Extensions of Bondy's Theorem on Cycles in 2-Connected Graphs.

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Abstract

A simple, undirected 2-connected graph G of order n belongs to the class $\mathcal{B}(n,\theta)$, $\theta \geq 0$ if $2(d(x)+d(y)+d(z)) > 3(n-1-\theta)$ holds for all independent triples $\{x,y,z\}$ of vertices. It is known (Bondy's theorem for 2-connected graphs) that G is hamiltonian if $\theta = 0$. In this paper we give a full characterization of graphs G in $\mathcal{B}(n,\theta)$, $\theta \leq 2$ in terms of their dual hamiltonian closure.

Keywords : Hamiltonian Cycle, Dual Closure.

1 Introduction

We consider throughout only simple 2-connected graphs $G = (V,E)$. We let $\alpha(G)$, $\nu(G)$, $\omega(G)$ denote respectively the independence number, the matching number and the number of components of the graph G. A graph G is 1-tough if $|S| \geq \omega(G-S)$ is true for any subset $S \subset V$ with $\omega(G-S) > 1$. For $k \leq \alpha(G)$ we set σ_k to be the minimum of $\sum_{x \in S} d(x)$ taken over all $k-$stables S. We use the term stable to mean independent set. A 2-connected graph G of order n belongs to the class $\mathcal{B}(n,\theta)$, $\theta \geq 0$ if $3(n-\theta) \geq 2\sigma_3 > 3(n-1-\theta)$. By this definition $\mathcal{B}(n,\theta) \cap \mathcal{B}(n,\theta+j) = \varnothing$ whenever $j \neq 0$. A graph G of order n belongs to class $\mathcal{O}(n,\varphi)$, $\varphi \geq 0$ if $\sigma_2 = n - \varphi$. As we shall see later, a strong link exists between the class $\mathcal{B}(n,\theta)$ and the class $\mathcal{O}(n,\varphi)$, $\varphi \geq 0$ of graphs. It is well known that G is hamiltonian if $G \in \mathcal{O}(n,0)$ ([15]) or $G \in \mathcal{B}(n,0)$ ([8]). Jung ([10]) proved that a 1-tough graph $G \in \mathcal{O}(n,\varphi)$ is hamiltonian if $\varphi \leq 4$ and $n \geq 11$. Faßbinder ([9]) generalized this result by proving that a 1-tough graph of order $n \geq 13$

and satisfying the condition $2\sigma_3 \geq 3n - 14$ is hamiltonian. In our terminology, Faßbinder'result states that any 1-tough graph in $\mathcal{B}(n,\theta)$ is hamiltonian if $\theta \leq 4$ and $n \geq 13$. However, assuming G to be 1-tough is a strong condition which is not easy to verify since recognizing tough graphs is NP-Hard ([12]). Using the concept of dual-closure we proved in [6] that $H = K_n$ if $G \in \mathcal{B}(n, 0)$. In this paper we go a step further by studying graphs in $\mathcal{B}(n,\theta)$, $\theta \leq 2$. At the same time, we provide a method for considering larger values of θ.

2 Preliminary results

A vertex of degree $n-1$ is a dominating vertex and Ω will denote the set of dominating vertices. If $D \subset V$, we shall say that $x \in V\backslash D$ (resp. $X \subset V\backslash D$) is D-dominated if $N_D(x) = D$ (resp. each vertex of X is D-dominated). The circumference $c(G)$ of G is the length of its longest cycle. For $u \in V(G)$, let $N_H(u)$ denote the set and $d_H(u)$ the number of neighbors of u in H, a subgraph of G. If $H = G$ we will write simply $N(u)$ for $N_G(u)$ and $d(u)$ for $d_G(u)$. For convenience, we extend this notation as follows. Given a subset $S \subset V$, we define the degree of a vertex x with respect to S as $d_S(x)$ to be the number of vertices of S adjacent to x. For $X \subset V$, put $N(X) = \cup_{u \in X} N(u)$. If $X, Y \subset V$, let $E(X,Y)$ denote the set of edges joining vertices of X to vertices of Y. As we need very often to refer to a presence or not of an edge, we write xy to mean that $xy \in E$ and $\overline{xy}$ to mean $xy \notin E$ For each pair (a,b) of nonadjacent vertices we associate

$$G_{ab} := G - N(a) \cup N(b),\ \gamma_{ab} := |N(a) \cup N(b)|,\ \lambda_{ab} := |N(a) \cap N(b)|$$
$$T_{ab} := V\backslash (N[a] \cup N[b]),\ t_{ab} := |T_{ab}|,\ \overline{\alpha}_{ab} := 2 + t_{ab} = |V(G_{ab})|$$
$$\delta_{ab} := \min\{d(x) \mid x \in T_{ab}\} \text{ if } T_{ab} \neq \varnothing,\ \lambda^*_{ab} := \max\{\lambda_{ab} - 2, 0\}$$
$$\delta^*_{ab} := d^T_{1+\lambda^*_{ab}},\ \alpha_{ab} = \alpha(G_{ab}),\ \nu_{ab} = \nu(G_{ab}),\ \omega(T_{ab}) = \omega(G[T_{ab}]).$$

In this paper there is a specially chosen pair (a,b) of vertices. To remain simple, we omit the reference to a, b for all parameters defined above. Moreover we understand T as the set, the graph induced by its vertices and its edge set. Our proofs are all based on the concept of the hamiltonian closure ([13], [1], [2]). These two conditions are both generalizations of Bondy-Chvàtal's closure. To state the condition under which our closure is based we define a binary variable ε_{ab} associated with (a,b).

Definition 2.1 *Let $\varepsilon_{ab} \in \{0,1\}$ be a binary variable, associated with a pair (a,b) of nonadjacent vertices. We set $\varepsilon_{ab} = 0$ if and only if*

1. $\varnothing \neq T$ and all vertices of T have the same degree $1 + t$. Moreover $\lambda_{ab} \leq 1$ if $N(T)\backslash T \subseteq N(a) \bigtriangleup N(b)$ (where $\bigtriangleup$ denotes the symmetric difference).

2. one of the following two local configurations holds

(a) T is a clique (possibly with one element), $\lambda_{ab} \leq 2$ and there exist $u, v \notin T$ such that $T \subset N(u) \cap N(v)$.

(b) T is an independent set (with at least two elements), $\lambda_{ab} \leq 1 + t$ and either $N(T) \subseteq D$ or there exists a vertex $u \in N(a) \triangle N(b)$ such that $|N_T(u)| \geq |T| - \max(\lambda_{ab} - 1, 0)$. Moreover T is a clique in G^2, the square of G.

Lemma 2.2 (the main closure condition) *Let G be a 2–connected graph and let (a, b) be a pair of nonadjacent vertices satisfying the condition*

$$\overline{\alpha}_{ab} \leq \max\{\lambda_{ab} + \nu_{ab}, \delta_{ab} + \varepsilon_{ab}\} \qquad \text{(ncc)}$$

Then $c(G) = p$ if and only if $c(G + ab) = p$, $p \leq n$.

The 0-dual neighborhood closure $nc_0^*(G)$ (the 0–dual closure for short) is the graph obtained from G by successively joining (a, b) satisfying the condition (ncc) until no such pair remains. Throughout we denote $nc_0^*(G)$ by H.

The first condition of (ncc) is a relaxation of the condition $\alpha_{ab} \leq \max\{\lambda_{ab}, 2\}$ given in [1]. Since by definition $\overline{\alpha}_{ab}$ is the order of G_{ab}, it follows that $\alpha_{ab} \leq \overline{\alpha}_{ab}$. As α_{ab} is not easy to compute we developed many upper bounds of α_{ab}, computable in polynomial time ([5]). One of these upper bounds is precisely ν_{ab}. It is known that for any graph H, $\alpha(H) + \nu(H) \leq n(H)$. We note that $\nu_{ab} = \nu(G_{ab}) = \nu(T)$ since a, b are isolated vertices in G_{ab}, we see that $\overline{\alpha}_{ab} - \nu_{ab} \leq \alpha_{ab}$ and hence $\alpha_{ab} \leq \lambda_{ab}$ implies $\overline{\alpha}_{ab} \leq \lambda_{ab} + \nu_{ab}$. We note that $\overline{\alpha}_{ab} \leq \lambda_{ab} + \nu_{ab}$ is stronger than Bondy-Chvàtal's hamiltonian closure condition ([13]) since $d(a) + d(b) \geq n \Leftrightarrow \overline{\alpha}_{ab} \leq \lambda_{ab}$. The second part of the condition (ncc) is a relaxation of a strongest one given in [2], improved in ([4]).The condition $\overline{\alpha}_{ab} \leq \delta_{ab} + \varepsilon_{ab}$, especially with the addition of the term ε_{ab} will prove to be a most useful tool in obtaining the main properties of the dual closure of any graph $G \in \mathcal{B}(n, 2)$. The condition $\overline{\alpha}_{ab} \leq \lambda_{ab} + \nu_{ab}$ is only used in very particular cases. Note that $\overline{\alpha}_{ab} \leq \delta_{ab} + \varepsilon_{ab} \Leftrightarrow \gamma_{ab} + \delta_{ab} + \varepsilon_{ab} \geq n$ and $\overline{\alpha}_{ab} \leq \lambda_{ab} + \nu_{ab} \Leftrightarrow d(a) + d(b) + \nu_{ab} \geq n$.

All closures based on the above conditions are well defined. Moreover, it is shown in ([5], [4]) that it takes a polynomial time to construct H and to exhibit a longest cycle in G whenever a longest cycle is known in H.

As a direct consequence of Lemma 2.2 we have.

Corollary 2.3 *Let G be a 2-connected graph. Then G is hamiltonian if and only if H is complete.*

3 Results

Theorem 3.1 *Let* $G \in \mathcal{B}(n,\theta)$, $\theta = 1,2$ *and let* $H := nc_0^*(G)$. *If* $H \neq K_n$ *then*

(i) $H = (m+1)K_1 + K_m$, $m \geq 2$

(ii) $H = (K_r \cup K_s \cup K_\varphi) + K_2$, $1 \leq r \leq s \leq \varphi$, $1 \leq r \leq \theta \leq \varphi \leq 3\theta$ *and* $5 \leq n \leq 10$.

(iii) $H = (m+2)\,K_1 + K_m$, $m \geq 2$

(iv) $H = (mK_1 \cup K_2) + K_m$, $m \geq 3$

(v) $H = (K_1 \cup K_s \cup 2K_2) + K_3$, $1 \leq s \leq 2$

(vi) $H = (K_1 \cup K_s \cup P_3) + K_2$, $1 \leq s \leq 2$.

Proof. By Lemma 4.3, $\theta \leq \varphi \leq 3\theta$ with $\theta = 1,2$. Choose (a,b) as specified in section 4. If $\epsilon_{ab} = 0$ then combining Lemma 5.2 and Lemma 4.4 we get (i) and (ii). If $\epsilon_{ab} = 1$ then $\varphi = 2,3$ by Proposition 5.1. If $\varphi = 2$ we have (iii) and (iv) by Lemma 5.3 and if $\varphi = 3$ we have (v) and (vi) by Lemma 5.4. ∎

Corollary 3.2 *Let* $G \in \mathcal{B}(n,\theta)$, $\theta = 1,2$. *Then*

(i) *G is nonhamiltonian if and only if* $\omega(H - \Omega) > |\Omega|$

(ii) *G is hamiltonian if and only if* $H = K_n$.

Corollary 3.3 *Let* $G \in \mathcal{B}(n,\theta)$, $\theta = 1,2$ *be* 1-*tough. Then* $H = K_n$.

Corollary 3.4 *If* $G \in \mathcal{B}(n,1)$ *and* $H \neq K_n$ *then (i)* $\varphi = 1$ *and* $H = (m+1)\,K_1 + K_m$, $m \geq 2$, *(ii)* $\varphi = 2$ *and* $H = (K_1 \cup K_s \cup K_2) + K_2$, $1 \leq s \leq 2$, *(iii)* $\varphi = 3$ *and* $H = (2K_1 \cup K_3) + K_2$.

Corollary 3.5 *If* $G \in \mathcal{B}(n,2)$ *is* 3-*connected and* $H \neq K_n$ *then (i)* $H = (m+2)\,K_1 + K_m$, $m \geq 3$, *(ii)* $H = (mK_1 \cup K_2) + K_m$ *with* $m \geq 3$.

Corollary 3.6 *If* $G \in \mathcal{B}(n,\theta)$, *and* $H \neq K_n$ *then* $c(G) \geq n-2$ *and* $c(G) = n-1$ *if* $n \geq 9$.

4 General Lemmas

Throughout we assume $G \in \mathcal{B}(n,\theta)$, $\theta \geq 0$, $H \neq K_n$ and all neighborhood sets and degrees are understood under H, unless otherwise stated. With each pair (a,b) we adopt the following decomposition of V by setting $A := N(a) \backslash N(b)$, $A^+ := A \cup \{a\}$, $B := N(b) \backslash N(a)$, $B^+ := B \cup \{b\}$, $D := N(a) \cap N(b)$,

$T := T_{ab}$ where $t = |T|$. For most of cases $t \leq 3$ and therefore we refer to x, y, z as possible vertices of T in that case. Also we set $T_i := \{x \in T \mid d_T(x) = i\}$, $i \geq 0$. We point out that $T \neq \varnothing$ by (ncc) whenever $H \neq K_n$ since H is 2-connected. For an ordered pair (x, y) of nonadjacent vertices we set $N(x, y) := N(x) \backslash N(y)$ and $n(x, y) := |N(x, y)|$. We shall say that $H \neq K_n$ is *well-shaped* if $E(A \cup B, T) \cup E(A, B) = \varnothing$ and $\Omega = D$.

Throughout, a, b are chosen as follows:
(i) $\overline{ab}$ and $d(a) + d(b) = \sigma_2 = n - \varphi$,
(ii) $x \in T$ is chosen so that $d(a) + d(b) + d(x) = \sigma_3$
(iii) subject to (i), λ_{ab} is minimum.

Moreover we always assume $d(a) \leq d(b) \leq d(x)$ for any $x \in T$. This choice implies immediately:

Lemma 4.1 *[3] If $H \neq K_n$ and $\varphi \geq 1$ then*

$(L1)$ $2 + t = \lambda_{ab} + \varphi$.

$(L2)$ $\forall p, q \in V, \overline{pq} \Rightarrow \max\{n(p, q), n(q, p)\} + \varepsilon_{pq} < \varphi$.

$(L3)$ $|A| \leq |B| < \varphi - \varepsilon_{ab}$.

$(L4)$ $T = \cup_{j=0}^{\varphi-1} T_j$. *Furthermore either* $T_{\varphi-1} = \varnothing$ *or* $E(A \cup B, T) \cup E(A, B) = \varnothing$.

$(L5)$ *if* $u \in A$ *then* $d_{A \cup T}(u) + \varepsilon_{bu} \leq \varphi - 2 + d(a) - \delta_{bu}$. *Similarly if* $v \in B$ *then* $d_{B \cup T}(v) + \varepsilon_{av} \leq \varphi - 2 + d(b) - \delta_{av}$.

$(L6)$ *if* $A \cup B = \varnothing$ *then* $\overline{xy} = \varnothing \Rightarrow d_T(x) + d_T(y) + \nu_{xy} < \varphi$ *for all* $x, y \in T$

The following constraints involving θ and φ will be useful to study the links between the classes $\mathcal{B}(n, \theta)$ and $\mathcal{O}(n, \varphi)$.

Lemma 4.2 *If $H \neq K_n$ then*

$(L7)$ $3(n - 1 - \theta) + 2\varphi < 2\delta_{ab} \leq n - 3\theta + 2\varphi$,

$(L8)$ $n - 3\theta + 2\varepsilon_{ab} \leq 2\lambda_{ab}$,

$(L9)$ $\varepsilon_{ab} = 1 \Rightarrow |A| + |B| + \varphi + 2(t - \delta_{ab}) \leq 3\theta - 2$.

Proof. $(L7)$ This is a consequence of the fact that $3(n - 1 - \theta) < 2\sigma_3 \leq 3(n - \theta)$ by definition of $\mathcal{B}(n, \theta)$ and $\sigma_3 = n - \varphi + \delta_{ab}$.

$(L8)$ Since $\overline{ab}$, we have $\gamma_{ab} + d(x) + \varepsilon_{ab} \leq n - 1$ by (ncc). This is equivalent to $\sigma_3 + \varepsilon_{ab} \leq n - 1 + \lambda_{ab}$. Thus $3(n - 1 - \theta) < 2(n - 1 + \lambda_{ab} - \varepsilon_{ab})$.

$(L9)$ Clearly $d(a) + d(b) = n - \varphi = |A| + |B| + 2\lambda_{ab}$. Combining with $(L1)$, we obtain $|A| + |B| + 2t = n + \varphi - 4$. By $(L7)$ $3(n - 1 - \theta) + 2\varphi < 2\delta_{ab}$. Combining the two results we get $(L9)$. ■

Lemma 4.3 *If $H \neq K_n$ then $\theta \leq \varphi \leq 3\theta$. Moreover $|A| < \theta$.*

Proof. If $\theta > \varphi$ then $d(a) + d(b) = n - \varphi \geq n - \theta + 1$. Similarly $d(a) + d(x) \geq n - \theta + 1$ and $d(b) + d(x) \geq n - \theta + 1$. Adding these inequalities we get $2\sigma_3 \geq 3(n - \theta) + 3$, a contradiction to the definition of $\mathcal{B}(n, \theta)$. If $\varphi \geq 3\theta + 1$ then $d(a) + d(b) = n - \varphi \leq n - 3\theta - 1$. Moreover $d(b) + d(x) \leq n - 1$ and $d(x) + d(a) \leq n - 1$. Summing the inequalities we get $2\sigma_3 \leq 3(n - 1 - \theta)$, another contradiction. By $(L3)$, we have $|A| \leq |B| < \varphi - \varepsilon_{ab}$. To prove $|A| < \theta$, suppose by contradiction $|A| \geq \theta$. Then $d(a) + d(b) + d(x) = \sigma_3 \geq 3(\lambda_{ab} + \theta)$. By $(L8)$, $2\lambda_{ab} \geq n - 3\theta$. Therefore $\sigma_3 \geq n + \lambda_{ab} \Leftrightarrow \gamma_{ab} + d(x) \geq n$ and hence ab by (ncc). The proof is now complete. ■

Lemma 4.4 *If $H \neq K_n$, $E(A \cup B, T) = \varnothing$ and $\varepsilon_{ab} = 0$ then either (i) $H = (m+1)K_1 + K_m$, $m \geq 2$, in which case $H \in \mathcal{B}(n, 1) \cap \mathcal{O}(n, 1)$ or (ii) $H = (K_r \cup K_s \cup K_\varphi) + K_2$, $1 \leq r \leq \theta$, $r \leq s \leq \varphi$, in which case $H \in \mathcal{B}(n, \theta) \cap \mathcal{O}(n, \varphi)$, $\theta \leq \varphi \leq 3\theta$ and $3\theta + 2 \leq n \leq 3\theta + 4$.*

Proof. Since $\varepsilon_{ab} = 0$ then either T is a clique or a stable. Suppose first that T is a clique. Then there exist two vertices, e, f say such that $e, f \in N(a) \cup N(b)$ and $T \subseteq N(e) \cap N(f)$. Since $E(A \cup B, T) = \varnothing$ then $e, f \in D$. For any $x \in T$, we have $\overline{\alpha}_{ax} = |\{a, b, x\}| + |B| = 3 + d(b) - \lambda_{ab}$. Moreover $\delta_{ax} = d(b)$ by the choice of a, b. By (ncc), $d(b) < \overline{\alpha}_{ax}$ and hence $\lambda_{ab} \leq 2$. Therefore $D = \{e, f\}$ and any vertex $v \in \{b\} \cup B$ satisfies the condition $N[v] = N[b]$. Similarly any vertex $u \in \{a\} \cup A$ satisfies the condition $N[u] = N[a]$. Clearly ef since $T_{ef} = \varnothing$ and $H = (K_r \cup K_s \cup K_\varphi) + K_2$. By $(L1)$, $t = \varphi$ and by Lemma 4.3, $1 \leq r < \theta$. Also $1 \leq r \leq s \leq \varphi$ by the choice of a, b. Moreover $H \in \mathcal{B}(n, \theta) \cap \mathcal{O}(n, \varphi)$, $\theta \leq \varphi \leq 3\theta$. It remains to prove that $3\theta + 2 \leq n \leq 3\theta + 4$. As $\sigma_3 = n - \varphi + d(x) = n - 1 + \lambda_{ab}$, we derive $\sigma_3 = n + 1$. As $H \in \mathcal{B}(n, \theta)$ and by definition $3(n - 1 - \theta) < 2(n + 1) \leq 3n + 3\theta$. Rearranging, we obtain the required bounds of n.

Suppose next that T is a stable. Then $d(x) = 1 + t$ and $N(x) \subseteq D$ for all $x \in T$. It follows that $N(w) = D$ for all $w \in V \backslash D$ by the choice of a, b, x. Moreover $|V \backslash D| = 2 + t$, $|D| = 1 + t$ and hence $|D| = \frac{n-1}{2}$. By $(L1)$ we get $\theta = 1$ and by (ncc) we easily check that D is a clique. Therefore $H = (m+1)K_1 + K_m$, $m \geq 2$ with $m = \lambda_{ab}$ and $H \in \mathcal{B}(n, 1) \cap \mathcal{O}(n, 1)$. The proof is now complete. ■

5 Application to graphs in $\mathcal{B}(n, \theta), \theta \leq 2$

Throughout we shall assume $H \neq K_n$.

Proposition 5.1 *If $\theta \leq 2$ and $\varepsilon_{ab} = 1$ then $\theta = 2$ and $\delta_{ab} = t$. Furthermore either $\varphi = 2$ and $A \cup B = \emptyset$ or $\varphi = 3$, $A = \emptyset$ and $B \subseteq \{v\}$.*

Proof. By contradiction suppose $H \in \mathcal{B}(n, 1)$ and $\varepsilon_{ab} = 1$. Then $\delta_{ab} \leq t$. By $(L9)$, $|A| + |B| + \varphi + 2(t - \delta_{ab}) \leq 1$. By Lemma 4.3, $1 \leq \varphi \leq 3$. Therefore $\varphi = 1$, $\delta_{ab} = t$ and $\lambda_{ab} = 1 + t$ by $(L1)$. This is a contradiction since then $d(a) = \lambda_{ab} > \delta_{ab}$. Thus $\theta = 2$ and $\varphi \geq 2$ by Lemma 4.3. Next we prove that $\delta_{ab} = t$. Because $\overline{ab}$ and $\varepsilon_{ab} = 1$, we have $\delta_{ab} \leq t$ by (ncc). By contradiction suppose $\delta_{ab} \leq t - 1$. By $(L9)$, $|A| + |B| + \varphi \leq 2$ and hence $\varphi = 2$, $A \cup B = \emptyset$. By $(L1)$, $t = \lambda_{ab}$, a contradiction since then $d(a) = \lambda_{ab} = t > \delta_{ab}$. Therefore $\delta_{ab} = t$ as claimed and the remaining follows by $(L9)$. ■

Lemma 5.2 *If $\theta \leq 2$ then H is (a, b)–well-shaped.*

Proof. If $\lambda_{ab} = 0$ then $n \leq 6$ by $(L8)$ and it is not difficult to see that $H = K_6$. For the following we assume $\lambda_{ab} \geq 1$. We need to consider two cases.

Case 1. $\varepsilon_{ab} = 0$

By Lemma 4.4, it suffices to prove that $E(A \cup B, T) = \emptyset$. Suppose by contradiction vx for some $(v, x) \in B \times T$. Then $n(v, a) \geq 2$ and hence $\varphi \geq 3$ for otherwise we contradict $(L2)$. By $(L1)$ and Proposition 5.1, $t = \lambda_{ab} + \varphi - 2 \geq \varphi - 1$. If T is a clique then $N(v) \supset T$ and $n(v, a) > (t - 1) + 1 \geq \varphi$, a contradiction to $(L2)$. Suppose next that T is a stable. By Definition 2.1(2.b), $d_T(v) \geq t - (\lambda_{ab} - 1) = t + 1 - \lambda_{ab} = \varphi - 1$ by $(L1)$. It follows that $n(v, a) \geq \varphi$ since vb. This is a contradiction. The same contradiction arises if ux for some $(u, x) \in A \times T$.

Case 2. $\varepsilon_{ab} = 1$

By Proposition 5.1, $\delta_{ab} = t$, $\varphi \in \{2, 3\}$ and $E(A \cup B, T) \cup E(A, B) = \emptyset$ unless $\varphi = 3$, $B = \{v\}$ and vx for some $x \in T$. Then $N_T(v) = \{x\}$ for otherwise $n(v, a) \geq \varphi = 3$. Moreover $t = \lambda_{ab} + \varphi - 2 \geq \varphi$ and hence $T_{av} = T \backslash \{x\}$ contains at least two vertices. Since $n(v, a) = 2 = \varphi - 1$ then $\varepsilon_{av} = 0$ and T_{av} is either a stable or a clique. If T_{av} is a stable then $N(y) = D \cup \{x\}$ and $N(z) = D \cup \{x\}$ for $y, z \in T_{av}^2$ since $d(y) = d(z) \geq d(b)$. This is a contradiction since now $n(x, a) \geq 3 = \varphi$. If T_{av} is a clique then there exist $e, f \in D$ such that $N(e) \cap N(f) \supset T_{av}$. Moreover $N(x) = D \cup \{v\}$ since $d(x) \geq d(b)$. As $d(x) = d(y) = \lambda_{ab} + 1 = t_{av} + 1 = t$ we get $\lambda_{ab} = t = \varphi - 1 = 2$. Now choosing (a, y) instead of (a, b) we see easily that H is (a, y)-*well-shaped*, a contradiction to our choice of (a, b). To complete the proof, we assume $E(A \cup B, T) \cup E(A, B) = \varnothing$ and we show that $\Omega = D$. If $\varphi = 2$ then $N_D(x) = D$ is true for all $x \in D$ since $n(e,x) \geq |\{a,b\}| = \varphi$. Moreover D is a clique by (ncc). It remains to assume $\varphi = 3$. If $B = \{v\}$ then $N_D(v) = D$ by the choice of (a,b) and the fact that $N_T(v) = \varnothing$. Suppose, by contradiction $\overline{ex}$ for some $(e,x) \in D \times T$. If $B = \{v\}$ we have a contradiction since then $n(e,x) \geq 3 = \varphi$. So we assume $A \cup B = \varnothing$. Since $\delta_{ab} = t$ then $\delta_{ab} = \lambda_{ab} + 1$ by $(L1)$. Now $\overline{\alpha}_{ex} \leq t + \lambda_{ab} - d(x) \leq \lambda_{ab} - 1 \leq \delta_{ex} - 1$, a contradiction by (ncc). Finally we note that D is a clique by (ncc) and $\Omega = D$ as claimed. ∎

Lemma 5.3 *If* $\theta = 2$, $\varphi = 2$ *and* $\varepsilon_{ab} = 1$ *then (i)* $H = (m+2)\,K_1 + K_m$, $m \geq 2$, *(ii)* $H = (mK_1 \cup K_2) + K_m, m \geq 3$.

Proof. As a first step we prove that H is $(a,b)-$*well-shaped*. By $(L2)$, $E(A \cup B, T) = \varnothing$ for if, for instance vx, $(v,x) \in B \times T$ then $n(v,a) \geq 2 = \varphi$, a contradiction. Moreover $E(A,B) = \varnothing$ for if uv, $(u,v) \in A \times B$ then $n(u,x) \geq 2 = \varphi$. Also $N_D(x) = D$ is true for all $x \in D$ since $n(e,x) \geq |\{a,b\}| = \varphi$ holds for all $x \in D$. It is now easy to see that D is a clique by (ncc) and $\Omega = D$. Therefore H is $(a,b)-$*well-shaped*. as claimed. Furthermore $\nu_{ab} \leq 1$ by (ncc). If $\nu_{ab} = 0$ then $H = (m+2)\,K_1 + K_m$, $m = \lambda_{ab} \geq 2$ since $t = \lambda_{ab} = m$ by $(L1)$. If $\nu_{ab} = 1$ and $t > 2$ then (ii) holds. If $\nu_{ab} = 1$ and $t = 2$ then $\delta_{ab} = 3 = t + 1$, a contradiction to our assuption $\varepsilon_{ab} = 1$. ∎

Lemma 5.4 *If* $\theta = 2$, $\varphi = 3$ *and* $\varepsilon_{ab} = 1$ *then (i)* $H = (K_1 \cup K_s \cup 2K_2) + K_3$, $1 \leq s \leq 2$, *(ii)* $H = (K_1 \cup K_s \cup P_3) + K_2$, $1 \leq s \leq 2$.

Proof. By Proposition 5.1, $A = \varnothing$, $B \subseteq \{v\}$ and $\delta_{ab} = t$. By $(L1)$, $t = \lambda_{ab} + 1$ and hence $T_0 = \varnothing$. By Lemma 5.2, H is $(a,b)-$well-shaped and hence $\delta_{ab} > d(a) = \lambda_{ab}$ and $\Omega = D$. This implies $T = T_1 \cup T_2$ by $(L4)$. Let us consider two cases.

Case 1. $T_2 = \varnothing$

Thus $T = pK_2, 2p = t \geq 3$ since obviously $\lambda_{ab} \geq 2$. By (ncc), $\nu_{av} \leq 2$ and hence $p \leq 2$. It follows that $p = 2$ and $H = (K_1 \cup K_s \cup 2K_2) + K_3$, $1 \leq s \leq 2$, since $t = 4 \Rightarrow \lambda_{ab} = m = 3$.

Case 2. $T_2 \neq \varnothing$

Choose $z \in T_2$, set $N(z) = \{x,y\}$ and suppose first $T \setminus \{x,y,z\} \neq \varnothing$. Considering (a,z) we may write $\overline{\alpha}_{az} = |\{a,b\}| + |B| + t - d_T(z) = |B| + t$. If $B = \varnothing$ then $\delta_{az} = d(b) = t - 1$. Therefore az by (ncc) since $d(w) \geq t$ for all $w \in T_{az} \setminus \{b\}$. With this contradiction, assume $B = \{v\}$, in which case $\overline{\alpha}_{az} = 1 + t$ with $\delta_{az} = t$. It follows that $\varepsilon_{az} = 0$ and T_{az} must be a clique (av is an edge of T_{az}). This is a contradiction since $\omega(T_{az}) \geq 2$. Therefore $T = \{x,y,z\}$ and $\lambda_{ab} = t - 1 = 2$. Moreover if xy then $T = T_2$ and hence $\delta_{ab} = t + 1$. This means $\varepsilon_{ab} = 0$, a contradiction to our assumption. We have just proved that $T = P_3$ and $H = (K_1 \cup K_s \cup P_3) + K_2$, $1 \leq s = |B \cup \{b\}| \leq 2$. ∎

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