ON H-CHROMATIC UNIQUENESS OF TWO LINEAR UNIFORM CYCLES HAVING SOME EDGES IN COMMON

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ABSTRACT. In this paper it is proved the h-chromatic uniqueness of the linear h-hypergraph consisting of two cycles of lengths p and q having r edges in common when $p = q$, $2 \leq r \leq p-2$ and $h \geq 3$. We also obtain the chromatic polynomial of a connected unicyclic linear h-hypergraph and show that every h-uniform cycle of length three is not chromatically unique for $h \geq 3$.

Keywords: *hypergraph, cycle, chromatic polynomial, h-chromatic uniqueness*

1. Notation and preliminary results

A *simple hypergraph* $H = (X, \mathbb{E})$, with *order* $|X|$ and *size* $m = |\mathbb{E}|$, consists of a *vertex-set* $V(H) = X$ and an *edge − set* $\mathbb{E}(H) = \mathbb{E}$, where $E \subseteq X$ and $|E| \geq 2$ for each $E \in \mathbb{E}$. H is linear if no two edges intersect in more than one vertex, and H is *h-uniform*, or is an *h−hypergraph*, if $|E| = h$ for each $E \in \mathbb{E}$. The number of edges containing a vertex x is its *degree* $d_H(x)$.

A *path* of *length* k joining vertices u and v in H is a subhypergraph consisting of $k+1$ distinct vertices $x_0 = u, x_1, ..., x_k = v$

*Research partially supported by the School of Mathematical Sciences, Lahore and by the Higher Education Commission of Pakistan.

and k distinct edges $E_1, ..., E_k$ of H such that $x_{i-1}, x_i \in E_i$ for each i $(1 \leq i \leq k)$.
Similarly, a *cycle* C of length k in H [1] is a subhypergraph comprising k distinct vertices $x_1, ..., x_k$ and k distinct edges $E_1, ..., E_k$ of H such that $x_i, x_{i+1} \in E_i$ for each i, $1 \leq i \leq k-1$ and $x_1, x_k \in E_k$. C is *elementary* if $d_C(x_i) = 2$ for each i and $d_C(y) = 1$ for each other vertex y in $\cup_{i=1}^{k} E_i$. Two vertices u, v of H are in the same *component*, if there is a path joining them. If H has only one component then it is *connected*; otherwise it has $r \geq 2$ *connected components*. We shall denote an elementary $h-uniform$ cycle with p edges by C_p^h; clearly it has order $p(h-1)$. An *h-uniform hypertree* with p edges denoted by T_p^h is a connected linear $h-hypergraph$ without cycles.

If H is a hypergraph and $\lambda \in \mathbb{N}$, a $\lambda-coloring$ of H is a function $f : V(H) \rightarrow \{1, ..., \lambda\}$ such that for each edge E of H there exist x, y in E for which $f(x) \neq f(y)$. The number of $\lambda-colorings$ of H is given by a polynomial $P(H, \lambda)$ of degree $|V(H)|$ in λ, called the *chromatic polynomial* of H. Two hypergraphs H and G are said to be *chromatically equivalent* or χ-equivalent, written $H \sim G$, if $P(H, \lambda) = P(G, \lambda)$. A simple hypergraph H is *chromatically unique* if $H' \cong H$ for every simple hypergraph H' such that $H' \sim H$; that is, the structure of H is uniquely determined up to isomorphism by its chromatic polynomial. The notion of χ-unique graphs was first introduced and studied by Chao and Whitehead [3](see also [5]).

The notion of χ-uniqueness in the class of h-hypergraphs may be defined as follows: an h-hypergraph H is said to be h-chromatically unique if H is isomorphic to H' for every h-hypergraph H' such that $H' \sim H$ (see [7]). A connected linear h-hypergraph H is said to be *minimally connected* if for every edge $E \in \mathbb{E}(H)$, $H - E$ is not connected. Let $c_g(H)$ denote the number of cycles of length g in H. We recall some results which will be used in the next section.

Lemma 1.1.[6]: *Let H be a hypergraph of order n. Then*

$$P(H,\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_2\lambda^2 + a_1\lambda,$$

where

$$a_i = \sum_{j\geq 0}(-1)^j N(i,j) \qquad for \qquad 1 \leq i \leq n-1,$$

where $N(i,j)$ represents the number of subhypergraphs of H with n vertices, i components and j edges.

Lemma 1.2.[4]: *If T_p^h is an h-uniform hypertree with p edges, then*

$$P(T_p^h,\lambda) = \lambda(\lambda^{h-1}-1)^p$$

and

$$P(C_p^h,\lambda) = (\lambda^{h-1}-1)^p + (-1)^p(\lambda-1).$$

Lemma 1.3. [6]: *If simple h-hypergraphs H and G are χ-equivalent and H is linear then G is linear too.*

Lemma 1.4. [6]: *Let H be a nonacyclic linear h-hypergraph of order n, size m, and girth $g(H) = g$. If the h-hypergraph G is chromatically equivalent to H and $(g,h) \neq (3,3)$ then G has the same order, size, and girth as H and $c_g(G) = c_g(H)$.*

2 Main Results

Theorem 2.1: *The chromatic polynomial of a connected unicyclic linear h-hypergraph H containing an elementary cycle C_p^h and m edges is*

$$P(H,\lambda) = [(\lambda^{h-1}-1)^p + (-1)^p(\lambda-1)](\lambda^{h-1}-1)^{m-p}.$$

Proof: Let H be a connected unicyclic linear h-hypergraph consisting of an elementary h-uniform cycle C_p^h of size p and distinct h-hypertrees of sizes $m_1, m_2, \ldots, m_k$ which have each a vertex common with distinct vertices of C_p^h. We shall use induction on the number k of hypertrees to prove the result. If $k = 0$, then $H = C_p^h$ so $m = p$ and the chromatic polynomial of H follows from Lemma 1.2. Let $k \geq 1$ and suppose the result is true for any connected unicyclic linear h-hypergraph consisting of C_p^h and $k-1$ distinct hypertrees having each a vertex common with distinct vertices of C_p^h. Let H be a connected unicyclic linear h-hypergraph defined by C_p^h and k hypertrees $T_1, \ldots, T_k$ of sizes $m_1, m_2, \ldots, m_k$, respectively. By deleting the edges of T_k, the resulting hypergraph H' has only $k-1$ hypertrees. Since T_k and H' have only a common vertex, we can write

$$P(H,\lambda) = \frac{P(H',\lambda)P(T_k,\lambda)}{\lambda}$$

and applying Lemma 1.2 and the induction hypothesis for H' yields

$$P(H,\lambda) = [(\lambda^{h-1}-1)^p + (-1)^p(\lambda-1)](\lambda^{h-1}-1)^{m-p}$$

since $m_1 + \ldots + m_k = m - p$. □

Note that the result expressed by Theorem 2.1 was also obtained very recently by Borowiecki and Lazuka [2]. They also proved that both uniform hypertrees and uniform unicyclic hypergraphs are chromatically characterized in the class of linear hypergraphs.

Theorem 2.2: *Let H be a minimally connected linear h-hypergraph. If G is another h-hypergraph such that $H \sim G$, then G is also connected.*

Proof: Let H be of order n, size m and $G \sim H$. By lemma 1.3, G must be linear, having the same order n. Suppose that G is not connected, further assume that it consists of $k \geq 2$ components. Since the chromatic polynomial of a hypergraph consisting of $k \geq 2$ components $C_1, ..., C_k$ is $\prod_{i=1}^{k} P(C_i, \lambda)$ and by making use of Lemma 1.1 it is easy to see that the last term in $P(G, \lambda)$ will have the degree at least k, which is different for all $k \geq 2$ from $(-1)^m \lambda$, the last term in $P(H, \lambda)$, which contradicts the fact that $G \sim H$. Note that $(-1)^m \lambda$ cannot be canceled by terms corresponding to other edge selections, since H is minimally connected. □

Remark: The above result is not valid if h-hypergraph H is not minimally connected. Consider the linear uniform 3-hypergraph H as shown in figure 1, there exists an edge E_4 whose deletion

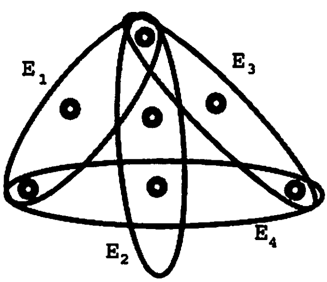


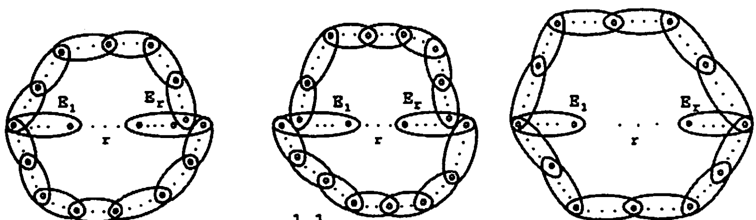
Figure 2

gives a connected subhypergraph. Using Lemma 1.1 we deduce that the chromatic polynomial of H is

$$P(H,\lambda) = \lambda^7 - 4\lambda^5 + 6\lambda^3 - 3\lambda^2.$$

Here we may note that the last term vanishes so the least non-vanishing term of the chromatic polynomial gives no information about the connectedness of a hypergraph G, such that $G \sim H$. □

Let $B^{1,1}_{p,p}$, $B^{1,2}_{p,p}$ and $B^{2,2}_{p,p}$ denote three kinds of bicycles defined as follows: Consider an elementary h-uniform cycle with p edges C^h_p $(p \geq 4, h \geq 3)$ and an elementary h-uniform path P^h_{p-r} with $p-r$ $(r \geq 2)$ edges having extremities x and y such that $V(C^h_p) \cap V(P^h_{p-r}) = \emptyset$. $B^{i,j}_{p,p}$ is obtained by identifying x and y with distinct vertices of degree i and j of E_1 and E_r of C^h_p $(1 \leq i,j \leq 2)$, where C^h_p is defined by edges $E_1, ..., E_p$ in this order. $B^{1,1}_{p,p}$ and $B^{1,2}_{p,p}$ are shown in the following figure 2.



$B^{1,2}_{p,p}$ $B^{1,1}_{p,p}$ $B^{2,2}_{p,p}$

Figure 2

Theorem 2.3: *$B^{1,1}_{p,p}$ and $B^{1,2}_{p,p}$ are not χ-equivalent for every $r \geq 2$ and $h \geq 3$.*

Proof: It is clear that both have the same size $m = 2p-r$, order $n = (2p-r)(h-1)-1$ and girth p. The number $c(G)$ of components in a subhypergraph G of both hypergraphs having n vertices and j edges satisfies $c(G) = n-j(h-1)$, when $0 \leq j \leq p-1$.

Moreover, if $p \leq j \leq m-1$ then $c(G) = n - j(h-1)$ or $c(G) = n-j(h-1)+1$, depending on whether G is or is not cycle-free [6]. If $j = m$ then $c(G) = 1$. We may note that if $r \geq 2$ and $h \geq 3$ then the coefficient of λ^{h-2} in $P(B^{1,2}_{p,p}, \lambda)$ is $(-1)^{r-1}$ while as the coefficient of λ^{h-2} in $P(B^{1,1}_{p,p}, \lambda)$ is $2(-1)^{r-1}$, since there are only one selection and two selections of $2p-r-1$ edges in $B^{1,2}_{p,p}$ and $B^{1,1}_{p,p}$ respectively, yielding subhypergraphs having $h-2$ components. So, $B^{1,2}_{p,p}$ and $B^{1,1}_{p,p}$ are not chromatically equivalent. □

Next result is a generalization of the result proved in [6].

Theorem 2.4: *$B^{2,2}_{p,p}$ is h-chromatically unique for every $r \geq 2, p \geq 4$ and $h \geq 3$.*

Proof: It is clear that $B^{2,2}_{p,p}$ has size $m = 2p - r$, order $n = (2p-r)(h-1)-1$ and girth $g(B^{2,2}_{p,p}) = p \geq 4$. Let G be a simple h-hypergraph such that $G \sim B^{2,2}_{p,p}$. From Lemma 1.3 it follows that G is linear. Because $(p,h) \neq (3,3)$ from Lemma 1.4 it follows that G has the same order and size as $B^{2,2}_{p,p}$, $g(G) = p$ and $c_p(G) = 2$. It follows that G is isomorphic to $B^{2,2}_{p,p}$, $B^{1,2}_{p,p}$ or $B^{1,1}_{p,p}$. We have seen in the proof of Theorem 2.3 that the coefficients of λ^{h-2} in the chromatic polynomial of $B^{1,2}_{p,p}$ and $B^{1,1}_{p,p}$ is $(-1)^{r-1}$ or $2(-1)^{r-1}$, respectively but in $P(B^{2,2}_{p,p}, \lambda)$ this coefficient vanishes. It follows that $G \cong B^{2,2}_{p,p}$. □

In [6] it was shown that every h-uniform cycle with p edges C^h_p is h-chromatically unique. We shall prove in the next lemma that it is not chromatically unique at least for $p = 3$.

Lemma 2.1: *C^h_p is not chromatically unique for $p = 3$ and for every $h \geq 3$.*

Proof: Let G be the hypergraph of order $n = 3h - 3$ and size 4 as shown in figure 4, where A, B, C are pairwise disjoint

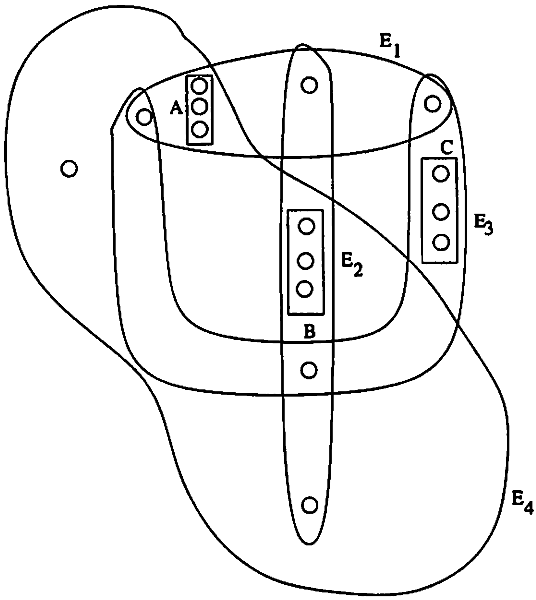


Figure 4

sets such that $| A |=| B |=| C |= h - 3$. By Lemma 1.1 it is easy to see that

$$P(G, \lambda) = \lambda^{3h-3} - 3\lambda^{2h-2} + 3\lambda^{h-1} - \lambda = P(C_3^h, \lambda).$$

□

An open problem is to study the chromaticity of h-uniform cycles C_p^h for $p \geq 4$ and $h \geq 3$.

Acknowledgement

I am grateful to my supervisor Ioan Tomescu for his valuable guidance and suggestions in preparing this paper.

I thank the referee for his comments which improved the presentation of the papaer.

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