More maps of p-gons with a ring of q-gons

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Abstract

Deza and Grishukhin studied 3-valent maps $M_n(p,q)$ consisting of a ring of n q-gons whose inner and outer domains are filled by p-gons. They described the conditions for n, p, q under which such map may exist and presented several infinite families of them. We extend their results by presenting several new maps concerning mainly large values of n and q.

1 Introduction

Throughout this paper, we consider connected plane graphs without loops or multiple edges. We use, however, more specialized notation: A *configuration* $C = (V(C), E(C))$ consists of a vertex set $V(C)$ and an edge set $E(C)$. Every edge of $E(C)$ has two ends and every end may or may not be incident to a vertex of $V(C)$ (however, an edge not incident with any vertex is not permitted). An edge with an end not incident to a vertex of $V(C)$ is called a *half-edge*. Thus, the configuration without half-edges is a graph (cf. also [2]).

The configurations we consider are embedded in the plane without crossing the edges; moreover, in the embedding, all half-edges are contained in the unbounded region of the plane. As for plane graphs, we also call this unbounded region the *outerface* of a plane configuration.

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In [1], Deza and Grishukhin introduced the special kind of plane maps: given integers n, p, q, $M_n(p,q)$ is a plane cubic graph having only p-gonal and q-gonal faces such that the q-gonal faces form a ring R_n of n q-gons (thus, the dual of $M_n(p,q)$ is a plane triangulation consisting of vertices of degrees p and q such that all q-valent vertices form an induced cycle); the parts of $M_n(p,q)$ that consist of the inner and outer domains of R_n (with the common boundary of R_n) are denoted by I_n and O_n. The authors studied the existence of $M_n(p,q)$ according to the values of n, p, q; they resolved the case $q = 4$ (it gives $n = p$ with the unique $M_p(p,4)$) and partially the case $q \geq 5$ (it implies $p \leq 7$ and if $p \geq 6$ then $q = 5$); in the latter case, the maps that are completely described are $M_{28}(7,5)$, $M_{12}(6,5)$, $M_6(5,5)$, $M_5(5,5)$, $M_5(5,6)$, $M_6(5,6)$, $M_8(5,6)$, $M_{10}(5,6)$, $M_4(5,7)$, $M_{10}(5,7)$, $M_{20}(5,7)$, $M_{12}(5,7)$, $M_{16}(5,7)$, $M_2(5,10)$, $M_3(5,8)$, $M_4(5,q), q \equiv 2,3 \pmod 5, q \neq 9$ and $M_2(4,8), M_3(4,6), M_4(4,q)$, $M_2(3,6)$. The open cases are, in particular, $M_n(7,5)$ with $n > 28$ and $M_n(5,7)$ for $17 \leq n \leq 19$ and $M_n(5,q)$ for $q \geq 8$ not listed above. Hajduk and Soták ([3]) showed that there exist maps $M_n(7,5)$ for $n = 30, 32, 36, 42$; they also constructed examples of $M_n(7,5)$ with large ring of 7-gons, namely, for $n = 28 + 12k, 48 + 8k, 52 + 20k, 56 + 28k, k \geq 0$.

The aim of this paper is to show the existence of $M_n(5,q)$ for several new values of n and q (including also graphs when n or q is large). We prove

Theorem 1.1 *The graph $M_n(5,q)$ exists for the following particular values:*

a) $n = 6,\ q \equiv 0,1,4 \pmod 5,\ q \geq 5,\ q \neq 9,$

b) $n = 8,\ q \equiv 1,4 \pmod 5,\ q \geq 5,$

c) $n = 10,\ q \equiv 0 \pmod{10},\ q \geq 20,$

d) $n = 12 + 4k,\ k \geq 0, q \equiv 2,3 \pmod 5,\ q \geq 13,$

e) $n = 14 + 4k,\ k \geq 0,\ q = 10.$

Together with the results of [1], it gives that for each even n there exists an integer q such that at least one map $M_n(5,q)$ exists.

2 The proof

In each of the cases below, the proof is based on the following argument: we consider certain plane configuration S (whose inner faces are pentagons) having n half–edges incident with its outerface. Assume that the vertices incident with these half–edges divide the boundary of the outerface of S into n parts containing (in the cyclical ordering) $n_1, n_2, \ldots, n_n$ vertices (the vertices incident with the half–edges are not counted). Assume further that there exists an integer $i \in \{1, \ldots, n\}$ and an sequence $n'_1 n'_2 \ldots n'_n$ of length n such that either for each $j \in \{1, \ldots, n\}$, $n'_j = n_{j+i}$ or, for each $j \in \{1, \ldots, n\}$, $n'_j = n_{n-i-j}$ (thus $n'_1 n'_2 \ldots n'_n$ is just an shifted sequence created from $n_1 n_2 \ldots n_n$ or from its reverse $n_n n_{n-1} \ldots n_1$)) and that, for each $j \in \{1, \ldots, n\}$, $n'_j + n_j = c$. Then it is possible to join the half–edges of two copies of S in such a way that the resulting plane graph is cubic having only pentagons and $(c+4)$-gons that form a ring of n faces. Thus, it shows that $M_n(5, c+4)$ exists. Instead of two copies of a configuration S, we may use two configurations S_1, S_2 that are joined in the way described above to form the suitable graph; these configurations are chosen in such a way that adding their asociated sequences (involving shifting of reversing) results in a sequence consisting of the same numbers.

The configurations used for this construction are built from certain smaller configurations by succesive concatenations. First, we define five basic configurations:

Consider a dodecahedron D and let $a_1 a_2 a_3 a_4 a_5$ be a 5-cycle bounding its outerface. The configuration A is obtained from D by splitting the edges $a_1 a_2, a_3 a_4$ into four half–edges $e_{a_1}, e_{a_2}, e_{a_3}, e_{a_4}$ (see Fig. 1,A). The half–edges e_{a_1}, e_{a_4} in the configuration A are called *input* half–edges (where e_{a_1} is the *upper* and e_{a_4} the *lower* input half–edge), the half–edges e_{a_2}, e_{a_3} are called *output* half–edges (where e_{a_2} is the upper and e_{a_3} is the lower one).

For $r \geq 1$, the configuration B_r is obtained in the following way: take two paths $b_0 x b_1 b_2 \ldots b_r$ and $c_0 c_1 c_2 \ldots c_{r-1} y c_r$ and r independent edges $w_1 z_1, w_2 z_2, \ldots, w_r z_r$. Next, add new edges $b_0 w_1, z_r c_r, c_i w_{i+1}$ for $i \in \{0, \ldots, r-1\}$, $b_i z_i$ for $i \in \{1, \ldots, r\}$ and $z_i w_{i+1}$ for $i \in \{1, \ldots, r-1\}$; finally, add the edges $b_0 c_0, b_r c_r, xy$ and split them into six half–edges $e_{b_0}, e_{c_0}, e_{b_r}, e_{c_r}, e_x, e_y$ (see Fig. 1,B_r). In this configuration, the half–edge e_{b_0} is the upper input,

e_{c_0} is the lower input, e_{b_r} is the upper output and e_{c_r} is the lower output half–edge.

The configuration H consists of the single edge gh and the four half–edges e_g, e'_g, e_h, e'_h; e_g is the upper input, e_h is the lower input, e'_g is the upper output and e'_h is the lower output half–edge (see Fig. 1,H).

The configuration P consists of the 5-cycle $p_1p_2p_3p_5p_5$ and the five half–edges e_{p_i}, $i = 1, \dots, 5$ incident with p_i; e_{p_1} is the upper input, e_{p_4} is the lower input, e_{p_2} is the upper output and e_{p_3} is the lower output half–edge (see Fig. 1,P).

The configuration K is obtained from the configuration A by deleting three vertices a_1, a_4, a_5 and attaching three new half–edges to all vertices of degree 2 in the resulting configuration (see Fig. 1,K). The output half–edges of K are the original output edges of A, the upper input half–edge is e, the lower input half–edge is e'.

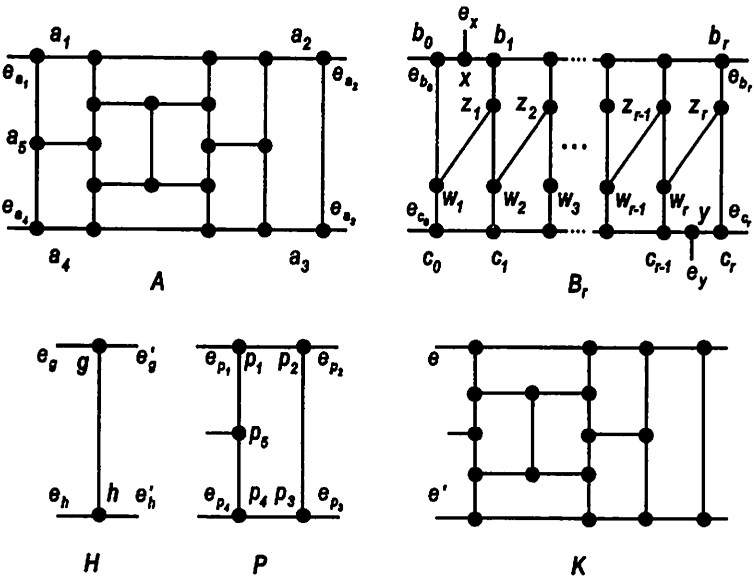

Figure 1: The configurations A, B_r, H, P and K

Next, we define the operation $\oplus$ (the concatenation) and the unary operation * (the mirror image) on a subset of plane configurations.

The operation $\oplus$ is defined in the following way: let M, N be two plane configurations such that e_u^X, e_l^X are declared as input half–edges of $X \in \{M, N\}$ (with e_u^X the upper and e_l^X the lower one), f_u^X, f_l^X are the corresponding output half–edges (again with f_u^X the upper and f_l^X the lower one). Then the configuration $M \oplus N$ is obtained from M, N by joining the half–edge f_u^M with e_u^N and f_l^M with e_l^N in the way that all inner faces of the resulting configuration are pentagons and all half–edges which were not joined lie in the outerface (so, for example, $A \oplus P$, $A \oplus H$ or $B_r \oplus A$ are not allowed). Thus, the input half–edges of $M \oplus N$ are e_u^M, e_l^M and the output half–edges are f_u^N, f_l^N.

Suppose that a configuration X, $X \in \{A, B, H, P, K\}$ is embedded in the plane in the fixed way. Then the mirror image X^* of X is the configuration obtained from X by the (geometric) reflection by the axis which crosses the output half–edges. According to this transformation, the input and output half–edges of X^* are exchanged comparing to X. The mirror image of a concatenation is defined by $(M \oplus N)^* = N^* \oplus M^*$.

We will use the notation rX for the succesive concatenation of $r \geq 0$ copies of X; $0X$ results in the empty configuration.

Case a) For $n = 6$, $q \equiv 1 \pmod 5$, $q \geq 5$ choose non-negative integers r, s such that $q = 5r + 5s + 6$. Then $M_6(5, q)$ is constructed from

$$S_1 = H \oplus rA \oplus B_{5s+2} \oplus rA^* \oplus H$$

$$S_2 = H \oplus sA \oplus B_{5r+2} \oplus sA^* \oplus H.$$

The corresponding sequences for S_1, S_2 are

$$(0(5r+1)(5r+5s+2))^2 \text{ and } ((5r+5s+2)(5s+1)0)^2.$$

Figure 2 illustrates the cases $q = 6, 11$.
For $n = 6$, $q \equiv 4 \pmod 5$, $q \geq 5$ choose integers $r \geq 1, s \geq 1$ such that $q = 5r + 5s + 4$. Then $M_6(5, q)$ is constructed from

$$S_1 = rA \oplus B_{5s} \oplus rA^*$$

$$S_2 = sA \oplus B_{5r} \oplus sA^*.$$

The corresponding sequences for S_1, S_2 are

$$(1(5r)(5r+5s-1))^2 \text{ and } ((5r+5s-1)(5s)1)^2.$$

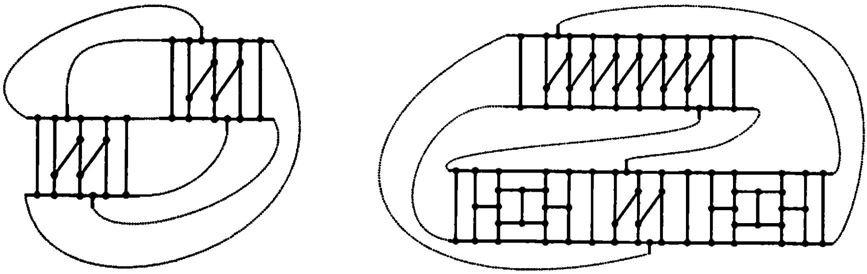

Figure 2: The maps $M_6(5,6)$ and $M_6(5,11)$

For $n = 6$, $q \equiv 0 \pmod 5$, $q \geq 5$ choose non–negative integers r, s and put $q = 5r + 5s + 5$. Then $M_6(5,q)$ is constructed from

$$S_1 = rA \oplus B_{5s+1} \oplus H \oplus rA$$

$$S_2 = sA \oplus B_{5r+1} \oplus H \oplus sA$$

whose corresponding sequences are

$1(5r)(5r+5s+1)0(5r+1)(5r+5s)$ and $(5r+5s)(5s+1)0(5r+5s+1)(5s)1$.

In all cases above, if $r \neq s$, then $I_n \neq O_n$; observe that for some q we can construct maps with $I_n = O_n$ as well as the ones with $I_n \neq O_n$.

Case b) For $n = 8$, $q \equiv 1 \pmod 5$, $q \geq 6$, put $q = 5r + 6$. Then $M_8(5,q)$ is constructed from

$$S = P \oplus rA \oplus B_1 \oplus rA^* \oplus P^*$$

and the corresponding sequences are

$$(0^2(5r+2)^2)^2 \text{ and } ((5r+2)^2 0^2)^2.$$

For $n = 8, q \equiv 4 \pmod 5$, $q \geq 9$ put $q = 5r + 9$. Then $M_8(5,q)$ is constructed from

$$S = K \oplus rA \oplus B_1 \oplus rA^* \oplus K^*$$

and the corresponding sequences are

$$(1^2(5r+4)^2)^2 \text{ and } ((5r+4)^2 1^2)^2.$$

Case c) For $n = 10$, $q \equiv 0 \pmod{10}$, $q \geq 20$, put $q = 10r + 20$. Then $M_{10}(5, q)$ is constructed from

$$S = H \oplus (r+2)A \oplus B_{5r+6} \oplus H \oplus rA \oplus B_3 \oplus H \oplus B_6 \oplus H \oplus (2r+2)A$$

and the corresponding sequences are

$$0(5r+11)(10r+8)5(10r+16)0(10r+11)8(5r+5)(10r+16)$$

and

$$(10r+16)(5r+5)8(10r+11)0(10r+16)5(10r+8)(5r+11)0.$$

Case d) In all subcases of this case, we consider

$$I = \alpha_0 A \oplus k(B_\beta \oplus H \oplus B_\gamma \oplus H \oplus \alpha A) \oplus B_\beta \oplus H \oplus B_1$$

$$I' = B_1 \oplus H \oplus B_\beta \oplus k(H \oplus \alpha A \oplus B_\gamma \oplus H \oplus B_\beta) \oplus \alpha_0 A^*$$

where $\alpha, \alpha_0, \beta, \gamma$ are specified separately.

For $n = 12 + 8k, q \equiv 3 \pmod 5), q \geq 13$, put $q = 5\delta + 8, \delta \geq 1$. Then $M_n(5, q)$ is constructed from

$$S = I \oplus H \oplus \delta A \oplus I'$$

with $\beta = 2, \gamma = 3, \delta = \alpha_0 = \alpha + 1 \geq 1$; the corresponding sequences are

$$(1(5\delta)(4(5\delta))^k 4(5\delta+3)3((5\delta-1)5)^k(5\delta+1))^2$$

and

$$((5\delta+3)4((5\delta)4)^k(5\delta)(5\delta+1)(5(5\delta-1))^k 3)^2$$

For $n = 12 + 8k, q \equiv 2 \pmod 5), q \geq 17$, we take $q = 5\delta + 7, \delta \geq 2$ and

$$S = H \oplus I \oplus H \oplus \delta A \oplus I' \oplus H$$

with $\beta = 5, \gamma = 4, \delta = \alpha_0 + 1 = \alpha + 2 \geq 2$; the corresponding sequences are

$$(0(5\delta-4)(7(5\delta-4))^k 7(5\delta+3)3((5\delta-3)6)^k(5\delta))^2$$

and

$$((5\delta+3)7((5\delta-4)7)^k(5\delta-4)0(5\delta)(6(5\delta-3))^k 3)^2.$$

For $n = 16 + 8k, q \equiv 3 \pmod 5, q \geq 13$, we take $q = 5\delta + 8, \delta \geq 1$ and

$$S = K \oplus I \oplus H \oplus \delta A \oplus B_1 \oplus \delta A^* \oplus H \oplus I' \oplus K^*$$

with $\beta = 3, \gamma = 2, \delta = \alpha_0 + 1 = \alpha + 1 \geq 1$; the corresponding sequences are

$$(1^2(5\delta - 1)(5(5\delta - 1))^k 5(5\delta + 3)^2 3((5\delta)4)^k(5\delta + 1))^2$$

and

$$((5\delta + 3)^2 5((5\delta - 1)5)^k(5\delta - 1)1^2(5\delta + 1)(4(5\delta))^k 3)^2.$$

For $n = 16 + 8k, q \equiv 2 \pmod 5, q \geq 17$, we take $q = 5\delta + 7, \delta \geq 2$ and

$$S = P \oplus I \oplus H \oplus \delta A \oplus B_1 \oplus \delta A^* \oplus H \oplus I' \oplus P^*$$

with $\beta = 4, \gamma = 5, \delta = \alpha_0 + 1 = \alpha + 2 \geq 2$; the corresponding sequences are

$$(0^2(5\delta - 3)(6(5\delta - 3))^k 6(5\delta + 3)^2 3((5\delta - 4)7)^k(5\delta))^2$$

and

$$((5\delta + 3)^2 6((5\delta - 3)6)^k(5\delta - 3)0^2(5\delta)(7(5\delta - 4))^k 3)^2.$$

Case e) For $n = 14 + 4k,\ k \geq 0,\ q = 10$, we take

$$S = H \oplus (k+1)(B_1 \oplus H) \oplus B_3 \oplus H \oplus B_4 \oplus H \oplus B_3 \oplus (k+1)(H \oplus B_1) \oplus H;$$

the corresponding sequences are

$$(013^{k+1}5653^k1)^2 \text{ and } (653^{k+1}1013^k5)^2.$$

□

3 Concluding remarks

While it is proved that $M_n(5, q)$ exists for all even n, no example is known for n odd. Also, it would be desirable to find $M_n(5, q)$ for infinitely many q if $n = 14 + 4k$ as well as to show the existence of $M_n(5, q)$ for other even n and the corresponding remaining residual classes modulo 5.

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