On Regular Cayley Maps with Alternating Power Functions

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Abstract

Two classes of regular Cayley maps, balanced and antibalanced, have long been understood, see [12,11]. A recent generalization is that of an e-balanced map, see [7,2,5,8]. These maps can be described using the power function introduced in [4]; e-balanced maps are the ones with constant power functions on the generating set. In this paper we examine a further generalization to the situation where the power function alternates between two values.

1 Introduction

The first steps toward the classification of regular Cayley maps appear in the book of Nathan Biggs and Arthur White [1]. The work of Josef Širáň and Martin Škoviera [12,11] gave complete descriptions of two classes of inverse distributions of regular Cayley maps, namely balanced and antibalanced maps. For a balanced map either all the generators are involutions or

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each dart labeled with a generator is symmetric across the vertex from the dart labeled with its inverse. In an antibalanced map the darts labels and their inverses are symmetric across a line through the vertex. A recent generalization of these maps is an e-balanced map. They are introduced in the paper by the first author, Michelle Schultz, and Martin Škoviera [7] and also appear in the papers [2,5,8]. In particular the inverse distributions of e-balanced maps are discussed in [5, 8]. A balanced map is a 1-balanced map and an antibalanced map is a (-1)-balanced map.

In [4], Robert Jajcay and Josef Širáň introduce the concepts of skew-morphisms and power functions to unify the study of regular Cayley maps. An e-balanced map is one where the power function is constant on the generating set. In this paper we examine the situation where the power function alternates between two values on the generating set. Section 2 contains preliminary definitions. In Section 3 we introduce a necessary and sufficient condition for the power function to repeat. Section 4 studies the automorphism groups of regular Cayley maps with alternating power functions. Finally Section 5 describes their inverse distributions.

Most of the results presented herein originally appeared in the second author's dissertiaton [13].

2 Preliminaries

Intuitively a map of a graph is a "drawing" of the graph onto an orientable, closed surface so that edges intersect only at their vertices. We follow the theoretical background and terminology of Nedela and Škoviera, set forth in [9]. A *graph* is a quadruple $K = (D, V; S, L)$, where the *dart set* $D = D(K)$ and the *vertex set* $V = V(K)$ are disjoint nonempty finite sets, the function $S : D \to V$ assigns to each dart its initial vertex, and L is a permutation on D of order 2 that determines the edges of the graph (dart δ and $L(\delta)$ together correspond to one and the same edge of the graph; the dart $L(\delta)$ is the *reverse* of δ). Under this definition, there are three possible kinds of edges: links, loops, and semiedges. Links are incident with two distinct vertices ($SL(\delta) \neq S(\delta)$). Loops and semiedges are incident with a single vertex ($SL(\delta) = S(\delta)$); when $L(\delta) = \delta$, the dart δ corresponds to a semiedge, and when $L(\delta) \neq \delta$, the dart δ corresponds to a loop. Thus a link or loop of a graph gives rise to two darts, while a semi-edge gives rise to only one dart.

An *oriented map* M is a 2-cell embedding of a graph K in an oriented surface; all maps in this paper will be oriented. Specifically, M is an ordered triple $(D; R, L)$, where D is the set of darts, L is the permutation of D of order 2 that determines the edges of the graph, and R is a permutation of D that specifies the cyclic ordering (as induced by the orientation of the surface) of darts at each initial vertex, i.e., $S(R(\delta)) = S(\delta)$, for all $\delta \in D$. The cycles of R determine the vertices of the graph, and the cycles of RL constitute the region boundaries.

An *automorphism* of the map $M = (D; R, L)$ is a bijection $\Theta : D \to D$ that commutes, under the operation of composition, with R and L, i.e., $\Theta R = R\Theta$ and $\Theta L = L\Theta$. By this definition, a map automorphism preserves a map's vertices, edges, and oriented boundary regions. The set of all map automorphisms of M forms the automorphism group of M, which is denoted by Aut M. Since an automorphism of M is uniquely determined by specifying the image of one dart, $|\text{Aut } M| \leq |D|$. When $|\text{Aut } M| = |D|$, the map M has the largest possible automorphism group and is called a *regular* map. Recently, certain classes of regular maps have become known [3, 7, 11, 1] through the study of Cayley maps.

To define Cayley graphs and Cayley maps, we turn to the terminology of Richter, Širáň, Jajcay, Tucker, and Watkins, set forth in [10]. Let G be a group and X be a finite sequence $\langle x_1, x_2, \ldots, x_k \rangle$ of elements of G that generate G and is closed under inverses; it is possible that $x_i = x_j$ though $i \neq j$, and it is possible that X includes e, the identity element of G. Let $\Delta = \{1, 2, \ldots, k\}$, and let $\tau : \Delta \to \Delta$ be an involution with the property that $x_{\tau(i)} = x_i^{-1}$ for each $i \in \Delta$. The *Cayley graph* $CG(G, X, \tau)$ has vertex set G, and for each $g \in G$ and $i \in \Delta$ there is a dart (g, i) incident from vertex g. The reverse of dart (g, i) is $L(g, i) = (gx_i, \tau(i))$. Aside from semi-edges, when $x_i = e$ and $\tau(i) = i$, the dart pair (g, i) and $L(g, i)$ together determine one and the same undirected edge; thus the Cayley graph $CG(G, X, \tau)$ is an undirected graph (possibly with semi-edges) whose darts are labeled by Δ. The group G is the *Cayley group*, and the involution τ is the *inverse distribution*. The dart set of the Cayley graph is $G \times \Delta$, and the initial vertex function S is defined by $S(g, i) = g$. Since a Cayley graph is a connected graph, S is a surjection.

The ordering of the sequence X determines a rotation scheme on the darts. Let $\sigma = (1, 2, \ldots, k) \in S_k$. The rotation defined for all g and i by the formula $(g, i) \mapsto (g, \sigma(i))$ determines a embedding of $CG(G, X, \tau)$ into a closed, orientable surface. Thus the sequence X yields not only a Cayley graph, but a Cayley map $CM(G, X, \tau) = M(G \times \Delta; R, L)$. The *Cayley map* $M = CM(G, X, \tau) = M(G \times \Delta; R, L)$ is the 2-cell embedding of the Cayley graph $CG(G, X, \tau)$ with $R(g, i) = (g, \sigma(i))$. Again, an *automorphism* of M is a bijection $\Theta : G \times \Delta \to G \times \Delta$ such that $\Theta R = R\Theta$ and $\Theta L = L\Theta$, and M is *regular* if $|\text{Aut } M| = |G||\Delta|$.

A map automorphism of a regular Cayley map that we consider in particular is translation by an element of the group. If $e, g \in G$ and $i \in X$, with e the identity element of G, then the map automorphism $A_g : D(M) \to D(M)$ is defined as, for every $h \in G, i \in \Delta$, $A_g(h, i) = (gh, i)$, and in particular, $A_g(e, i) = (g, i)$. This is a left G-action on $D(M)$: $A_e(h, i) =$

$(eh, i) = (h, i)$ and $A_{g_1 g_2}(h, i) = (g_1 g_2 h, i) = A_{g_1}(g_2 h, i) = A_{g_1} A_{g_2}(h, i)$. Hence the function $\Phi : G \to \text{Aut } M$, defined by $\Phi(g) = A_g$, is an injective group homomorphism, and shows that G is isomorphic to a subgroup of Aut M, or, loosely speaking, $G \leq \text{Aut } M$.

Another map automorphism of a regular Cayley map is the first rotary automorphism. The *first rotary automorphism* is the automorphism of M defined on the neighborhood of the identity by $\phi : G \times \Delta \to G \times \Delta$, where

$$\phi(e, i) = (e, \sigma(i)).$$

Consider (e, i), a specific but arbitrary dart incident from the identity vertex e. Since a map automorphism is determined by where it sends (e, i), we can take the dart (e, i) to dart (g, j) by first rotating dart (e, i) to (e, j) by means of ϕ^{j-i}, then send that dart to (g, j) by means of A_g. Hence, we see that G and ϕ together generate all of Aut M, that is, any automorphism of M can be uniquely written as $A_g \phi^m$ for some $g \in G$ and $m \in \Delta$, i.e., Aut $M = G\langle\phi\rangle$.

The following theorem was discovered independently by the two groups Richter, Širáň, Jajcay, Tucker, and Watkins; and the first author, Schultz, and Škoviera.

Theorem 2.1 ([10], [7]). *Let $M = CM(G, X, \tau)$ be a regular Cayley map with $k = |X|$. Then there is a right Aut M-action on the set Δ.*

Since Aut M is the group $G\langle\phi\rangle$, for any $\alpha \in \text{Aut } M$ and $i \in \Delta$, there is some $h \in G$ and $j \in \Delta$ such that $(\phi^i)\alpha = A_h \phi^j$. Consequently, we may define the action of α on Δ by $(i)\alpha = j$.

The action of the automorphisms on Δ induces a homomorphism

$$\lambda : \text{Aut } M \to S_k.$$

The image of Aut M under λ is generated by σ and τ, Im $\lambda = \langle\sigma, \tau\rangle$ [10].

In [4], Jajcay and Širáň show that a Cayley map is regular if and only if there exists a *skew-morphism* $\mu : G \to G$ and a *power function* $\pi : G \to \Delta$ such that the first rotary automorphism is given by

$$\phi(g, i) = (\mu(g), \sigma^{\pi(g)}(i))$$

and for $g, h \in G$, $i \in \Delta$,

$$\mu(gh) = \mu(g)\mu^{\pi(g)}(h),$$

$$\mu(x_i) = x_{\sigma(i)},$$

and

$$\pi(gh) = \sum_{i=0}^{\pi(g)-1} \pi(\mu^i(h)),$$

where the sum is taken in $\mathbb{Z}_k$. Hence the first rotary automorphism, skew-morphism, and power function are all determined by the values of the power function on X. Note since ϕ commutes with R, the power function is evaluated on x_i, not on i.

Consequently, for $i \in \Delta$,

$$\tau(\sigma(i)) = \sigma^{\pi(x_i)}(\tau(i)), \quad (1)$$

since $L\phi = \phi L$.

Finally in [6] (and implicit in [10]) is the following theorem.

Theorem 2.2 ([6]). *A group A is the automorphism group of a regular Cayley map if and only if both $A = \langle s, t\rangle$, t an involution, and there is a subgroup $G \leq A$ such that $A = \{gs^i \mid g \in G\}$ and $G \cap \langle s\rangle = 1$.*

The regular Cayley map is constructed from the group A in the following manner. The vertex set is the group G. Let s have order k. For each $1 \leq i \leq k$, $s^i t = g_i s^j$ for some $g_i \in G$ and some $1 \leq j \leq k$. The set $X = \{g_1, g_2, \ldots, g_k\}$ generates G and is the set of dart labels. Furthermore, $g_j = g_i^{-1}$, and A is the automorphism group of the Cayley map $CM(G, X, \tau)$, where $\tau(i) = j$.

3 Power Functions with Repeated Blocks

Since σ and τ generate the image of Aut M under λ we are interested in the relationship between σ and τ. Equation 1 tells us this is determined by the power function π on X. If for each i, $\pi(x_i) = a_i$, we will write

$$\pi = a_1, a_2, \ldots, a_k.$$

We begin our analysis by observing that the values of π on X are in fact determined by τ.

Proposition 3.1. *Let $CM(G, X, \tau)$ be a regular Cayley map with power function π, then for $i \in \Delta$,*

$$\pi(x_i) = \tau(i+1) - \tau(i),$$

where the difference is in $\mathbb{Z}_k$.

Proof. Since $\sigma^{\pi(x_i)}\tau(i) = \tau(i) + \pi(x_i)$ and $\tau(\sigma(i)) = \tau(i+1)$, we see from Equation (1) that, for each i, $\pi(x_i) = \tau(i+1) - \tau(i)$. $\square$

Hence the power function is determined by the image of Aut M under λ in S_k.

We are particularly interested in power functions that exhibit a repeated pattern or block when written in the above manner, i.e.,

$$\pi = a_1, a_2, \ldots, a_d, a_1, a_2, \ldots, a_d, \ldots, a_1, a_2, \ldots, a_d.$$

Clearly d divides k.

The existence of such a repeated pattern is easily identifiable by the group structure of the image of Aut M under λ.

Theorem 3.2. *Let $CM(G, X, \tau)$ be a regular Cayley map such that $k = |\Delta|$ is a multiple of some d; let $A = \langle\sigma, \tau\rangle = \lambda(Aut\ M)$. Then $\langle\sigma^d\rangle$ is normal in A if and only if the power function π has the repeated pattern $a_1, a_2, ..., a_d, a_1, a_2, ..., a_d, ..., a_1, a_2, ..., a_d$.*

Proof. ($\Rightarrow$)

Let $l = \frac{k}{d}$. If $\langle\sigma^d\rangle \triangleleft A = \langle\sigma, \tau\rangle$, then $\tau(\sigma^d)\tau^{-1} = \sigma^{dc}$ for some c, $1 \le c \le l$. For every dart i in the dart rotation σ, $\tau(i+1) = \pi(x_i) + \tau(i)$, where $\pi(x_i)$ comes from $\mathbb{Z}_k$, and so $\tau(i+d) = \pi(x_i) + \pi(x_{i+1}) + \cdots + \pi(x_{i+d-1}) + \tau(i)$. But $\tau(i+d) = \tau\sigma^d(i) = \sigma^{dc}\tau(i) = \tau(i) + dc$, so

$$\pi(x_i) + \pi(x_{i+1}) + \cdots + \pi(x_{i+d-1}) = dc$$

for every i, $1 \le i \le k$. Thus in particular, $\pi(x_{i+1}) + \pi(x_{i+2}) + \cdots + \pi(x_{i+d}) = dc$, and so $\pi(x_{i+d}) - \pi(x_i) = dc - dc = 0 \bmod k$, or

$$\pi(x_{i+d}) = \pi(x_i), \text{ for every } i.$$

Thus we see that every dth element of π repeats. Let $\pi(x_i) = a_i$ for $1 \le i \le d$; then $\pi = a_1, a_2, ..., a_d, a_1, a_2, ..., a_d, ..., a_1, a_2, ..., a_d$. In particular, $a_1 + a_2 + ... + a_d = dc$.

($\Leftarrow$)

Let $\pi = a_1, a_2, \ldots, a_d, a_1, a_2, \ldots, a_d, \ldots, a_1, a_2, \ldots, a_d$ be the power function for a regular Cayley map. Then by Proposition 3.1 $\tau(i+d) = \tau(i) + a_1 + a_2 + \cdots + a_d$. Consequently, $\tau(i+k) = \tau(i) + l(a_1 + a_2 + \cdots + a_d)$. Thus $l(a_1 + a_2 + \cdots + a_d) = 0 \bmod k$ and so

$$a_1 + a_2 + \cdots + a_d = dc \tag{2}$$

for some c. Therefore,

$$\tau\sigma^d(i) = \tau(i+d) = \tau(i) + dc = \sigma^{dc}\tau(i),$$

for every $i \in \Delta$. Hence

$$\tau\sigma^d = \sigma^{dc}\tau \quad \Rightarrow \quad \tau\sigma^d\tau = \sigma^{dc} \in \langle\sigma^d\rangle, \tag{3}$$

it follows that $\langle\sigma^d\rangle$ is normal in A. □

The above result is closely related to the following.

Theorem 3.3 **([6]).** *Let $M = CM(G, X, \tau)$ be a regular Cayley map with first rotary automorphism ϕ and valence k. Then the Cayley graph $CG(G, X, \tau)$ has $\frac{k}{d}$ multiple edges between any two vertices if and only if $\langle\phi^d\rangle$ is a normal subgroup of Aut M, where d is a divisor of k.*

Consequently, if the power function for a regular Cayley map M has a repeated block, then the Cayley map M/K obtained by identifying vertices in the same right coset of $K = \text{Ker } \lambda$ has multiple edges. Note $K \leq G \leq$ Aut M [7].

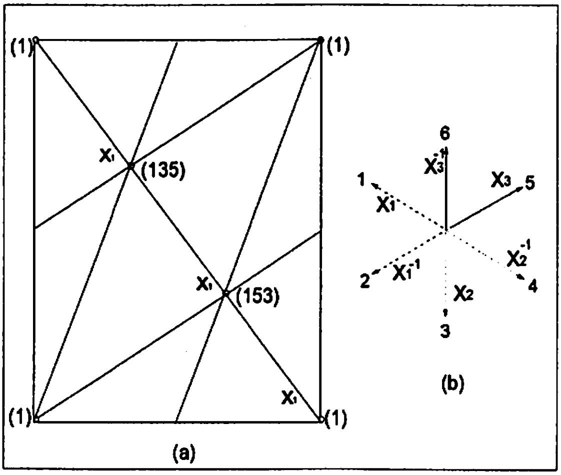

Figure 1:

Example 3.4. If $A = \langle(1,2,3,4,5,6),(1,2)(3,4)(5,6)\rangle$, then Theorem 2.2 produces a regular Cayley map $M = CM(G, X, \tau)$, where $G = \{(1),(135),(153)\} \cong \mathbb{Z}_3$, $X = \langle(153),(135),(153),(135),(153),(135)\rangle$, $\tau = (1,2)(3,4)(5,6)$, the power function $\pi = 1,3,1,3,1,3$, and Aut $M = A \cong \mathbb{Z}_3 \rtimes D_3$ (see Theorem 4.2 below). In this case $K = \text{Ker } \lambda = 1$, since A is already a subgroup of S_6. The map $M = M/K$ embeds into a torus, and is shown in Figure 1(a). Notice the multiple edges between the vertices. Figure 1(b) shows the dart rotation at each of the three vertices.

4 The Image of Aut M under λ for Alternating Pairs

In this section we concern ourselves with the image of λ : Aut $M \to S_k$ for power functions with an alternating pattern of $\pi = a, b, a, b, \ldots, a, b$. A regular Cayley map is e-balanced if and only if $\pi = e, e, \ldots, e$, and in this case the image is found in [7]. (The regular Cayley maps called "balanced" are those for which $e = 1$, and the ones called "anti-balanced" are those for which $e = -1$.)

Theorem 4.1 ([7]). *Let $M = CM(G, X, \tau)$ be a regular Cayley map of valence k. Then M is e-balanced if and only if Im λ, the image of Aut M under λ, is isomorphic to either $\mathbb{Z}_k$, for $e = 1$ or $\mathbb{Z}_k \rtimes \mathbb{Z}_2$, for $e \neq 1$.*

In the latter case where Im $\lambda \cong \mathbb{Z}_k \rtimes \mathbb{Z}_2$, the $\mathbb{Z}_2$-action is to raise the generator of $\mathbb{Z}_k$ to the eth power.

Recall that by Proposition 3.1 $\pi(x_i) = \tau(i+1) - \tau(i)$. Thus the power function is determined by the inverse distribution τ and $\tau \in$ Im λ. Therefore, the power function can be recovered from Im λ.

From now on we will restrict ourselves to the situation that $a \neq b$ in the power function.

Theorem 4.2. *Let $M = CM(G, X, \tau)$ be a regular Cayley map of valence $k = 2l$ with power function $\pi = a, b, a, b, \ldots, a, b$, $a \neq b$. Then there is a short exact sequence*

$$1 \to \mathbb{Z}_l \to Im\ \lambda \to D_m \to 1,$$

where D_m is the dihedral group of order $2m$.

Furthermore, if l is odd, then the short exact sequence splits, i.e.,

$$Im\ \lambda \cong \mathbb{Z}_l \rtimes D_m.$$

Note we consider $\mathbb{Z}_2 \times \mathbb{Z}_2$ to be D_2. We postpone the proof until later to assemble the necessary lemmas.

Remark 4.3. By equation 2, $a + b = 2c \mod k$, and in the above semidirect product if $\mathbb{Z}_l = \langle z \rangle$ and $D_m = \langle x, y \rangle$, where x and y are both involutions, then the action of D_m on $\mathbb{Z}_l$ is given by $x^{-1}zx = z$ and $y^{-1}zy = z^c$.

Remark 4.4. When l is even the short exact sequence need not split. In particular, it does not split for the group in Example 5.5.

In the special case where $a + b = 2 \mod k$ the short exact sequence will split whether l is even or odd, and $m = l$. Thus Im $\lambda \cong \mathbb{Z}_l \times D_l \cong (\mathbb{Z}_l \times \mathbb{Z}_l) \rtimes \mathbb{Z}_2 = \mathbb{Z}_l \wr \mathbb{Z}_2$. This observation along with the one that the image of Aut M under λ determines π yields the result:

Theorem 4.5. *Let $M = CM(G, X, \tau)$ be a regular Cayley map of valence $k = 2l$. The map M has the power function $\pi = a, b, a, b, \ldots, a, b$, $a \neq b$, $a + b = 2 \mod k$ if and only if $Im\ \lambda \cong \mathbb{Z}_l \wr \mathbb{Z}_2$.*

The proof of this theorem will be given after that of Theorem 4.2.

We now assemble necessary lemmas for the proofs of the main theorems.

Lemma 4.6. *Let $M = CM(G, X, \tau)$ be a regular Cayley map of valence $k = 2l$, where l is an odd, positive integer, and with power function $\pi = a, b, a, b, ..., a, b$ where $a, b \in \mathbb{Z}_k$ are each nonzero. Let $\sigma \in S_k$ where $\sigma = (1, 2, \ldots, k)$. Then $\langle \sigma^2 \rangle \cap \langle \sigma^l, \tau \rangle = 1$.*

Proof. Since σ^l and τ are both involutions, $\langle \sigma^l, \tau \rangle$ is a dihedral group. An element of a dihedral group is either a rotation $(\tau\sigma^l)^i$ or a flip $(\tau\sigma^l)^i\tau$ for some exponent i.

Suppose $(\sigma^2)^j \in \langle \sigma^2 \rangle \cap \langle \sigma^l, \tau \rangle$. All flips are of order 2 and since the order of $(\sigma^2)^j$ is odd, $(\sigma^2)^j$ is a rotation. Thus for some i,

$$(\sigma^2)^j = (\tau\sigma^l)^i = (\tau\sigma^l)^{i-1}\tau\sigma^l.$$

Therefore, $\sigma^{2j-l} = (\tau\sigma^l)^{i-1}\tau$ is a flip and hence of order 2. Thus $2j - l = l \mod k$, so $2j = 2l = 0 \mod k$ and $j = 0 \mod l$. □

Lemma 4.7. *Let $M = CM(G, X, \tau)$ be a regular Cayley map of valence $k = 2l$, where l is an odd positive integer, and with power function $\pi = a, b, a, b, ..., a, b$ where $a, b \in \mathbb{Z}_k$ are each nonzero. The permutation $\tau\sigma^2\tau = \sigma^{2c}$.*

Proof. This is an immediate corollary of Theorem 3.2; note that $a + b = 2c \mod k$, and see equations 2 and 3 in the proof of Theorem 3.2. □

Lemma 4.8. *Let $M = CM(G, X, \tau)$ be a regular Cayley map of valence $k = 2l$, and with power function $\pi = a, b, a, b, ..., a, b$ where $a, b \in \mathbb{Z}_k$ are each nonzero and $a + b = 2 \mod k$. Then $\langle \tau\sigma, \sigma\tau \rangle \cong \mathbb{Z}_l \times \mathbb{Z}_l$.*

Proof. It follows immediately that $\tau\sigma$ and $\sigma\tau$ commute by Lemma 4.7. We need only determine the orders of $\tau\sigma$ and $\sigma\tau$. Suppose the order of $\tau\sigma$ is m and the order of $\sigma\tau$ is n. Since $(\sigma\tau)(\tau\sigma) = \sigma^2$, l is the least common multiple of m and n. Observe that $(\tau\sigma)^{n+1} = \tau(\sigma\tau)^n\sigma = \tau\sigma$ or $(\tau\sigma)^n = 1$, so m divides n. Similarly n divides m. Thus $m = n$ and they are both equal to l. □

We are now ready to prove the main theorems.

Proof of Theorem 4.2. It is easily verified by direct computation that in order for $\pi = a, b, a, b, \ldots, a, b$, $a \neq b$ to occur we must have $k \geq 6$. Let $A = \text{Im } \lambda$. Define $\phi_1 : \mathbb{Z}_l \to A$ as $\phi_1(i) = (\sigma^2)^i$; $\phi_1(\mathbb{Z}_l) = \langle \sigma^2 \rangle \subseteq A$. Since, by Theorem 3.2, $\langle \sigma^2 \rangle$ is normal in A, there is an induced homomorphism with $\langle \sigma^2 \rangle$ as the kernel that sends A to $A/\langle \sigma^2 \rangle$. Define ϕ_2 as this quotient homomorphism.

The element $\phi_2(\sigma)$ is an involution in $\phi_2(A)$, since $(\phi_2(\sigma))^2 = \phi_2(\sigma^2) = 1$; the element $\phi_2(\tau)$ is an involution because τ is an involution. Hence, $\phi_2(A)$ is generated by two involutions. Thus $\phi_2(A)$ is a dihedral group. Therefore, we have the short exact sequence.

When l is odd, $\phi_2(\sigma^l) = \phi_2(\sigma)$, and $\phi_2(A) = \langle \phi_2(\sigma^l), \phi_2(\tau) \rangle$; by Lemma 4.6, $\langle \sigma^2 \rangle \cap \langle \sigma^l, \tau \rangle = 1$. Since $\langle \sigma^2 \rangle = \phi_1(\mathbb{Z}_l)$ is a direct summand of A, the sequence splits. Let $x = \sigma^l$, $y = \tau$, and $z = \sigma^2$. The action of x on z is $\sigma^{-l}(\sigma^2)\sigma^l = \sigma^2$ and the action of y on z is $\tau(\sigma^2)\tau = (\sigma^2)^c$ by Lemma 4.7. □

Proof of Theorem 4.5. As before we have $k \geq 6$. We will use the presentation

$$\mathbb{Z}_l \wr \mathbb{Z}_2 = \langle x, y, z \mid x^l = y^l = z^2, xy = yx, zxz = y \rangle.$$

($\Rightarrow$)

Let $\pi = a, b, a, b, \ldots, a, b$ with $a \neq b$. By Lemma 4.8, $\langle \sigma\tau, \tau\sigma \rangle \cong \mathbb{Z}_l \times \mathbb{Z}_l$. Furthermore, since $\tau(\sigma\tau)\tau = \tau\sigma$, we have $\text{Im } \lambda = \langle \sigma, \tau \rangle \cong \mathbb{Z}_l \wr \mathbb{Z}_2$.

($\Leftarrow$)

Assume that $f : \text{Im } \lambda \to \mathbb{Z}_l \wr \mathbb{Z}_2$ is an isomorphism. An arbitrary element of $\mathbb{Z}_l \wr \mathbb{Z}_2$ is of the form $x^i y^j z^\epsilon$, where $0 \leq i, j \leq l-1$ and $\epsilon = 0$ or 1. Since σ has order k, it follows $f(\sigma) = x^i y^j z$ for some choice of i and j. Thus $f(\sigma^2) = x^{i+j} y^{i+j}$, from whence it follows, since σ^2 has order l, that $f(\sigma^2)$ generates the diagonal of $\mathbb{Z}_l \times \mathbb{Z}_l$, which is the center of $\mathbb{Z}_l \wr \mathbb{Z}_2$. Therefore, $\langle \sigma^2 \rangle \lhd \text{Im } \lambda$.

Theorem 3.2 now implies that

$$\pi = a, b, a, b, \ldots, a, b.$$

□

5 Inverse Distributions for Alternating Pairs

Let $M = CM(G, X, \tau)$ be a Cayley map with $X = \langle x_1, x_2, \ldots, x_k \rangle$. Recall X is a sequence of generators for G, $x_{\tau(i)} = x_i^{-1}$, and X determines a rotation and hence a embedding into a closed, orientable surface. We will call the induced rotation $p = (x_1, x_2, \ldots, x_k)$. In the accompanying examples, we will use certain letters in p to emphasize the relation between x_i and $x_{\tau(i)}$. If $\tau(i) \neq i$, then we will replace x_i with a letter from the end of the alphabet, e.g., y, and $x_{\tau(i)}$ with its inverse, y^{-1}. When $\tau(i) = i$ we will replace x_i with a letter from the beginning of the alphabet, e.g., a. Thus, if $k = 6$ and $\tau = (1,4)(3,5)$, then $p = (x, a, y, x^{-1}, y^{-1}, b)$. We will also call a rotation p of a map of valence k a k-rotation. For the discussion in this section we number the darts emanating from a vertex with 1 through k in the counterclockwise direction. Recall from Section 2 that these numbers are the dart labels.

If the power function $\pi = a, b, a, b, \ldots, a, b$, then k is even and the elements of $\mathbb{Z}_k$ naturally bifurcate into even and odd elements. By Proposition 3.1,

$$\tau(i+1) = \tau(i) + \pi(x_i).$$

Then by induction and since $a + b = 2c$ (equation 2) if $\tau(1) = v$, then

$$\begin{aligned} \tau(2i) &= v + 2c(i-1) + a, \\ \tau(2i+1) &= v + 2ci, \end{aligned} \tag{4}$$

where $1 \leq i \leq \frac{k}{2}$.

Before we can continue our analysis we need the following proposition.

Proposition 5.1. *Let $M = CM(G, X, \tau)$ be a Cayley map with power function $\pi = a, b, a, b, \ldots, a, b$. Then both a and b are odd.*

Proof. Let $\tau(1) = v$. If $v = 2i$ for some i, then $1 = \tau(v) = v + 2c(i-1) + a$. Since both v and $2c(i-1)$ are even, a must be odd, and since $a + b = 2c$, b must also be odd.

On the other hand, suppose v is odd. By equation 4, $\tau(2) = v + a$. If a is even, then $v + a$ is odd. Let $v + a = 2j + 1$. Then $2 = \tau(v + a) = v + 2cj$ and $v + 2cj$ is odd, and this is a contradiction. Therefore, a is odd and hence b is also odd. □

If $v = \tau(1)$ is odd, then τ preserves parity by equation 4, otherwise τ switches parity. We will divide our analysis into two cases depending on the parity of $\tau(1)$. As we will see the inverse distribution is an amalgam of two e-balanced inverse distributions. We refer the reader to [5] and [8] for the inverse distributions of e-balanced Cayley maps.

Case 1: $\tau(1)$ odd.

We first consider the simpler case of $\tau(1)$ being odd.

Theorem 5.2. *Let $M = CM(G, X, \tau)$ be a regular Cayley map with power function $\pi = a, b, a, b, \ldots, a, b$, $k = |X|$, and $\tau(1)$ odd. If $a + b = 2c$ mod k, then the inverse distribution is an overlay of two c-balanced inverse distributions: one for the odd dart labels and the other for the even dart labels.*

Proof. The permutation $\sigma^2 = (1, 3, 5, ..., k-1)(2, 4, 6, ..., k)$ consists of two cycles, $\sigma_o = (1, 3, 5, \ldots, k-1)$ and $\sigma_e = (2, 4, 6, \ldots, k)$. By Lemma 4.7, $\tau\sigma^2\tau = (\sigma^2)^c$, so

$$(\tau\sigma_o\tau)(\tau\sigma_e\tau) = \tau\sigma_o\sigma_e\tau = \tau\sigma^2\tau = (\sigma_o\sigma_e)^c = (\sigma_o)^c(\sigma_e)^c.$$

Since τ preserves parity,

$$\tau\sigma_o\tau = (\sigma_o)^c \text{ and } \tau\sigma_e\tau = (\sigma_e)^c.$$

Therefore, the even dart labels have a c-balanced inverse distribution and the odd dart labels also have a c-balanced inverse distribution. □

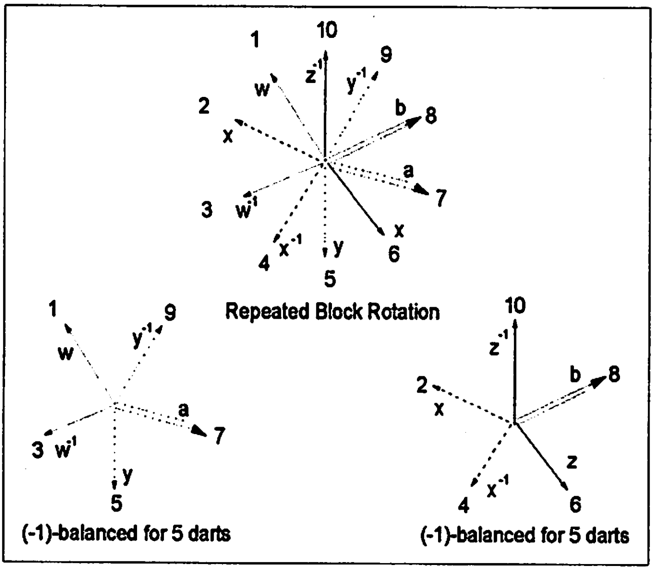

Figure 2:

Example 5.3. In Figure 2, we show a k-rotation such that $\tau(1)$ is odd. Set $k = 10$, $c = -1$, $\tau(1) = 3$; $\tau = (1, 3)(2, 4)(5, 9)(6, 10)$, $\pi = 1, 7, 1, 7, 1, 7, 1, 7, 1, 7$, $p = (w, x, w^{-1}, x^{-1}, y, z, a, b, y^{-1}, z^{-1})$, and Im $\lambda \cong \mathbb{Z}_5 \rtimes D_{10}$. The transpositions of the k-rotation separate easily into the permutations $\tau_e = (2,4)(6,10)$ and $\tau_o = (1,3)(5,9)$; both of which describe a (-1)-balanced inverse distribution for 5 darts. A regular Cayley map exists with this Im λ by Theorem 2.2.

Case 2: $\tau(1)$ even. We now turn to the more complicated case of $\tau(1)$ even.

Theorem 5.4. *Let $M = CM(G, X, \tau)$ be a regular Cayley map with power function $\pi = a, b, a, b, \dots, a, b$, $k = |X|$, and $\tau(1)$ even. The inverse distribution forms a c-balanced inverse distribution if the dart label $1 + 2i$ is identified with the dart label $\tau(1) + 2i$.*

Proof. Let $v = \tau(1)$. As in the proof of Theorem 5.2

$$\tau\sigma_o\tau = (\sigma_e)^c \text{ and } \tau\sigma_e\tau = (\sigma_o)^c,$$

except that the parity is reversed. Thus we have a c-balanced inverse distribution after the identification.

We still need to show that the inverse distribution is well defined after the identification, i.e., $\tau(1 + 2i)$ is identified with $\tau(v + 2i)$. An odd dart label d is identified with the even dart label $d + (v - 1) \mod k$. By equation 4 $\tau(r + 2i) = \tau(r) + 2ci$. So $\tau(1 + 2i) = v + 2ci$ and $\tau(v + 2i) = 1 + 2ci$. Hence $\tau(1 + 2i) - \tau(v + 2i) = v - 1$, so the dart labels are identified. □

Example 5.5. Figure 3 shows a k-rotation such that $\tau(1)$ is even. Set $k = 16$, $c = 3$, and $\tau(1) = 2$; then $\tau = (1,2)(3,8)(4,7)(5,14)(6,13)(9,10)(11,16)(12,15)$, $\pi = 15,7,15,7,15,7,15,7,15,7,15,7,15,7,15,7$, and $p = (a, a^{-1}, x_1, x_2, y_1, y_2, x_2^{-1}, x_1^{-1}, b, b^{-1}, z_1, z_2, y_2^{-1}, y_1^{-1}, z_2^{-1}, z_1^{-1})$. Note $\tau(1) - 1 = 1$. For $0 \le i \le l - 1$, relabeling $\{2i+1, 2i+2\}$ as $i + 1$, we get the distribution $\overline{\tau} = (2,4)(3,7)(6,8)$, which is a 3-balanced inverse distribution for 8 darts, and the rotation $\overline{p} = (a, x, y, x^{-1}, b, z, y^{-1}, z^{-1})$. Again a regular Cayley map exists with this Im λ by Theorem 2.2.

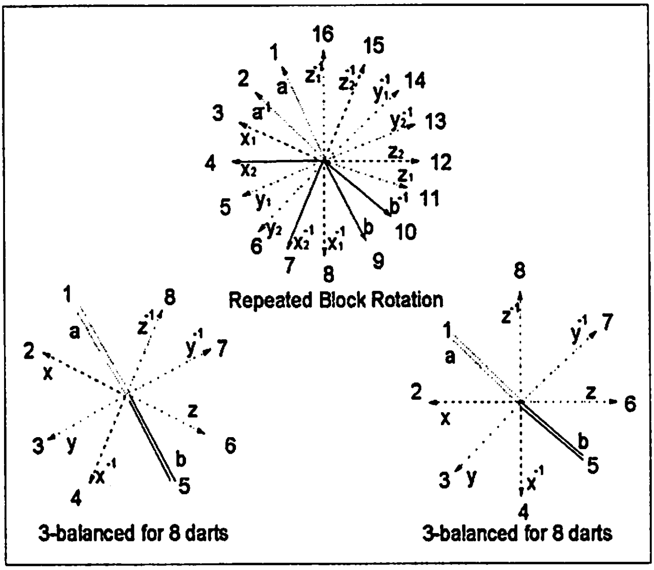

Figure 3:

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