NORMS of CIRCULANT and SEMICIRCULANT MATRICES WITH HORADAM'S NUMBERS

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Abstract

In this paper, we obtain the spectral norm and eigenvalues of circulant matrices with Horadam's numbers. Furthermore, we define the semicirculant matrix with these numbers and give the Euclidean norm of this matrix.

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1. Introduction

The second-order linear recurrence sequence $(h_n(a,b;p,q))_{n\geq 0}$, or briefly $(h_n)_{n\geq 0}$, is defined by

$$h_{n+2} = ph_{n+1} + qh_n, \tag{1.1}$$

with given $h_0 = a, h_1 = b$ and $n \geq 0$. This sequence was introduced, in 1965, by Horadam [2, 3], and it generalizes many sequences (see [1, 4]). Examples of such sequences are Fibonacci numbers sequence $(F_n)_{n\geq 0}$, Lucas numbers sequence $(L_n)_{n\geq 0}$, and Pell numbers sequence $(P_n)_{n\geq 0}$, when one has $p = q = b = 1$, $a = 0$; $p = q = b = 1$, $a = 2$; and $p = 2$, $q = b = 1$, $a = 0$; respectively.

An explicit formula for h_n can be presented as

$$h_n = A\alpha^n + B\beta^n = A\left(\frac{p+\sqrt{p^2+4q}}{2}\right)^n + B\left(\frac{p-\sqrt{p^2+4q}}{2}\right)^n, \quad (1.2)$$

where

$$A = \frac{b-a\beta}{\sqrt{p^2+4q}}, \quad B = \frac{a\alpha-b}{\sqrt{p^2+4q}}.$$

Let $x \in \mathbb{C}^n$, $x = (x_0, x_1, \ldots, x_{n-1})^T$. The $n \times n$ *circulant matrix* $C(x) = (c_{ij})$ given by $c_{ij} = x_{j-i \ (mod\ n)}$. The elements of each row of $C(x)$ are identical to those of the previous row, but are moved one position to the right and wrapped around. Let $x \in \mathbb{C}^n$, $x = (x_1, x_2, \ldots, x_n)^T$. The $n \times n$ *semicirculant matrix* $S(x) = (s_{ij})$ given by

$$s_{ij} = \begin{cases} x_{j-i+1} & i \le j, \\ 0 & \text{otherwise.} \end{cases}$$

We have occasion to use the $n \times n$ *Fourier matrix* $F = \left(\frac{1}{\sqrt{n}}\omega^{(i-1)(j-1)}\right)$. Circulant matrices are an especially tractable class of matrices since inverses, conjugate transpose, products, and sums are also circulant matrices and hence both straightforward to construct and normal. In addition the eigenvalues of such matrices can easily be found exactly and circulant matrices are diagonalized by Fourier matrices.

The eigenvalues $\lambda_j(x)$ and the eigenvectors $y^{(j)}$ of $C(x)$ are the solutions of $C(x)y = \lambda y$ or, equivalently, of the n difference equations

$$\sum_{k=0}^{j-1} x_{n-j+k}y_k + \sum_{k=j}^{n-1} x_{k-j}y_k = \lambda y_j, \quad j = 0, 1, \ldots, n-1. \quad (1.3)$$

One can solve difference equations as one solves differential equations by guessing. Since the equation is linear with constant coefficients a reasonable guess is $y_k = \rho^k$. Substitution into (1.3), then we have an eigenvalue and corresponding eigenvector

$$\lambda_j(x) = \sum_{k=0}^{n-1} x_k\omega^{-jk}, \qquad y^{(j)} = \frac{1}{\sqrt{n}}\left(1, \omega^{-j}, \omega^{-2j}, \ldots, \omega^{-(n-1)j}\right)^T \quad (1.4)$$

for all $j = 0, 1, \ldots, n-1$, where ω is the nth primitive root of the unity.

Theorem 1.1. *(see [6]) Let $C(x)$ be an $n \times n$ general circulant matrix. Then*

$$C(x) = F^* diag(\lambda_0(x), \ldots, \lambda_{n-1}(x))F,$$

where $\lambda_j(x) = \sum_{k=0}^{n-1} x_k \omega^{-jk}$, $j = 0, 1, \ldots, n-1$, A^* *is the conjugate transpose of matrix* A, *and* ω *is the nth primitive root of the unity.*

Theorem 1.2. *(see* [5, Section 3.1, Exercise 19, page 157]*) Let* A *be an* $n \times n$ *matrix with eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$. A *is a normal matrix if and only if the eigenvalues of* AA^* *are* $|\lambda_1|^2, |\lambda_2|^2, \ldots, |\lambda_n|^2$, *where* A^* *is the conjugate transpose of matrix* A.

Let $A = (a_{ij})$ be an $n \times n$ matrix, the *Euclidean* (or *Frobenius*) norm of matrix A is

$$\|A\|_E = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}},$$

the *spectral norm* of matrix A is

$$\|A\|_2 = \left(\max_{1 \leq i \leq n} \lambda_i(A^*A) \right)^{\frac{1}{2}},$$

where $\lambda_i(A^*A)$ is the eigenvalue of the matrix A^*A. The maximum column sum matrix norm of $n \times n$ matrix A is

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

and the maximum row sum matrix norm is

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$

In 1970 Lind [7] defined determinant of circulant and skew circulant matrices with the Fibonacci numbers and some generalization of Fibonacci numbers. More recently, Solak [8] defined circulant matrix with the Fibonacci and Lucas numbers and give lower and upper bounds for the spectral norm of these matrices. The author gives also some corollaries related to norms of Kronecker and Hadamard products of these matrices. In this article, we obtain the best bounds for the spectral norm of circulant matrix with Horadam's numbers and give the eigenvalues of these matrices. Afterwards we define semicirculant matrix with the Horadam's numbers and obtain some matrix norms of these matrices.

2. Main results

Theorem 2.1. *Let the $n \times n$ circulant matrix with Horadam's numbers be $C(H) = (c_{ij})$ such that $c_{ij} = h_{j-i \pmod n}$. The eigenvalues of $C(H)$ are*

$$\lambda_j(C(H)) = \frac{h_n + (pa - b + qh_{n-1})\omega^{-j} - a}{q\omega^{-2j} + p\omega^{-j} - 1}.$$

Proof. Using the eigenvalue of circulant matrices and (1.2), we obtain

$$\begin{aligned}\lambda_j(C(H)) &= \sum_{k=0}^{n-1} h_k\omega^{-jk} = \sum_{k=0}^{n-1}(A\alpha^k + B\beta^k)\omega^{-jk} \\ &= A\sum_{k=0}^{n-1}(\alpha\omega^{-j})^k + B\sum_{k=0}^{n-1}(\beta\omega^{-j})^k \\ &= \frac{A(\alpha^n - 1)}{\alpha\omega^{-j} - 1} + \frac{B(\beta^n - 1)}{\beta\omega^{-j} - 1} \\ &= \frac{\left[\begin{array}{c} -(A\alpha^n + B\beta^n) - q\omega^{-j}(A\alpha^{n-1} + B\beta^{n-1}) - (A\beta + B\alpha)\omega^{-j} \\ +(A+B) \end{array}\right]}{\alpha\beta\omega^{-2j} - (\alpha+\beta)\omega^{-j} + 1}.\end{aligned}$$

Using the following facts $\alpha\beta = -q$, $\alpha+\beta = p$, $A+B = a$, and $A\beta+B\alpha = pa-b$ we get the desired result. □

Theorem 2.2. *Let $C(H)$ be the $n \times n$ circulant matrix with the Horadam's numbers. Then*

$$\|C(H)\|_2 = \left(\max_{0\le j\le n-1}\left|\frac{h_n + (pa - b + qh_{n-1})\omega^{-j} - a}{q\omega^{-2j} + p\omega^{-j} - 1}\right|^2\right)^{\frac{1}{2}},$$

where $\|\cdot\|_2$ is the spectral norm of $C(H)$ and h_n is the nth Horadam's numbers.

Proof. Using Theorem 1.1 and Theorem 1.2, we have

$$\|C(H)\|_2 = \left(\max_{0\le j\le n-1}\lambda_j(C(H)C(H)^*)\right)^{\frac{1}{2}} = \left(\max_{0\le j\le n-1}|\lambda_j(C(H))|^2\right)^{\frac{1}{2}}.$$

Thus, Theorem 2.1 gives

$$\|C(H)\|_2 = \left(\max_{0\le j\le n-1}\left|\frac{h_n + (pa - b + qh_{n-1})\omega^{-j} - a}{q\omega^{-2j} + p\omega^{-j} - 1}\right|^2\right)^{\frac{1}{2}}.$$

□

As an application of the above theorem we obtain the following result.

Corollary 2.3. *Let* $V = (h_0(a,1;p,q), h_1(a,1;p,q), \ldots, h_{n-1}(a,1;p,q))^T$, *where* $p, q \geq 1$. *Then*

$$\|C(V)\|_2 = \frac{1}{p+q-1}(h_n + qh_{n-1} - 1 + a(p-1)).$$

Proof. Theorem 2.2 gives that

$$\|C(H)\|_2 = \left(\max_{0 \leq j \leq n-1} \left| \frac{h_n + (pa - b + qh_{n-1})\omega^{-j} - a}{q\omega^{-2j} + p\omega^{-j} - 1} \right|^2 \right)^{\frac{1}{2}}.$$

assume that $H = V$. If $j = 0$ then the eigenvalue is maximum. Therefore

$$\|C(V)\|_2 = \frac{1}{p+q-1}(h_n + qh_{n-1} - 1 + a(p-1)),$$

as claimed. □

In particular, Corollary 2.3 with $p = 1$ gives

$$\|C((h_0(0,1;1,q), h_1(0,1;1,q), \ldots, h_{n-1}(0,1;1,q))^T)\|_2 = \frac{1}{q}(h_{n+1} - 1)$$

(In the case $a = 0$ and $q = 1$ (resp. $a = 2$ and $q = 1$; $a = 0$ and $q = 2$) we get that $h_{n+1} = F_{n+1}$ the $(n+1)$th Fibonacci number (resp. $h_{n+1} = L_{n+1}$ the $(n+1)$th Lucas number; $h_{n+1} = P_{n+1}$ the $(n+1)$th Pell number).

Theorem 2.4. *The maximum column (or row) sum matrix norm of* $C(H)$ *is*

$$\|C(H)\|_1 = \|C(H)\|_\infty = \sum_{k=0}^{n-1} h_k.$$

Moreover if $p + q \neq 1$ *and* $p, q \geq 0$ *then*

$$\|C(H)\|_1 = \|C(H)\|_\infty = \frac{1}{p+q-1}(h_n + qh_{n-1} + pa - a - b).$$

As an application for the above theorem we obtain that

$$\|C(F)\|_1 = \|C(F)\|_\infty = F_{n+1} - 1, \|C(L)\|_1 = \|C(L)\|_\infty = L_{n+1} - 1$$

and

$$\|C(P)\|_1 = \|C(P)\|_\infty = \frac{1}{2}(P_n + P_{n-1} - 1),$$

where F and L are given in the statement of Corollary 2.3.

Now, let us consider the case of the Euclidean norm of the semicirculant matrix with the Horadam's numbers, namely $\|S(H)\|_E$. To do that we need the following lemma.

Lemma 2.5. *For all $n \geq 3$,*

$$\sum_{j=1}^{n}\sum_{k=1}^{j} h_k^2 = \frac{A^2\alpha^4}{(1-\alpha^2)^2}(\alpha^{2n}-1) + \frac{B^2\beta^4}{(1-\beta^2)^2}(\beta^{2n}-1) - \frac{2q^2AB}{(1+q)^2}(1-(-q)^n) + \left(\frac{\beta^2B^2}{1-\beta^2} - \frac{2qAB}{1+q} + \frac{\alpha^2A^2}{1-\alpha^2}\right)n.$$

Proof. Using (1.2), we obtain that

$$\begin{aligned}\sum_{j=1}^{n}\sum_{k=1}^{j} h_k^2 &= \sum_{j=1}^{n}\sum_{k=1}^{j}(A\alpha^k + B\beta^k)^2 \\ &= \sum_{j=1}^{n}\sum_{k=1}^{j}(A^2\alpha^{2k} + 2AB(-q)^k + B^2\beta^{2k}).\end{aligned}$$

Using the fact that $\sum_{k=1}^{j} t^k = \dfrac{t - t^{j+1}}{1-t}$ we get the desired result. □

The above lemma and (1.2) together with the definition of the Euclidean norm of the semicirculant matrix proves the following result.

Theorem 2.6. *We have*

$$\|S(H)\|_E^2 = h_{2n}(a', b'; p, q) - a' - \frac{2q^2AB}{(1+q)^2}(1-(-q)^n) + \left(\frac{\beta^2B^2}{1-\beta^2} - \frac{2qAB}{1+q} + \frac{\alpha^2A^2}{1-\alpha^2}\right)n,$$

where

$$\begin{aligned}\alpha &= \frac{p+\sqrt{p^2+4q}}{2}, \quad \beta = \frac{p-\sqrt{p^2+4q}}{2}, \quad A = \frac{b-a\beta}{\sqrt{p^2+4q}}, \quad B = \frac{a\alpha - b}{\sqrt{p^2+4q}}, \\ a' &= \frac{\alpha^4A^2}{(1-\alpha^2)^2} + \frac{\beta^4B^2}{(1-\beta^2)^2}, \quad \textit{and} \quad b' = \frac{\alpha^5A^2}{(1-\alpha^2)^2} + \frac{\beta^5B^2}{(1-\beta^2)^2}.\end{aligned}$$

Now let us give three applications for the above theorem:

- Let $a = 0$ and $p = q = b = 1$. So

$$\begin{aligned}\|S(F)\|_E &= \sqrt{\frac{1}{5}\left(\alpha^{2n+2} + \beta^{2n+2} - 3 + \frac{1-(-1)^n}{2}\right)} \\ &= \sqrt{\frac{1}{5}\left(L_{2n+2} - 3 + \frac{1-(-1)^n}{2}\right)}.\end{aligned}$$

- Let $a = 2$ and $p = q = b = 1$. So

$$\begin{aligned}\|S(L)\|_E &= \sqrt{\alpha^{2n+2} + \beta^{2n+2} - 3 + \frac{1-(-1)^n}{2}} \\ &= \sqrt{L_{2n+2} - 3 - 2n + \frac{1-(-1)^n}{2}}.\end{aligned}$$

- Let $a = 0$, $b = q = 1$, and $p = 2$. So

$$\|S(P)\|_E = \frac{1}{4}\sqrt{\frac{1}{(2+\sqrt{2})^2}(1+\sqrt{2})^{2n+4} - \frac{1}{(2-\sqrt{2})^2}(1-\sqrt{2})^{2n+4} - 2 - (-1)^n}.$$

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The Graphs $C_9^{(t)}$ are Graceful for $t \equiv 0, 3$ (mod 4) *

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Abstract

Let C_n denote the cycle with n vertices, and $C_n^{(t)}$ denote the graphs consisting of t copies of C_n with a vertex in common. Koh et al. conjectured that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0, 3 \pmod 4$. The conjecture has been shown true for $n = 3, 5, 6, 7, 4k$. In this paper, the conjecture is shown to be true for $n = 9$.
Keywords: *graceful graph, vertex labeling, edge labeling*

1 Introduction

Let C_n denote the cycle with n vertices, and $C_n^{(t)}$ denote the graphs consisting of t copies of C_n with a vertex in common. Koh et al. [4] conjectured that the graphs $C_n^{(t)}$ are graceful if and only if $nt \equiv 0, 3 \pmod 4$, and proved that the graphs $C_{4k}^{(t)}$ and $C_6^{(2t)}$ are graceful for $t \geqslant 1$. Qian [7] proved that the graphs $C_{2k}^{(2)}$ are graceful. Bermond et al. [1, 2] proved that the graphs $C_3^{(t)}$ (i.e, the friendship graph or Dutch t-windmill) are graceful if and only if $t \equiv 0$ or $1 \pmod 4$. The first author [6, 8] of this paper proved that the graphs $C_5^{(t)}$ are graceful for $t \equiv 0, 3 \pmod 4$, and $C_7^{(t)}$ are graceful for $t \equiv 0, 1 \pmod 4$. So the conjecture has been shown true

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for $n = 3, 5, 6, 7, 4k$. In this paper, the conjecture is shown to be true for $n = 9$.

For the literature on graceful graphs we refer to [3] and the relevant references given in it.

2 The graphs $C_9^{(t)}$

Now, we consider the graphs $C_9^{(t)}$. Let $v_0^i, v_1^i, v_2^i, v_3^i, v_4^i, v_5^i, v_6^i, v_7^i, v_8^i$ be the vertices of the i-th cycle, $v_0^i = v$ for all i. Then we have

Theorem 2.1. The graphs $C_9^{(t)}$ are graceful for $t \equiv 0, 3 \pmod 4$.

Proof. Case 1. $t \equiv 0 \pmod 4$, say $t = 4k$, i.e. $C_9^{(4k)}$.

For $k = 1$, we give a vertex labeling of $C_9^{(4)}$ as the one shown in Figure 1.

	36	7	32	11	28	18	21	14
	35	5	31	12	27	19	20	24
0	34	3	30	8	26	15	10	23
	33	1	29	9	25	16	4	6

Figure 1: The graceful labeling of $C_9^{(4)}$.

By the definition of graceful graph, it is clear that $C_9^{(4)}$ is a graceful graph.

For $k > 1$, we define a vertex labeling f as follows.

$$\begin{array}{lcll} f(v) & = & 0, & \\ f(v_1^i) & = & 36k+1-i, & 1 \le i \le 4k, \\ f(v_2^i) & = & 8k+1-2i, & 1 \le i \le 4k, \\ f(v_3^i) & = & 32k+1-i, & 1 \le i \le 4k, \\ f(v_4^i) & = & \begin{cases} 10k+i, & 1 \le i \le 2k, \\ 6k-1+i, & 2k+1 \le i \le 4k, \end{cases} \\ f(v_5^i) & = & 28k+1-i, & 1 \le i \le 4k, \\ f(v_6^i) & = & \begin{cases} 16k+1+i, & 1 \le i \le 2k, \\ 12k+i, & 2k+1 \le i \le 4k, \end{cases} \\ f(v_7^i) & = & \begin{cases} 20k+2-i, & 1 \le i \le 2k, \\ 10k, & i = 2k+1, \\ 24k+2-i, & 2k+2 \le i \le 3k, \\ 24k+1-i, & 3k+1 \le i \le 4k-1, \\ 4k, & i = 4k, \end{cases} \\ f(v_8^i) & = & \begin{cases} 12k+1+i, & 1 \le i \le k, \\ 23k+1, & i = k+1, \\ 12k+i, & k+2 \le i \le 2k, \\ 20k+i, & 2k+1 \le i \le 3k, \\ 20k+1+i, & 3k+1 \le i \le 4k-1, \\ 6k, & i = 4k. \end{cases} \end{array}$$

Now we prove that f is a graceful labeling of $C_9^{(4k)}$ as follows.

Denote by

$$S_j = \{f(v_j^i) \,|1 \le i \le 4k\}, \quad 0 \le j \le 8.$$

Then

$$\begin{array}{rl} S_0 = & \{0\}, \\ S_1 = & \{36k, 36k-1, \ldots, 32k+1\}, \\ S_2 = & S_{21} \cup S_{22} \cup S_{23} \\ = & \{8k-1, 8k-3, \ldots, 6k+1\} \cup \{6k-1, 6k-3, \ldots, 4k+1\} \\ & \cup\{4k-1, 4k-3, \ldots, 1\}, \\ S_3 = & \{32k, 32k-1, \ldots, 28k+1\}, \\ S_4 = & S_{41} \cup S_{42} \\ = & \{10k+1, 10k+2, \ldots, 12k\} \cup \{8k, 8k+1, \ldots, 10k-1\}, \\ S_5 = & \{28k, 28k-1, \ldots, 24k+1\}, \\ S_6 = & S_{61} \cup S_{62} \\ = & \{16k+2, 16k+3, \ldots, 18k+1\} \cup \{14k+1, 14k+2, \ldots, 16k\}, \\ S_7 = & S_{71} \cup S_{72} \cup S_{73} \cup S_{74} \cup S_{75} \\ = & \{20k+1, 20k, \ldots, 18k+2\} \cup \{10k\} \cup \{22k, 22k-1, \ldots, 21k+2\} \\ & \cup\{21k, 21k-1, \ldots, 20k+2\} \cup \{4k\}, \end{array}$$

$$\begin{aligned}S_8 = & S_{81} \cup S_{82} \cup S_{83} \cup S_{84} \cup S_{85} \cup S_{86} \\ = & \{12k+2, 12k+3, \ldots, 13k+1\} \cup \{23k+1\} \\ & \cup\{13k+2, 13k+3, \ldots, 14k\} \cup \{22k+1, 22k+2, \ldots, 23k\} \\ & \cup\{23k+2, 23k+3, \ldots, 24k\} \cup \{6k\}.\end{aligned}$$

Hence, $S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8$ is the set of labels of all vertices, and

$$\begin{aligned}& S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8 \\ = & S_0 \cup S_{23} \cup S_{75} \cup S_{22} \cup S_{86} \cup S_{21} \cup S_{42} \cup S_{72} \cup S_{41} \cup S_{81} \cup S_{83} \\ & \cup S_{62} \cup S_{61} \cup S_{71} \cup S_{74} \cup S_{73} \cup S_{84} \cup S_{82} \cup S_{85} \cup S_5 \cup S_3 \cup S_1\end{aligned}$$

$$\begin{aligned}= \{ & 0, \quad 1, 3, \ldots, 4k-1, \quad 4k, \quad 4k+1, 4k+3, \ldots, 6k-1, \quad 6k, \\ & 6k+1, 6k+3, \ldots, 8k-1, \quad 8k, 8k+1, \ldots, 10k-1, \quad 10k, \\ & 10k+1, 10k+2, \ldots, 12k, \quad 12k+2, 12k+3, \ldots, 13k+1, \\ & 13k+2, 13k+3, \ldots, 14k, \quad 14k+1, 14k+2, \ldots, 16k, \\ & 16k+2, 16k+3, \ldots, 18k+1, \quad 18k+2, 18k+3, \ldots, 20k+1, \\ & 20k+2, 20k+3, \ldots, 21k, \quad 21k+2, 21k+3, \ldots, 22k, \\ & 22k+1, 22k+2, \ldots, 23k, \quad 23k+1, \quad 23k+2, 23k+3, \ldots, 24k, \\ & 24k+1, 24k+2, \ldots, 28k, \quad 28k+1, 28k+2, \ldots, 32k, \\ & 32k+1, 32k+2, \ldots, 36k\}.\end{aligned}$$

It is clear that the labels of each vertex are different, and $Max\{f(v_j^i) \mid 1 \leq i \leq 4k,\ 0 \leq j \leq 8\} = 36k = |E|$. We thus conclude that f is an injective mapping from the vertex set of G into $\{0, 1, \ldots, |E|\}$.

Denote by

$$D_j = \{g(v_j^i, v^i_{(j+1) \bmod 9}) \mid 1 \leq i \leq 4k\}, \quad 0 \leq j \leq 8,$$

$$g(v_j^i, v^i_{(j+1) \bmod 9}) = |f(v^i_{(j+1) \bmod 9}) - f(v_j^i)|, \quad 1 \leq i \leq 4k, \quad 0 \leq j \leq 8.$$

Now, we verify that g maps E onto $\{1, 2, \ldots, |E|\}$.

$$\begin{aligned}D_0 &= \{|f(v_1^i) - f(v_0^i)| \ 1 \leq i \leq 4k\} = \{36k+1-i | 1 \leq i \leq 4k\} \\ &= \{36k, 36k-1, \ldots, 32k+1\}, \\ D_1 &= \{28k+i | 1 \leq i \leq 4k\} = \{28k+1, 28k+2, \ldots, 32k\}, \\ D_2 &= \{24k+i | 1 \leq i \leq 4k\} = \{24k+1, 24k+2, \ldots, 28k\}, \\ D_3 &= D_{31} \cup D_{32} \\ &= \{22k+1-2i | 1 \leq i \leq 2k\} \cup \{26k+2-2i | 2k+1 \leq i \leq 4k\} \\ &= \{22k-1, 22k-3, \ldots, 18k+1\} \cup \{22k, 22k-2, \ldots, 18k+2\}, \\ D_4 &= D_{41} \cup D_{42} \\ &= \{18k+1-2i | 1 \leq i \leq 2k\} \cup \{22k+2-2i | 2k+1 \leq i \leq 4k\} \\ &= \{18k-1, 18k-3, \ldots, 14k+1\} \cup \{18k, 18k-2, \ldots, 14k+2\},\end{aligned}$$

$$
\begin{aligned}
D_5 &= D_{51} \cup D_{52} \\
&= \{12k - 2i | 1 \le i \le 2k\} \cup \{16k + 1 - 2i | 2k + 1 \le i \le 4k\} \\
&= \{12k - 2, 12k - 4, \ldots, 8k\} \cup \{12k - 1, 12k - 3, \ldots, 8k + 1\}, \\
D_6 &= D_{61} \cup D_{62} \cup D_{63} \cup D_{64} \cup D_{65} \\
&= \{4k + 1 - 2i | 1 \le i \le 2k\} \cup \{4k + 1 | i = 2k + 1\} \\
&\quad \cup \{12k + 2 - 2i | 2k + 2 \le i \le 3k\} \\
&\quad \cup \{12k + 1 - 2i | 3k + 1 \le i \le 4k - 1\} \cup \{12k | i = 4k\} \\
&= \{4k - 1, 4k - 3, \ldots, 1\} \cup \{4k + 1\} \cup \{8k - 2, 8k - 4, \ldots, 6k + 2\} \\
&\quad \cup \{6k - 1, 6k - 3, \ldots, 4k + 3\} \cup \{12k\}, \\
D_7 &= D_{71} \cup D_{72} \cup D_{73} \cup D_{74} \cup D_{75} \cup D_{76} \cup D_{77} \\
&= \{8k + 1 - 2i | 1 \le i \le k\} \cup \{4k | i = k + 1\} \\
&\quad \cup \{8k + 2 - 2i | k + 2 \le i \le 2k\} \cup \{12k + 1 | i = 2k + 1\} \\
&\quad \cup \{2i - 4k - 2 | 2k + 2 \le i \le 3k\} \cup \{2i - 4k | 3k + 1 \le i \le 4k - 1\} \\
&\quad \cup \{2k | i = 4k\} \\
&= \{8k - 1, 8k - 3, \ldots, 6k + 1\} \cup \{4k\} \cup \{6k - 2, 6k - 4, \ldots, 4k + 2\} \\
&\quad \cup \{12k + 1\} \cup \{2, 4, \ldots, 2k - 2\} \cup \{2k + 2, 2k + 4, \ldots, 4k - 2\} \\
&\quad \cup \{2k\}, \\
D_8 &= D_{81} \cup D_{82} \cup D_{83} \cup D_{84} \cup D_{85} \cup D_{86} \\
&= \{12k + 1 + i | 1 \le i \le k\} \cup \{22k + i | i = k + 1\} \\
&\quad \cup \{12k + i | k + 2 \le i \le 2k\} \cup \{20k + i | 2k + 1 \le i \le 3k\} \\
&\quad \cup \{20k + 1 + i | 3k + 1 \le i \le 4k - 1\} \cup \{2k + i | i = 4k\} \\
&= \{12k + 2, 12k + 3, \ldots, 13k + 1\} \cup \{23k + 1\} \\
&\quad \cup \{13k + 2, 13k + 3, \ldots, 14k\} \cup \{22k + 1, 22k + 2, \ldots, 23k\} \\
&\quad \cup \{23k + 2, 23k + 3, \ldots, 24k\} \cup \{6k\}.
\end{aligned}
$$

Let D be the set of labels of all edges, then we have

$$
\begin{aligned}
D &= D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6 \cup D_7 \cup D_8 \\
&= D_{61} \cup D_{75} \cup D_{77} \cup D_{76} \cup D_{72} \cup D_{62} \cup D_{73} \cup D_{64} \cup D_{86} \\
&\quad \cup D_{71} \cup D_{63} \cup D_{51} \cup D_{52} \cup D_{65} \cup D_{74} \cup D_{81} \cup D_{83} \cup D_{41} \\
&\quad \cup D_{42} \cup D_{31} \cup D_{32} \cup D_{84} \cup D_{82} \cup D_{85} \cup D_2 \cup D_1 \cup D_0 \\
&= \{1, 3, \ldots, 4k - 1, \quad 2, 4, \ldots, 2k - 2, \quad 2k, \quad 2k + 2, 2k + 4, \\
&\quad \ldots, 4k - 2, 4k, \quad 4k + 1, \quad 4k + 2, 4k + 4, \ldots, 6k - 2, \\
&\quad 4k + 3, 4k + 5, \ldots, 6k - 1, \quad 6k, 6k + 1, 6k + 3, \ldots, 8k - 1, \\
&\quad 6k + 2, 6k + 4, \ldots, 8k - 2, \quad 8k, 8k + 2, \ldots, 12k - 2, \\
&\quad 8k + 1, 8k + 3, \ldots, 12k - 1, \quad 12k, \quad 12k + 1, \quad 12k + 2, \\
&\quad 12k + 3, \ldots, 13k + 1, 13k + 2, 13k + 3, \ldots, 14k, \\
&\quad 14k + 1, 14k + 3, \ldots, 18k - 1, \; 14k + 2, 14k + 4, \ldots, 18k, \\
&\quad 18k + 1, 18k + 3, \ldots, 22k - 1, \; 18k + 2, 18k + 4, \ldots, 22k, \\
&\quad 22k + 1, 22k + 2, \ldots, 23k, \quad 23k + 1, \\
&\quad 23k + 2, 23k + 3, \ldots, 24k, \quad 24k + 1, 24k + 2, \ldots, 28k, \\
&\quad 28k + 1, 28k + 2, \ldots, 32k, \quad 32k + 1, 32k + 2, \ldots, 36k\} \\
&= \{1, 2, \ldots, 36k\}.
\end{aligned}
$$

It is clear that the labels of each edge are different. So, g maps E onto $\{1, 2, \ldots, |E|\}$. By the definition of graceful graph, we thus conclude that $C_9^{(4k)}$ are graceful.

Case 2. $t \equiv 3 \pmod 4$, say $t = 4k - 1$, i.e. $C_9^{(4k-1)}$.

For $k = 1$, we give a vertex labeling of $C_9^{(3)}$ as the one shown in Figure 2. By the definition of graceful graph, it is clear that $C_9^{(3)}$ is a graceful graph.

	27	5	24	9	21	13	15	18
0	26	3	23	7	20	11	16	17
	25	1	22	8	19	12	2	6

Figure 2: The graceful labeling of $C_9^{(3)}$.

For $k > 1$, we define a vertex labeling f as follows.

$$
\begin{aligned}
f(v) &= 0, \\
f(v_1^i) &= 36k - 8 - i, && 1 \le i \le 4k - 1, \\
f(v_2^i) &= 8k - 1 - 2i, && 1 \le i \le 4k - 1, \\
f(v_3^i) &= 32k - 7 - i, && 1 \le i \le 4k - 1, \\
f(v_4^i) &= \begin{cases} 10k - 2 + i, & 1 \le i \le 2k - 1, \\ 6k - 1 + i, & 2k \le i \le 4k - 1, \end{cases} \\
f(v_5^i) &= 28k - 6 - i, && 1 \le i \le 4k - 1, \\
f(v_6^i) &= \begin{cases} 16k - 4 + i, & 1 \le i \le 2k - 1, \\ 12k - 3 + i, & 2k \le i \le 4k - 1, \end{cases} \\
f(v_7^i) &= \begin{cases} 20k - 4 - i, & 1 \le i \le 2k - 1, \\ 24k - 6 - i, & 2k \le i \le 3k - 1, \\ 24k - 7 - i, & 3k \le i \le 4k - 3, \\ 18k - 4, & i = 4k - 2, \\ 2k, & i = 4k - 1, \end{cases} \\
f(v_8^i) &= \begin{cases} 24k - 6, & i = 1, \\ 12k - 4 + i, & 2 \le i \le k, \\ 23k - 5, & i = k + 1, \\ 12k - 5 + i, & k + 2 \le i \le 2k - 1, \\ 20k - 5 + i, & 2k \le i \le 3k - 1, \\ 20k - 4 + i, & 3k \le i \le 4k - 3, \\ 14k - 5, & i = 4k - 2, \\ 8k - 2, & i = 4k - 1. \end{cases}
\end{aligned}
$$

Similar to the proof in Case 1, it can be shown that this assignment provides a graceful labeling of $C_9^{(4k-1)}$.

According to the proof in Case 1 and Case 2, we thus conclude that $C_9^{(t)}$ are graceful for $t \equiv 0, 3 \pmod 4$. □

In Figure 3, we illustrate our graceful labeling for $C_9^{(12)}$ and $C_9^{(15)}$.

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0

108	23	96	31	84	50	61	38
107	21	95	32	83	51	60	39
106	19	94	33	82	52	59	40
105	17	93	34	81	53	58	70
104	15	92	35	80	54	57	41
103	13	91	36	79	55	56	42
102	11	90	24	78	43	30	67
101	9	89	25	77	44	66	68
100	7	88	26	76	45	65	69
99	5	87	27	75	46	63	71
98	3	86	28	74	47	62	72
97	1	85	29	73	48	12	18

$C_9^{(12)}$

0

135	29	120	39	105	61	75	90
134	27	119	40	104	62	74	46
133	25	118	41	103	63	73	47
132	23	117	42	102	64	72	48
131	21	116	43	101	65	71	87
130	19	115	44	100	66	70	49
129	17	114	45	99	67	69	50
128	15	113	31	98	53	82	83
127	13	112	32	97	54	81	84
126	11	111	33	96	55	80	85
125	9	110	34	95	56	79	86
124	7	109	35	94	57	77	88
123	5	108	36	93	58	76	89
122	3	107	37	92	59	68	51
121	1	106	38	91	60	8	30

$C_9^{(15)}$

Figure 3: The graceful labelings of $C_9^{(12)}$ and $C_9^{(15)}$.