Hyperbolic Modified Pell Functions

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ABSTRACT

In this paper, we define the hyperbolic modified Pell functions by the modified Pell sequence and classical hyperbolic functions. Afterwards, we investigate the properties of the modified Pell functions.

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1. Introduction

Modified Pell sequence is defined by Horadam. Modified Pell sequence $\{q_n\}$ for all n is defined by

$$q_{n+2} = 2q_{n+1} + q_n \qquad q_0 = 1,\ q_1 = 1.$$

It is clear that, the only difference between modified Pell and Pell sequence is the initial conditions. It is known that, the characteristic equation of modified Pell sequence is

$$x^2 - 2x - 1 = 0, \tag{1}$$

which has two real roots;

$$x_1 = \alpha = 1 + \sqrt{2}, \quad x_2 = \frac{-1}{\alpha} = 1 - \sqrt{2}.$$

Thus, Binet's formula for $\{q_n\}$ is given by

$$q_n = \frac{\alpha^n + \left(\frac{-1}{\alpha}\right)^n}{2}$$

where $n = 0,\ \pm 1,\ \pm 2,\ \ldots$. Modified Pell sequence is given as follows;

$$\{q_n\} = \{1, 1, 3, 7, 17, 41, 99, \ldots\}.$$

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In (5), the authors define a new class of hyperbolic functions by the Fibonacci and Lucas sequences and give symmetrical representation of the hyperbolic Fibonacci and Lucas functions. In (3), the authors define hyperbolic functions with the second order recurrence sequences. It' s known that, Binet's formula for modified Pell sequence is different from other recurrence sequences(e.g. Fibonacci, Pell and Jacobsthal). Therefore, the purpose of this article is to define hyperbolic functions with modified Pell sequence $\{q_n\}$ and investigate the properties of these functions.

Let q_n represent the n^{th} modified Pell numbers. It's known that, the terms of sequences q_n and q_{-n} coincide for the $n = 2k$, that is $q_{2k} = q_{-2k}$. Then, for the $n = 2k+1$, they are opposite by the sign , that is $q_{2k+1} = -q_{-2k-1}$. In this connection, Binet's formula for $\{q_n\}$ sequence may be written as follows

$$q_n = \begin{cases} \dfrac{\alpha^n - \alpha^{-n}}{2} & , n \ odd \\ \dfrac{\alpha^n + \alpha^{-n}}{2} & , n \ even \end{cases}$$

where α root of characteristic equation (1) and $n = 0, \ \pm 1, \ \pm 2, \ \ldots$.

Now, we define the hyperbolic functions with the modified Pell sequence. Hyperbolic modified Pell functions are as follows; the hyperbolic modified Pell sine

$$sq\left(x\right) = \frac{\alpha^{2x+1} - \alpha^{-2x-1}}{2} \tag{2}$$

The hyperbolic modified Pell cosine

$$cq\left(x\right) = \frac{\alpha^{2x} + \alpha^{-2x}}{2} \tag{3}$$

Note that, The correlations between modified Pell numbers and hyperbolic modified Pell functions are given by

$$cq\left(k\right) = q_{2k}; \quad sq\left(k\right) = q_{2k+1}$$

where $k = 0, \ \pm 1, \pm 2, \ldots$.

It's known that, the classical hyperbolic functions are

$$sh\left(x\right) = \frac{e^x - e^{-x}}{2}, \quad ch\left(x\right) = \frac{e^x + e^{-x}}{2}. \tag{4}$$

Now, we consider symmetrical representation of the hyperbolic modified Pell functions.

2. Symmetrical Representation of the Hyperbolic Modified Pell Functions

Based on classical hyperbolic functions (4) and Binet's formula for the modified Pell numbers, we can give the definitions of the hyperbolic modified Pell functions that are different from definitions (2) and (3): Symmetrical modified Pell sine

$$sqs\,(x) = \frac{\alpha^x - \alpha^{-x}}{2}$$

Symmetrical modified Pell cosine

$$cqs\,(x) = \frac{\alpha^x + \alpha^{-x}}{2}$$

Symmetrical hyperbolic modified Pell functions are connected with the classical hyperbolic functions by the following correlations

$$sqs\,(x) = sh\,((\ln\alpha)\,x)\,, \qquad cqs\,(x) = ch\,((\ln\alpha)\,x)\,.$$

The modified Pell numbers are determined with the symmetrical hyperbolic modified Pell functions as follows,

$$q_n = \begin{cases} cqs\,(n) & ,\ n \text{ even} \\ sqs\,(n) & ,\ n \text{ odd} \end{cases} .$$

The hyperbolic modified Pell functions transform the modified Pell numbers theory into a continuous theory because every identity for the hyperbolic modified Pell functions has its discrete analogy in the framework of the modified Pell numbers.

It is easy to construct the graphs for the symmetrical modified Pell functions (Figure 1). Their graphs have a symmetrical form and are analogous to the graphs of the classical hyperbolic functions.

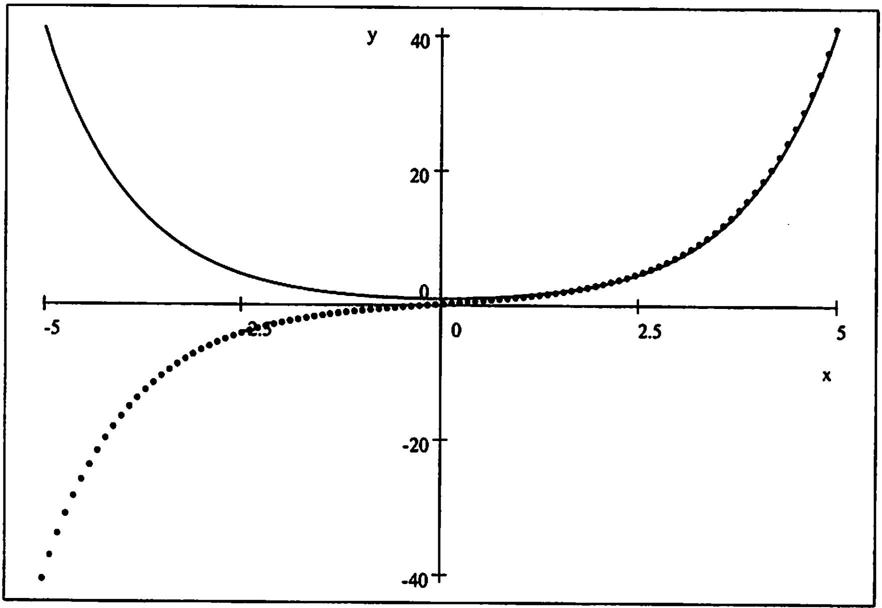

Figure 1. Symmetrical Hyperbolic Modified Pell Function

3. Properties of the Symmetrical Hyperbolic Modified Pell Functions

Now, we give some properties of the symmetrical hyperbolic modified Pell functions.

Theorem 1 *The recurrence equation of q_n, $q_{n+2} = 2q_{n+1} + q_n$ is valid for the symmetrical hyperbolic modified Pell functions:*

$$sqs\,(x+2) = 2cqs\,(x+1) + sqs\,(x)$$

and

$$cqs\,(x+2) = 2sqs\,(x+1) + cqs\,(x)\,.$$

Proof. Using definitions of symmetrical hyperbolic modified Pell functions, we have

$$\begin{aligned} 2cqs(x+1)+sqs(x) &= 2\left(\frac{\alpha^{x+1}+\alpha^{-x-1}}{2}\right)+\left(\frac{\alpha^{x}-\alpha^{-x}}{2}\right) \\ &= \frac{\alpha^{x}(2\alpha+1)+\alpha^{-x}\left(\frac{2}{\alpha}-1\right)}{2} \\ &= \frac{\alpha^{x}\alpha^{2}-\alpha^{-x}\alpha^{-2}}{2}=\frac{\alpha^{x+2}-\alpha^{-x-2}}{2} \\ &= sqs(x+2). \end{aligned}$$

■

Theorem 2 *The equation* $q_n^2 - q_{n+1}q_{n-1} = 2(-1)^n$ *is valid for the symmetrical hyperbolic modified Pell functions;*

$$\left[sqs(x)^2\right] - cqs(x+1)\,cqs(x-1) = -2$$

and

$$\left[cqs(x)^2\right] - sqs(x+1)\,sqs(x-1) = 2.$$

Proof. From the definitions of symmetrical hyperbolic modified Pell functions, we obtain

$$\begin{aligned} \left[\begin{array}{c} \left[sqs(x)^2\right] - \\ cqs(x+1)\,cqs(x-1) \end{array}\right] &= \left[\begin{array}{c} \left(\frac{\alpha^{x}-\alpha^{-x}}{2}\right)^2 - \\ \left(\frac{\alpha^{x+1}+\alpha^{-x-1}}{2}\right)\left(\frac{\alpha^{x-1}+\alpha^{-x+1}}{2}\right) \end{array}\right] \\ &= \frac{-\left(\alpha^2+\frac{1}{\alpha^2}+2\right)}{4} = -2 \end{aligned}$$

and

$$\begin{aligned} \left[\begin{array}{c} \left[cqs(x)^2\right] - \\ sqs(x+1)\,sqs(x-1) \end{array}\right] &= \left[\begin{array}{c} \left(\frac{\alpha^{x}+\alpha^{-x}}{2}\right)^2 - \\ \left(\frac{\alpha^{x+1}-\alpha^{-x-1}}{2}\right)\left(\frac{\alpha^{x-1}-\alpha^{-x+1}}{2}\right) \end{array}\right] \\ &= \frac{\left(\alpha^2+\frac{1}{\alpha^2}+2\right)}{4} = 2. \end{aligned}$$

■

Some other properties of the symmetrical hyperbolic modified Pell functions listed in the following table.

The identities for q_n numbers	The identities for the symmetrical Hyperbolic modified Pell functions
$q_{n+2} = 2q_{n+1} + q_n$	$sqs\,(x+2) = 2cqs\,(x+1) + sqs\,(x)$ $cqs\,(x+2) = 2sqs\,(x+1) + cqs\,(x)$
$q_n = (-1)^n\, q_{-n}$	$sqs\,(x) = -sqs\,(-x)$ $cqs\,(x) = cqs\,(-x)$
$q_n^2 - q_{n+1}q_{n-1} = 2\,(-1)^n$	$[sqs\,(x)]^2 - cqs\,(x+1)\, cqs\,(x-1) = -2$ $[cqs\,(x)]^2 - sqs\,(x+1)\, sqs\,(x-1) = 2$
$Q_{2n} = 2q_n$	$sQs\,(2x) = 4sqs\,(x)\, cqs\,(x)$ $sQs\,(2x) = 4cqs\,(x)\, sqs\,(x)$
$2P_n = q_{n-1} + q_n$	$2cPs\,(x) = cqs\,(x-1) + sqs\,(x)$ $2sPs\,(x) = sqs\,(x-1) + cqs\,(x)$

The symmetrical hyperbolic modified Pell functions have properties that are similar to the classical hyperbolic functions. Now, we give some hyperbolic properties of the symmetrical hyperbolic modified Pell functions.

Theorem 3 *The equation* $(ch\,(x))^2 + (sh\,(x))^2 = ch\,(2x)$ *is valid for the symmetrical hyperbolic modified Pell functions;*

$$cqs\,(2x) = [cqs\,(x)]^2 + [sqs\,(x)]^2 .$$

Proof.

$$\begin{aligned}[cqs\,(x)]^2 + [sqs\,(x)]^2 &= \left(\frac{\alpha^x + \alpha^{-x}}{2}\right)^2 + \left(\frac{\alpha^x - \alpha^{-x}}{2}\right)^2 \\ &= \frac{2\alpha^{2x} + 2\alpha^{-2x}}{4} = cqs\,(2x)\end{aligned}$$

■

Theorem 4 *The equation* $sh\,(2x) = 2sh\,(x)\, ch\,(x)$ *is valid for the symmetrical hyperbolic modified Pell functions;*

$$sqs\,(2x) = 2sqs\,(x)\, cqs\,(x) .$$

The proof is analogous to Theorem 3.

Theorem 5 *The equation* $ch(x+y) = ch(x)\,ch(y) + sh(x)\,sh(y)$ *is valid for the symmetrical hyperbolic modified Pell functions;*

$$cqs(x+y) = cqs(x)\,cqs(y) + sqs(x)\,sqs(y).$$

Proof.

$$\begin{aligned} cqs(x)\,cqs(y) + sqs(x)\,sqs(y) &= \left[\begin{array}{c} \left(\frac{\alpha^x + \alpha^{-x}}{2}\right)\left(\frac{\alpha^y + \alpha^{-y}}{2}\right) + \\ \left(\frac{\alpha^x - \alpha^{-x}}{2}\right)\left(\frac{\alpha^y - \alpha^{-y}}{2}\right) \end{array}\right] \\ &= \frac{\alpha^{x+y} + \alpha^{-x-y}}{2} = cqs(x+y) \end{aligned}$$

■

Theorem 6 *The equation* $ch(x-y) = ch(x)\,ch(y) - sh(x)\,sh(y)$ *is valid for the symmetrical hyperbolic modified Pell functions;*

$$cqs(x-y) = cqs(x)\,cqs(y) - sqs(x)\,sqs(y).$$

The proof is analogous to Theorem 5.

Theorem 7 *The following correlations are valid for the derivative symmetrical hyperbolic modified Pell functions;*

$$[ch(x)]^{(n)} = \begin{cases} sh(x) & for\ n\ odd \\ ch(x) & for\ n\ even \end{cases};\ [sh(x)]^{(n)} = \begin{cases} ch(x) & for\ n\ odd \\ sh(x) & for\ n\ even \end{cases}.$$

Proof.

$$\begin{aligned} [cqs(x)]' &= \left(\frac{\alpha^x + \alpha^{-x}}{2}\right)' = (\ln\alpha)\,sqs(x) \\ [sqs(x)]' &= \left(\frac{\alpha^x - \alpha^{-x}}{2}\right)' = (\ln\alpha)\,cqs(x) \\ [cqs(x)]'' &= ((\ln\alpha)\,sqs(x))' = (\ln\alpha)^2\,cqs(x) \\ [sqs(x)]'' &= ((\ln\alpha)\,cqs(x)) = (\ln\alpha)^2\,sqs(x) \end{aligned}$$

$$\cdots$$

$$\begin{aligned} [cqs(x)]^{(n)} &= \begin{cases} (\ln\alpha)^n\,sqs(x) & for\ n\ odd \\ (\ln\alpha)^n\,cqs(x) & for\ n\ even \end{cases} \\ [sqs(x)]^{(n)} &= \begin{cases} (\ln\alpha)^n\,cqs(x) & for\ n\ odd \\ (\ln\alpha)^n\,sqs(x) & for\ n\ even. \end{cases} \end{aligned}$$

■

Theorem 8 *The following equation that is similar to Moivre's equation is valid for the symmetrical hyperbolic modified Pell functions*

$$[cqs\,(x) \pm sqs\,(x)]^n = [cqs\,(nx) \pm sqs\,(nx)]$$

The proof is analogous to Theorem 3.

It's easy to prove the identities for the symmetrical hyperbolic modified Pell functions in the following table.

Classical Hyperbolic Functions	Symmetrical Hypebolic Modified Pell Functions
$[ch\,(x)]^2 - [ch\,(y)]^2 = 1$	$[cqs\,(x)]^2 - [sqs\,(x)]^2 = 1$
$[ch\,(x)]^2 + [ch\,(y)]^2 = ch\,(2x)$	$cqs\,(2x) = [cqs\,(x)]^2 + [sqs\,(x)]^2$
$ch\,(x \pm y) = \begin{bmatrix} ch\,(x)\,ch\,(y) \\ \pm sh\,(x)\,sh\,(y) \end{bmatrix}$	$cqs\,(x \pm y) = \begin{bmatrix} cqs\,(x)\,cqs\,(y) \\ \pm sqs\,(x)\,sqs\,(y) \end{bmatrix}$
$sh\,(x \pm y) = \begin{bmatrix} sh\,(x)\,ch\,(y) \\ \pm ch\,(x)\,sh\,(y) \end{bmatrix}$	$sqs\,(x \pm y) = \begin{bmatrix} sqs\,(x)\,cqs\,(y) \\ \pm cqs\,(x)\,sqs\,(y) \end{bmatrix}$
$sh\,(2x) = 2sh\,(x)\,ch\,(x)$	$sqs\,(2x) = 2sqs\,(x)\,cqs\,(x)$
$[ch\,(x)]^{(n)} = \begin{cases} sh\,(x) & for\ n\ odd \\ ch\,(x) & for\ n\ even \end{cases}$	$[cqs\,(x)]^{(n)} = \begin{cases} (\ln\alpha)^n\,sqs\,(x) & for\ n\ odd \\ (\ln\alpha)^n\,cqs\,(x) & for\ n\ even \end{cases}$
$[sh\,(x)]^{(n)} = \begin{cases} ch\,(x) & for\ n\ odd \\ sh\,(x) & for\ n\ even \end{cases}$	$[sqs\,(x)]^{(n)} = \begin{cases} (\ln\alpha)^n\,cqs\,(x) & for\ n\ odd \\ (\ln\alpha)^n\,sqs\,(x) & for\ n\ even. \end{cases}$
$\int\int_n ch\,(x)\,dx = \begin{cases} sh\,(x) & for\ n\ odd \\ ch\,(x) & for\ n\ even \end{cases}$	$\int\int_n cqs\,(x)\,dx = \begin{cases} (\ln\alpha)^{-n}\,sqs\,(x) & for\ n\ odd \\ (\ln\alpha)^{-n}\,cqs\,(x) & for\ n\ even \end{cases}$
$\int\int_n sh\,(x)\,dx = \begin{cases} ch\,(x) & for\ n\ odd \\ sh\,(x) & for\ n\ even \end{cases}$	$\int\int_n sqs\,(x)\,dx = \begin{cases} (\ln\alpha)^{-n}\,cqs\,(x) & for\ n\ odd \\ (\ln\alpha)^{-n}\,sqs\,(x) & for\ n\ even \end{cases}$

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