codes are listed in [13]. Two of the codes are self-dual and generated by

$$G_{5,4} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad G_{5,5} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix},$$

with Euclidean weight enumerators

$$\begin{aligned} W_E(G_{5,4}) &= 1 + 5X^4 + 10X^8 + 10X^{12} + 5X^{16} + X^{20}, \\ W_E(G_{5,5}) &= 1 + 9X^4 + 14X^8 + 6X^{12} + X^{16} + X^{20}, \end{aligned}$$

respectively. In addition to the two self-dual codes, there are another six isodual codes generated by

$$G_{5,6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \quad G_{5,7} = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix},$$

$$G_{5,8} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}, \quad G_{5,9} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix},$$

$$G_{5,10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}, \quad G_{5,11} = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

The Euclidean weight enumerators for these six isodual codes are

$$\begin{aligned} W_E(G_{5,6}) &= 1 + 4X^4 + 2X^5 + 6X^8 + 6X^9 + 4X^{12} + 6X^{13} + X^{16} + 2X^{17}, \\ W_E(G_{5,7}) &= 1 + 3X^4 + 8X^6 + 4X^8 + 8X^{10} + 4X^{12} + 3X^{16} + X^{20}, \\ W_E(G_{5,8}) &= 1 + 3X^4 + 4X^5 + 3X^8 + 8X^9 + 4X^{10} + X^{12} + 4X^{13} + 4X^{14}, \\ W_E(G_{5,9}) &= 1 + 2X^4 + 4X^5 + 2X^6 + 4X^7 + 2X^8 + 4X^9 + 4X^{10} + 4X^{11} \\ &\quad + 2X^{12} + 2X^{14} + X^{16}, \\ W_E(G_{5,10}) &= 1 + 3X^4 + 2X^5 + 4X^6 + 3X^8 + 6X^9 + 4X^{10} + X^{12} + 6X^{13} \\ &\quad + 2X^{17}, \\ W_E(G_{5,11}) &= 1 + 5X^4 + 8X^6 + 6X^8 + 8X^{10} + 2X^{12} + X^{16} + X^{20}, \end{aligned}$$

respectively. All six of these codes have linear binary images. Of the 51 inequivalent codes, all but two have linear binary images.

Length 6

There are three Euclidean distance optimal length 6 codes generated by $G_{6,1,2}, G_{6,1,4}$ and $G_{6,1,8}$ with Euclidean weight enumerators

$$\begin{aligned}
W_E(G_{6,1,2}) &= 1+32X^6+15X^8+15X^{16}+X^{24},\\
W_E(G_{6,1,4}) &= 1+24X^6+23X^8+8X^{14}+7X^{16}+X^{24},\\
W_E(G_{6,1,8}) &= 1+20X^6+27X^8+12X^{14}+3X^{16}+X^{24},
\end{aligned}$$

respectively. All three codes are isodual and have linear binary images.

Length 7

There are 14 optimal codes of length 7, none of which are FSD. The codes are listed in [13]. Nine of these codes have linear binary images, for example

$$G_{7,4} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1\\ 0 & 2 & 0 & 0 & 0 & 0 & 2\\ 0 & 0 & 2 & 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 2 & 0 & 0 & 2\\ 0 & 0 & 0 & 0 & 2 & 0 & 2\\ 0 & 0 & 0 & 0 & 0 & 2 & 2 \end{pmatrix},$$

with Euclidean weight enumerator

$$W_E(G_{7,4}) = 1+32X^6+21X^8+32X^{10}+35X^{16}+7X^{24}.$$

Length 8

There are four optimal length eight codes generated by $G_{8,1}$ given previously and

$$G_{8,7} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2\\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2\\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2\\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \end{pmatrix},$$

$$G_{8,8} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \end{pmatrix},$$

$$G_{8,9} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \end{pmatrix},$$

with weight enumerators

$$\begin{aligned} W_E(G_{8,1}) &= 1 + 128X^8 + 126X^{16} + X^{32}, \\ W_E(G_{8,7}) &= 1 + 156X^8 + 70X^{16} + 28X^{24} + X^{32}, \\ W_E(G_{8,8}) &= 1 + 140X^8 + 102X^{16} + 12X^{24} + X^{32}, \\ W_E(G_{8,9}) &= 1 + 132X^8 + 118X^{16} + 4X^{24} + X^{32}, \end{aligned}$$

All four codes are self-dual, and they correspond to O_8, K_8, K_8' and Q_8, respectively in [3]. All but $C_{8,1}$ has a linear binary image.

3 Summary

The problem of classifying optimal linear codes is more difficult than classifying self-dual codes as the significant structure of the latter codes can be utilized in that process. So, whereas self-dual codes have been classified up to length 16 [3][4][8], we had to stop the current classification at length 8.

Let $N_T(n)$, $N_S(n)$ and $N_F(n)$ denote the number of optimal codes, the number of optimal self-dual codes and the number of optimal formally self-dual codes which are not self-dual. In Tables 1 to 3 we list the values of $N_T(n)$, $N_S(n)$ and $N_F(n)$ for $1 \leq n \leq 8$. The number of optimal codes which have a linear binary image is given in brackets. Note that the self-dual codes presented here are the same as in [3] and [4], which provides some verification of our results.

Table 1: $N_T(n)$, $N_S(n)$ and $N_F(n)$ for Lee Distance and $1 \le n \le 8$.

n	d_L	$N_T(n)$	$N_S(n)$	$N_F(n)$
1	2	1(1)	1(1)	0(0)
2	2	3(3)	1(1)	2(2)
3	2	8(8)	1(1)	2(2)
4	4	2(2)	1(1)	1(1)
5	5	5(4)	0(0)	0(0)
6	6	46(22)	1(1)	15(9)
7	6	357(118)	1(0)	6(0)
8	6	1(0)	1(0)	0(0)

Table 2: $N_T(n)$, $N_S(n)$ and $N_F(n)$ for Hamming Distance and $1 \le n \le 8$.

n	d_H	$N_T(n)$	$N_S(n)$	$N_F(n)$
1	1	1(1)	1(1)	0(0)
2	2	1(1)	0(1)	1(1)
3	2	1(1)	0(0)	0(0)
4	2	9(5)	1(1)	5(3)
5	2	27(14)	0(0)	1(0)
6	3	5(0)	0(0)	5(0)
7	3	1(0)	1(0)	1(0)
8	4	9(0)	2(0)	7(0)

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Table 3: $N_T(n)$, $N_S(n)$ and $N_F(n)$ for Euclidean Distance and $1 \le n \le 8$.

n	d_E	$N_T(n)$	$N_S(n)$	$N_F(n)$
1	4	1(1)	1(1)	0(0)
2	4	2(2)	1(1)	1(1)
3	4	4(4)	1(1)	1(1)
4	4	13(13)	2(2)	6(6)
5	4	51(49)	2(2)	6(6)
6	6	3(3)	0(0)	3(3)
7	6	14(9)	0(0)	0(0)
8	8	4 (3)	4 (3)	0 (0)

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