ON BLOCKING SETS OF EXTERNAL LINES TO A QUADRIC IN PG(3, q), q PRIME

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ABSTRACT. Minimal blocking sets of class $[h, k]$ with respect to the external lines to an elliptic quadric of PG$(3, q)$, $q \geq 5$ prime, are characterized.

1. INTRODUCTION

A *blocking set* in a projective space $\mathbb{P} = \mathrm{PG}(d, q)$ is a subset of $\mathbb{P}$ which meets every line. Blocking sets have been investigated by several authors, from many points of view. The reader is referred to [2, 4] and papers cited there for a survey on this topic.

Now, let $\mathcal{F}$ be a family of lines of $\mathbb{P}$. A point-set B of $\mathbb{P}$ is a *blocking set with respect to* $\mathcal{F}$ (briefly, an $\mathcal{F}$*-blocking set*) if every line in $\mathcal{F}$ is incident with B. An $\mathcal{F}$-blocking set B is *minimal* if no proper subset of B is an $\mathcal{F}$-blocking set. If $t_1, t_2, ..., t_s$ are non-negative integers and, for any $L \in \mathcal{F}$, $|B \cap L| \in \{t_1, t_2, ..., t_s\}$, B is *of class* $[t_1, t_2, ..., t_s]$. Moreover, if B is a blocking set of class $[t_1, t_2, ..., t_s]$ and, for any $i = 1, 2, ..., s$, there exsists in $\mathcal{F}$ a t_i-secant to B, i.e. a line L such that $|B \cap L| = t_i$, B is *of type* $(t_1, t_2, ..., t_s)$. In [1] and [3] blocking sets with respect to the family of the external lines to a conic in PG$(2, q)$ are studied.

In this paper we are interested in blocking sets of PG$(3, q)$ with respect to the family $\mathcal{F}$ of all external lines to an elliptic quadric Ω. Obviously, for any plane π of PG$(3, q)$, $\pi \backslash \Omega$ is a minimal $\mathcal{F}$-blocking set. Any line in $\mathcal{F}$ not contained in $\pi \backslash \Omega$ meets $\pi \backslash \Omega$ precisely in one point, so $\pi \backslash \Omega$ is of type $(1, q+1)$ with respect to $\mathcal{F}$. We show that, if $q \geq 5$ is a prime, this is the only possibility for a minimal $\mathcal{F}$-blocking set of class $[h, k]$, $1 \leq h < k \leq q+1$, by proving the following

Theorem 1.1. *Let Ω be an elliptic quadric of $PG(3, q)$, $q \geq 5$ prime. If B is a minimal blocking set of class $[h, k]$, $1 \leq h < k \leq q+1$, with respect to the family $\mathcal{F}$ of the external lines to Ω, then $B = \pi \backslash \Omega$ for some plane π of $PG(3, q)$.*

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From now on, we denote by Ω an elliptic quadric of $\mathrm{PG}(3,q), q \geq 5$ prime and by $\mathcal{F}$ the family of the external lines to Ω. Moreover, B will denote a minimal $\mathcal{F}$-blocking set of class $[h,k]$, $1 \leq h < k \leq q+1$. Of course, since B is minimal, no point of Ω is in B.

If x and y are distinct points of $\mathrm{PG}(3,q)$, the line through x and y will be denoted by xy.

2. FIRST PROPERTIES

It is useful for the sequel to remaind that :

-the number of the external lines to Ω is

$$\frac{q^2(q^2+1)}{2}; \tag{2.1}$$

-the number of the external lines to Ω through a point not in Ω is

$$\frac{q^2+q}{2}. \tag{2.2}$$

Moreover, if π is a plane of $\mathrm{PG}(3,q)$ and Γ is a conic of π,

-the number of the lines in π external to Γ is

$$\frac{q(q-1)}{2}; \tag{2.3}$$

-the number of the lines in π external to Γ through a point in $\pi \backslash \Gamma$ is at least

$$\frac{q-1}{2}. \tag{2.4}$$

Now we prove the following

Proposition 2.1. *B is of type $(1,k)$, $2 \leq k \leq q+1$.*

Proof. Since B is minimal, for any $x \in B$ there exists a line in $\mathcal{F}$ meeting B exactly in x; so $h=1$. Now, let B be of type (1). By counting in two ways the incident point-line pairs (x,L), $x \in B$ and $L \in \mathcal{F}$, we have, by (2.1) and (2.2),

$$|B|\frac{q^2+q}{2} = \frac{q^2(q^2+1)}{2}.$$

It follows that $q+1$ divides 2, a contradiction. So, the statement follows. □

Proposition 2.2. *For any plane π , $B \cap \pi$ is a blocking set of class [1,k], $2 \leq k \leq q+1$, with respect to the family of the lines in π external to Ω .*

Proof. It immediately follows from Proposition 2.1. □

3. INTERSECTION WITH THE TANGENT PLANES

By Proposition 2.1, B is of type $(1, k)$, $2 \leq k \leq q+1$.

Let π be a tangent plane to $\mathcal{Q}$. Set $\{p_0\} = \mathcal{Q} \cap \pi$ and $B' = B \cap \pi$. By Proposition 2.2, B' is a blocking set of class $[1, k]$ with respect to the family $\mathcal{F}'$ of the lines in π not on p_0.

Proposition 3.1. *If B' is a minimal $\mathcal{F}'$-blocking set, then either*
$k = q+1$ and B' is a line in π not on p_0
or
$B' = L \backslash \{p_0\}$, L a line in π through p_0.

Proof. Let $x' \in B'$. If a denotes the number of the 1-secants to B' in $\mathcal{F}'$ through x', then

$$|B'| = (q-a)(k-1) + |B' \cap x'p_0|. \tag{3.1}$$

Two cases can occur:
(i) there exists a point $y \neq p_0$ in $x'p_0 \backslash B'$;
(ii) the points distinct from p_0 on the line $x'p_0$ all are in B'.

Case (i). If b denotes the number of the 1-secants to B' in $\mathcal{F}'$ through y, then

$$|B'| = b + (q-b)k + |B' \cap x'p_0|. \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$q = (k-1)(b-a). \tag{3.3}$$

Since q is prime, (3.3) implies that either $k-1 = 1$ and $b-a = q$ or $k-1 = q$ and $b-a = 1$. Since B' is minimal, $a \geq 1$; so the case $b = q+a$ cannot occur as $b \leq q$. Hence, $k = q+1$ and $b = a+1$. Since $k = q+1$ and $y \notin B'$, then no line in $\mathcal{F}'$ through y is k-secant to B'; so, $b = q$ which implies $a = q-1$. Therefore B' contains a line $L \in \mathcal{F}'$ through x' and no point not on $L \cup p_0x'$. Since L is an $\mathcal{F}'$-blocking set of π and B' is minimal, then $B' = L$.

Case (ii). Since $p_0x' \backslash \{p_0\}$ is an $\mathcal{F}'$-blocking set of π and B' is minimal, then $B' = p_0x' \backslash \{p_0\}$. So, the statement is completely proved. □

Proposition 3.2. *If B' is a non-minimal $\mathcal{F}'$-blocking set, then either $k = 2$ and $B' = (L \backslash \{p_0\}) \cup S$, where L is a line in π through p_0 and S is a non-empty set of points distinct from p_0 on a line in π through p_0 distinct from L*
or
$k = q+1$ and $B' = B = \pi \backslash \{p_0\}$.

Proof. Since B' is not minimal, there exists a point x' in B' such that the lines of π through x' and distinct from $x'p_0$ all intersect B' in exactly k

points. Therefore,

$$|B'| = q(k-1) + |B' \cap x'p_0|. \tag{3.4}$$

Two cases can occur:
(i) there exists a point $y \neq p_0$ in $x'p_0 \backslash B'$;
(ii) the points distinct from p_0 on the line $x'p_0$ all are in B'.

Case (i). Denote by a the number of the 1-secants to B' in $\mathcal{F}'$ through y. Since the k-secants to B' in $\mathcal{F}'$ through y are $q-a$, then

$$|B'| = a + (q-a)k + |B' \cap x'p_0|. \tag{3.5}$$

From (3.4) and (3.5) it follows that

$$q = a(k-1). \tag{3.6}$$

Since q is prime, (3.6) implies that either $a = 1$ and $k-1 = q$ or $a = q$ and $k-1 = 1$. If $a = 1$ and $k = q+1$, then there are $q-1$ lines through y contained in B', a contradiction since $y \notin B'$.
Now, let $a = q$ and $k = 2$. The lines in $\mathcal{F}'$ through x' all intersect $B' \backslash \{x'\}$ precisely in one point. Let z and w be two distinct points in $B' \backslash p_0x'$ and let $\{t\} = p_0x' \cap zw$. Since B' is of class [1,2], then $t \notin B'$. On the other hand, since $a = q$, the lines in $\mathcal{F}'$ through a point on $p_0x' \backslash B'$ all are 1-secants to B'. Hence, $t = p_0$. It follows that the q points in $B' \backslash p_0x'$ all are on a line L through p_0.

Case (ii). First, observe that (3.4) implies

$$|B'| = q(k-1) + q = qk. \tag{3.7}$$

Since $k \geq 2$, there exists a point $z \in B' \backslash p_0x'$. The lines in $\mathcal{F}'$ through z all have at least two points in B' and so they are k-secants to B'; therefore, by (3.7), $|p_0z \cap B'| = q$; so $p_0z \backslash \{p_0\} \subseteq B'$. Now, consider a point w in π not on $p_0x' \cup p_0z$. The lines in $\mathcal{F}'$ through w all have at least two points in B', so they are k-secants to B'. Hence, by (3.7), $w \in B'$ if, and only if, $p_0w \backslash \{p_0\} \subseteq B'$. It follows that B' is the union of k lines ($2 \leq k \leq q+1$) through p_0 but the point p_0.
Assume $3 \leq k \leq q$. A plane π' tangent to Ω at a point $p_0' \neq p_0$ intersects π in a line external to Ω and k-secant to B. By Proposition 3.1, $\pi' \cap B$ is a blocking set of π' with respect to the family of the lines in π' external to Ω which is not minimal. So, the previous argument implies that $\pi' \cap B$ is the union of k lines through p_0' but the point p_0'. Since any external line to Ω is on a tangent plane to Ω, then any external line to Ω is a k-secant to B, a contradiction.
Hence, $k = 2$ or $k = q+1$. If $k = q+1$, then $B' = \pi \backslash \{p_0\}$. Since $\pi \backslash \{p_0\}$ is an $\mathcal{F}$-blocking set of type $(1, q+1)$ and B is minimal, then $B = \pi \backslash \{p_0\}$ and the statement is completely proved. □

4. PROOF OF THEOREM 1.1

First we prove the following

Proposition 4.1. *It is* $k = q + 1$.

Proof. Assume, on the contrary, $k < q + 1$.

We divide the proof in several steps.

Step 1. It is $k = 2$.
Since B is of type $(1, k)$, there exists in $\mathcal{F}$ a k-secant to B, say M . Since a tangent plane to Ω through M exists, then the statement follows from Propositions 3.1 and 3.2.

Step 2. For any plane π, $|\pi \cap B| \leq 2q$.
If π is a tangent plane to Ω, the statement follows from Propositions 3.1 and 3.2. Now, let π be a secant plane. Set $\Gamma = \Omega \cap \pi$ and $B' = B \cap \pi$. Count in two ways the incident point-line pairs (x, L), $x \in B'$ and L a line in π external to Γ. By Step 1, (2.3) and (2.4)

$$|B'|\frac{q-1}{2} \leq 2\frac{q(q-1)}{2}, \tag{4.1}$$

follows, from which $|B'| \leq 2q$.

Step 3. If π is a secant plane to Ω, then $B \cap \pi$ does not contain the union of three tangent lines to $\pi \cap \Omega$ but the points of contact.
It immediately follows from Step 2, as $q \geq 5$.

Step 4. There exists a plane π_0 tangent to Ω such that $|\pi_0 \cap B| \leq 2q - 1$.
Since B is of type $(1, k)$, a 1-secant to B exists in $\mathcal{F}$, say L. Let π_0 be a tangent plane through L. By Propositions 3.1 and 3.2, the statement follows.

Step 5. $|B| \leq q^2 + 2q - 1$.
Let π_0 be a tangent plane as in Step 4. By Propositions 3.1 and 3.2, there exists in π_0 a tangent line L such that $L \backslash \Omega \subseteq B$. By Step 2, for any plane π through L distinct from π_0, $|\pi \cap B| \leq 2q$. By counting the points of B on all planes containing L, the statement follows.

Step 6. There exists a plane π secant to Ω in a conic Γ such that $\pi \cap B = (L_1 \cup L_2) \backslash \{p_1, p_2\}$, where L_1 and L_2 are two distinct lines tangent to Γ at the points p_1 and p_2 respectively.
Since B is of type $(1, k)$, a line L 1-secant to B exists in $\mathcal{F}$. Set $\{p\} = L \cap B$. Denote by π_1 and π_2 the tangent planes to Ω through L. If $\{p_i\} = \pi_i \cap \Omega$, then, by Propositions 3.1 and 3.2, there exists in π_i a line L_i through p tangent to Ω such that $L_i \backslash \{p_i\} \subseteq B$, $i = 1, 2$. Now, let π be the plane

through L_1 and L_2. Obviously, π is a secant plane to Ω. Set $\pi \cap \Omega = \Gamma$. Assume a point z exists in $\pi \cap B$ not on $L_1 \cup L_2$. Since $q \geq 5$, then, by (2.4), there exists in π an external line to Γ through z meeting $L_1 \cup L_2$ in two distinct points, a contradiction as, by Step 1, $k = 2$.

Now, consider a secant plane π satisfying the condition of Step 6. By Propositions 3.1 and 3.2, through any point x in $\Omega \setminus \pi$ there exists a line L tangent to Ω such that $L \setminus \{x\} \subseteq B$. Say $\mathcal{L}$ the set of all such lines. Obviously, any line in $\mathcal{L}$ meets π in a point of $(L_1 \cup L_2) \setminus \{p_1, p_2\}$. So, by Step 3, any point in $B \setminus \pi$ is on at most two lines of $\mathcal{L}$. Therefore, by counting in two ways the incident point-line pairs (x, L), $x \in B \setminus \pi$ and $L \in \mathcal{L}$,

$$(q-1)|\mathcal{L}| \leq 2|B \setminus \pi| \tag{4.2}$$

follows.

Since $|\mathcal{L}| \geq |\Omega \setminus \pi| = q^2 - q$ and, by Step 5, $|B \setminus \pi| \leq q^2$, (4.2) implies $q \leq 3$, a contradiction. So, $k = q + 1$ follows. □

Now, we can prove Theorem 1.1.

By Proposition 4.1, B is of type $(1, q+1)$; hence, there exists in $\mathcal{F}$ a line L contained in B. Since a line exists in $\mathcal{F}$ skew with L, then $B \setminus L \neq \emptyset$. Let $x \in B \setminus L$ and let π be the plane through L and x. If π is tangent to Ω, then, by Propositions 3.1 and 3.2, $B = \pi \setminus \Omega$; so, the statement follows.

Now, consider the case when π is secant to Ω in a conic Γ. Since $q \geq 5$, by (2.4) a line L' exists in π through x external to Γ. As $|B \cap L'| \geq 2$, then $L' \subseteq B$. Again, (2.4) and $q \geq 5$ imply that through any point p in π not on $\Gamma \cup L \cup L'$ a line exists in π external to Γ and intersecting both L and L' at distinct points; such a line is contained in B, so $p \in B$. Hence, $\pi \setminus \Gamma \subseteq B$. Since B is minimal, then $\pi \setminus \Gamma = B$ and the statement is completely proved.

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Some equitably 2-colourable cycle decompositions

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Abstract

Let G be a graph in which each vertex has been coloured using one of k colours, say $c_1, c_2, \ldots, c_k$. If an m-cycle C in G has n_i vertices coloured c_i, $i = 1, 2, \ldots, k$, and $|n_i - n_j| \leq 1$ for any $i, j \in \{1, 2, \ldots, k\}$, then C is equitably k-coloured. An m-cycle decomposition $\mathcal{C}$ of a graph G is equitably k-colourable if the vertices of G can be coloured so that every m-cycle in $\mathcal{C}$ is equitably k-coloured. For $m = 4, 5$ and 6, we completely settle the existence problem for equitably 2-colourable m-cycle decompositions of complete graphs and complete graphs with the edges of a 1-factor removed.

1 Introduction

Let G and H be graphs. A G-decomposition of H is a set $\mathcal{G} = \{G_1, G_2, \ldots, G_p\}$ such that G_i is isomorphic to G for $1 \leq i \leq p$ and $\mathcal{G}$ partitions the edge set of H. Most commonly, $H = K_v$, the complete graph on v vertices. Another popular choice for H is $K_v - F$, the complete graph with the edges of a 1-factor removed. The problem of determining all values of v for which there exists a G-decomposition of K_v is called the *spectrum problem* for G.

An m-cycle, denoted by $(x_1, x_2, \ldots, x_m)$, is the graph with vertex set $\{x_1, x_2, \ldots, x_m\}$ and edge set $\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_m, x_1\}\}$. The spectrum problem for m-cycles has recently been solved; see [3] and [9].

A *colouring* of an m-cycle decomposition $\mathcal{C}$ of a graph G is an assignment of colours to the vertices of G. A *k-colouring* of $\mathcal{C}$ is a colouring in which

k distinct colours are used. A k-colouring of an m-cycle decomposition $\mathcal{C}$ induces a colouring of each m-cycle in $\mathcal{C}$. If n_i is the number of vertices coloured c_i in an m-cycle $C \in \mathcal{C}$, then C is *equitably k-coloured* if $|n_i - n_j| \leq 1$ for any $i, j \in \{1, 2, \dots, k\}$ and an m-cycle decomposition $\mathcal{C}$ is equitably k-coloured if every $C \in \mathcal{C}$ is equitably k-coloured. An m-cycle decomposition is *equitably k-colourable* if it can be equitably k-coloured.

A 3-cycle decomposition of K_v is just a Steiner triple system of order v. There is extensive literature on colourings of Steiner triple systems and a fine survey can be found in [5]. Much of the literature on vertex colourings of Steiner triple systems has focused on colourings in which each triple is assigned at least two distinct colours.

In [4], Colbourn, Dinitz and Rosa consider colouring the vertices of a Steiner triple system using 3, 4 or 5 colours such that, in each triple of the system, exactly two colours are represented. In [8], Milici, Rosa and Voloshin consider colouring Steiner triple systems and also Steiner systems of the form $S(2,4,v)$, with the restriction that the blocks of each design must have specified colour patterns.

The existence question for equitably 3-colourable m-cycle decompositions of K_v and $K_v - F$, where $m \in \{4,5,6\}$, has been completely settled in [2]. Quattrocchi has also considered certain colourings of 4-cycle decompositions of K_v in [10]. Quattrocchi's paper differs from both this paper and [2] as it is a condition that each system must have at least one 4-cycle with 3 vertices of one colour.

We consider a natural extension of these colouring problems by investigating the existence of equitably 2-colourable m-cycle decompositions. (In this paper we use colours black and white and, unless otherwise stated, b and w are used to denote the number of black and white vertices in K_v or $K_v - F$ respectively. Futhermore, an edge which connects two black (white) vertices is said to be a one-coloured black (white) edge, and an edge which connects two differently coloured vertices is said to be a two-coloured edge.) A simple counting argument shows that the only equitably 2-colourable Steiner triple system is the trivial system of order 3. Moreover, it is easy to see that for $k \geq 3$, there exists an equitably k-colourable Steiner triple system of order v if and only if $v = k$ and $v \equiv 1$ or $3 \pmod 6$.

As we shall see, equitable k-colourings of m-cycle decompositions of K_v and $K_v - F$ are no longer trivial when $m \geq 4$. This paper is primarily concerned with equitably 2-colourable m-cycle decompositions of K_v and $K_v - F$ for $m = 4$, 5 and 6. We make frequent use of the following important result.

Lemma 1.1 ([3], [9]) *An m-cycle decomposition of K_v ($K_v - F$) exists for all admissible v, that is, for all odd (even) v such that $3 \leq m \leq v$ and m divides the number of edges in K_v ($K_v - F$).*

Our main result is the following theorem which completely settles the existence question for equitably 2-coloured m-cycle decompositions of K_v and $K_v - F$ for $m = 4$, 5 and 6.

Main Theorem

- No equitably 2-colourable m-cycle decomposition of K_v exists for m even.
- For all admissible v, there exists at least one 5-cycle decomposition of K_v which is equitably 2-colourable.
- For all admissible v, $v > 5$, there exists at least one 5-cycle decomposition of K_v which is *not* equitably 2-colourable.
- There exists an equitably 2-colourable m-cycle decomposition of $K_v - F$, with $m \in \{4, 5, 6\}$, for all admissible values of v.

Finally, we introduce some notation to be used throughout the paper. We use $K_{p(n)}$ to denote the multipartite graph with p parts with n vertices in each part. Let G and H be graphs. The join of G and H, denoted $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{u, v\} \mid u \in V(G) \text{ and } v \in V(H)\}$.

2 Equitably 2-colourable even-length cycle decompositions of K_v

We start with a trivial result.

Lemma 2.1 *No equitably 2-colourable m-cycle decomposition of K_v exists for m even.*

Proof. Every equitably 2-coloured cycle of even length contains an equal number of vertices of each colour. Hence, v must be even for an equitably 2-coloured even-length cycle decomposition of K_v to exist. However, it is not possible to decompose a complete graph of even order into cycles as each vertex is of odd degree and cycles have even degree. □

3 Equitably 2-colourable 4-cycle decompositions

Every equitably 2-coloured 4-cycle must contain two black and two white vertices, which may only be arranged in two formations; see Figure 1. From the previous section we know that no equitably 2-colourable 4-cycle decomposition of K_v exists. This prompts us to consider instead the decomposition of $K_v - F$ since this graph has the required even number of vertices and even degree.

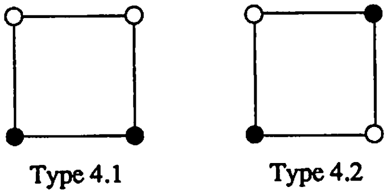


Figure 1: Possible equitable 2-colourings of 4-cycles.

We prove the following:

Theorem 3.1 *There exist equitably 2-colourable 4-cycle decompositions of $K_v - F$ if and only if $v \equiv 0 \pmod 2$, $v \geq 4$.*

Proof. From Lemma 1.1, a 4-cycle decomposition of $K_v - F$ exists if and only if $v \equiv 0 \pmod 2$, $v \geq 4$. As v is even, we colour $v/2$ vertices black and $v/2$ vertices white. The proof that these conditions are also sufficient requires the consideration of two cases. Note that there are $\frac{1}{8}v(v-2)$ 4-cycles in any 4-cycle decomposition of $K_v - F$.

Case 1: $v \equiv 0 \pmod 4$.

Let the vertex set of $K_v - F$ be $\bigcup_{i=1,2} \left\{0_i, 1_i, \ldots, \frac{1}{2}(v-2)_i\right\}$. Colour the vertices with subscript 1 black and colour the the vertices with subscript 2 white. Let the edges in F be $\left\{j_k, \left(j + \frac{v}{4}\right)_k\right\}$, where $j = 0, 1, \ldots, (v-4)/4$ and $k = 1, 2$. We obtain $v/4$ cycles of Type 4.2 (see Figure 1) by forming the 4-cycles

$$\left(j_1, j_2, \left(j + \frac{v}{4}\right)_1, \left(j + \frac{v}{4}\right)_2\right)$$

for $j = 0, 1, \ldots, (v-4)/4$. We generate the remaining $v(v-4)/8$ cycles of Type 4.1 (see Figure 1) by working modulo $v/2$ and forming the 4-cycles

$$(j_1, (j+l)_1, j_2, (j+l)_2)$$

for $j = 0, 1, \ldots, (v-2)/2$ and $l = 1, 2, \ldots, (v-4)/4$.

Case 2: $v \equiv 2 \pmod 4$.

Label and colour the vertices as for Case 1. Let the edges in F be $\{j_1, j_2\}$, where $j = 0, 1, \ldots, (v-2)/2$. Working modulo $v/2$, we obtain $v(v-2)/8$ cycles of Type 4.1 by forming the 4-cycles

$$(j_1, (j+l)_1, j_2, (j+l)_2)$$

for $j = 0, 1, \ldots, (v-2)/2$ and $l = 1, 2, \ldots, (v-2)/4$. □

4 Equitably 2-colourable 5-cycle decompositions

When considering cycles of odd length, if b vertices of a cycle are coloured black and w white, we cannot have $b = w$. Instead, each cycle within the decomposition must satisfy $|b - w| = 1$. The only ways of equitably 2-colouring a 5-cycle are shown in Figure 2. In this case, as we do not require equal numbers of black and white vertices, it is feasible to consider equitably 2-colourable decompositions of K_v as well as decompositions of $K_v - F$.

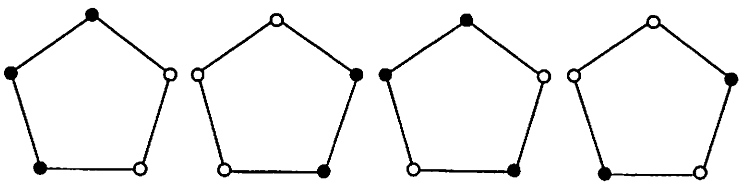


Figure 2: Possible equitable 2-colourings of 5-cycles.

4.1 Equitably 2-colourable 5-cycle decompositions of K_v

We begin with some definitions.

Definition A *pairwise balanced design* of order v, with block sizes in the set K and index λ, denoted PBD(v, K, λ), is a pair $(V, \mathcal{B})$. Here V is a set of elements such that $|V| = v$ and $\mathcal{B}$ is a collection of subsets of V, called blocks, such that for each $B \in \mathcal{B}$, $|B| \in K$ and any two elements in V occur together in precisely λ blocks.

Definition A *group divisible design* of order v, with block sizes in the set K, group sizes in the set M and index λ, denoted GDD$[K, \lambda, M; v]$, is a triple

$(V, \Gamma, \mathcal{B})$. Here V is a set of elements such that $|V| = v$, $\Gamma = \{G_1, G_2, \ldots\}$ is a partition of X with each class of the partition called a group, such that for each $G_i \in \Gamma$, $|G_i| \in M$ and $\mathcal{B}$ is a collection of subsets of V, called blocks, such that for each $B \in \mathcal{B}$, $|B| \in K$. The triple also satisfies the following properties:

1. for all $x, y \in V$ belonging to distinct groups, x and y occur together in precisely λ blocks; and
2. for all $x, y \in V$ belonging to the same group, x and y do not occur together in any blocks.

The following result is well-known, and is useful in proving Theorem 4.8 below.

Lemma 4.1 [6] *For all positive integers x there exists a PBD$(2x+1, 3, 1)$ or a PBD$(2x+1, \{3, 5^*\}, 1)$.*

Corollary 4.2 *For all positive integers x there exists a GDD$[3, 1, 2; 2x]$ or a GDD$[3, 1, \{2, 4^*\}; 2x]$.*

Proof: Take a PBD$(2x+1, 3, 1)$ or a PBD$(2x+1, \{3, 5^*\}, 1)$ (see Lemma 4.1), and remove an element e (if there is a block of size 5, remove an element occuring in this block). The result is a GDD$[3, 1, 2; 2x]$ or a GDD$[3, 1, \{2, 4^*\}; 2x]$ respectively, where the old blocks not containing e are blocks in the design and the old blocks containing e are now groups within the design. □

We also make use of the following existence results. Most of these and other designs in this paper were obtained using computational techniques.

Lemma 4.3 *There exists an equitably 2-coloured 5-cycle decomposition of K_5.*

Proof. Let the vertex set of K_5 be $\mathbb{Z}_5$. Colour the vertices 0, 1 and 2 black and colour the vertices 3 and 4 white. A suitable decomposition of K_5 is given by: $(0, 1, 2, 3, 4)$, $(0, 2, 4, 1, 3)$. □

Lemma 4.4 *There exists an equitably 2-coloured 5-cycle decomposition of K_{11}.*

Proof. Let the vertex set of K_{11} be $\mathbb{Z}_{11}$. Colour the vertices 1, 2, 4, 7, 9 and 10 black and colour the remaining vertices white. Working modulo 11, the starter cycle $(0, 1, 10, 2, 6)$ generates an equitably 2-coloured 5-cycle decomposition of K_{11}. □

Lemma 4.5 *There exists an equitably 2-coloured 5-cycle decomposition of* K_{21}.

Proof. Let the vertex set of K_{21} be $\bigcup_{i=1,\dots,7} \{0_i, 1_i, 2_i\}$. Colour the vertices with subscripts 4, 5, 6 and 7 black and colour the remaining vertices white. Fourteen equitably 2-coloured starter cycles are given by:

$(0_1, 1_1, 0_2, 0_4, 1_4)$, $(0_1, 0_2, 2_1, 1_4, 0_5)$, $(0_1, 0_3, 0_4, 0_5, 1_5)$, $(0_1, 1_3, 0_4, 1_5, 0_6)$,
$(0_1, 2_3, 0_4, 0_6, 2_5)$, $(0_1, 0_4, 1_2, 2_4, 1_6)$, $(0_1, 2_6, 0_2, 0_5, 0_7)$, $(0_1, 1_7, 0_2, 1_5, 2_7)$,
$(0_2, 1_2, 0_5, 0_6, 1_6)$, $(0_2, 0_3, 0_5, 2_7, 0_6)$, $(0_2, 1_3, 0_6, 2_4, 0_7)$, $(0_2, 2_3, 0_6, 0_7, 2_7)$,
$(0_3, 1_3, 0_7, 1_4, 1_7)$, $(0_3, 1_5, 2_3, 2_6, 0_7)$.

Each starter cycle is developed modulo 3 such that the subscripts remains fixed, thus ensuring that every cycle is equitably 2-coloured. For example, the first starter cycle develops to produce:

$$(0_1, 1_1, 0_2, 0_4, 1_4),\ (1_1, 2_1, 1_2, 1_4, 2_4),\ (2_1, 0_1, 2_2, 2_4, 0_4).$$

□

Lemma 4.6 *There exists an equitably 2-coloured 5-cycle decomposition of* $K_{3(5)}$.

Proof. Let the vertex set of $K_{3(5)}$ be $\bigcup_{i=1,2,3} \{0_i, 1_i, \dots, 4_i\}$, with the obvious vertex partition. Colour the vertices 0_i, 2_i and 4_i black for $i = 1, 2, 3$, and colour the remaining vertices white. A suitable decomposition of $K_{3(5)}$ is given by:

$(1_1, 1_2, 0_1, 0_2, 0_3)$, $(1_1, 3_2, 0_1, 2_2, 2_3)$, $(1_1, 1_3, 0_1, 4_2, 4_3)$, $(1_1, 3_3, 0_1, 0_3, 2_2)$,
$(3_1, 1_2, 2_1, 0_2, 2_3)$, $(3_1, 3_2, 2_3, 0_1, 4_3)$, $(3_1, 1_3, 2_1, 4_2, 0_3)$, $(3_1, 3_3, 2_1, 2_3, 4_2)$,
$(1_2, 1_3, 0_2, 4_3, 4_1)$, $(1_2, 3_3, 2_2, 2_1, 0_3)$, $(3_2, 1_3, 2_2, 4_3, 2_1)$, $(3_2, 3_3, 4_2, 4_1, 0_3)$,
$(1_1, 4_2, 1_3, 4_1, 0_2)$, $(3_1, 0_2, 3_3, 4_1, 2_2)$, $(1_2, 4_3, 3_2, 4_1, 2_3)$.

□

Lemma 4.7 *There exists an equitably 2-coloured 5-cycle decomposition of* $K_{5(5)}$.

Proof. Let the vertex set of $K_{5(5)}$ be $\bigcup_{i=1,\dots,5} \{0_i, 1_i, \dots, 4_i\}$, with the obvious vertex partition. Colour the vertices 0_i, 2_i and 4_i black for $i = 1, \dots, 5$, and colour the remaining vertices white. A suitable decomposition is given by developing the following ten starter cycles:

$(1_1, 1_2, 2_3, 0_4, 4_5)$, $(1_1, 3_2, 4_3, 4_4, 2_5)$, $(3_4, 3_1, 2_3, 4_5, 0_2)$, $(0_1, 0_2, 1_3, 4_4, 3_5)$,
$(1_4, 1_1, 2_3, 0_5, 4_2)$, $(1_4, 3_1, 4_3, 4_5, 2_2)$, $(2_1, 2_2, 3_3, 1_4, 0_5)$, $(0_4, 0_1, 1_3, 4_5, 3_2)$,
$(3_1, 3_2, 2_3, 4_4, 0_5)$, $(2_4, 2_1, 3_3, 1_5, 0_2)$.

Working modulo 5, each starter cycle is developed by incrementing the subscript by one. So, for example, the first starter cycle develops to produce the following 5-cycles:

$$(1_1,1_2,2_3,0_4,4_5),\ (1_2,1_3,2_4,0_5,4_1),\ (1_3,1_4,2_5,0_1,4_2),\ (1_4,1_5,2_1,0_2,4_3),$$
$$(1_5,1_1,2_2,0_3,4_4).$$

□

Theorem 4.8 *There exist equitably 2-colourable 5-cycle decompositions of K_v if and only if $v \equiv 1,5 \pmod{10}$, $v \geq 5$.*

Proof. By Lemma 1.1, a 5-cycle decomposition of K_v exists if and only if $v \equiv 1,5 \pmod{10}$, $v \geq 5$. We need to consider these two cases separately.

Case 1: $v \equiv 1 \pmod{10}$.

Let $v = 10x + 1$, $x \geq 1$. By Corollary 4.2, we can take either a GDD$[3,1,2;2x]$ or a GDD$[3,1,\{2,4^*\};2x]$ and simultaneously construct the complete graph K_v and its equitably 2-coloured 5-cycle decomposition as follows. Replace each element of the design with five vertices, colouring three vertices black and two vertices white. Adjoin a new vertex ∞, coloured white, which is adjacent to all other vertices.

By Lemma 4.6, we can place an equitably 2-coloured 5-cycle decomposition of $K_{3(5)}$ on each set of vertices arising from a block of the design. Furthermore, by Lemmas 4.4 and 4.5, we can place an equitably 2-coloured 5-cycle decomposition of K_{11} or K_{21} on $\{\infty\} \cup g$ for each set of vertices g arising from a group of the design of size 2 or 4 respectively. It is not difficult to check that the result is an equitably 2-coloured 5-cycle decomposition of K_v.

Case 2: $v \equiv 5 \pmod{10}$.

Let $v = 10x + 5$, $x \geq 0$. By Lemma 4.1, we can take either a PBD$(2x+1,3,1)$ or a PBD$(2x+1,\{3,5^*\},1)$ and, proceeding as for Case 1, replace each element of the design with five vertices, colouring three vertices black and two vertices white.

By Lemma 4.3, we can place an equitably 2-coloured 5-cycle decomposition of K_5 on each set of five new vertices. By Lemmas 4.6 and 4.7, we can place an equitably 2-coloured 5-cycle decomposition of $K_{3(5)}$ or $K_{5(5)}$ on each set of vertices arising from a block of the design of size 3 or 5 respectively. It is not difficult to check that the result is an equitably 2-coloured 5-cycle decomposition of K_v. □

4.2 Equitably 2-colourable 5-cycle decompositions of $K_v - F$

When proving Theorem 4.13, we again make use of Corollary 4.2 as well as the existence results in Lemma 4.6 and Lemmas 4.9 to 4.12.

Lemma 4.9 *There exists an equitably 2-coloured 5-cycle decomposition of $K_{10} - F$.*

Proof. Let the vertex set of $K_{10}-F$ be $\mathbb{Z}_{10}$. Colour the vertices $0, 1, \ldots, 5$ black and colour the vertices 6, 7, 8 and 9 white. Let the edges in F be $\{0,1\}$, $\{2,3\}$, $\{4,5\}$, $\{6,7\}$ and $\{8,9\}$. A suitable decomposition of $K_{10}-F$ is given by:

$$(4,0,3,6,8),\quad (0,2,4,6,9),\quad (1,3,4,7,8),\quad (4,1,2,7,9),$$
$$(5,2,6,0,7),\quad (5,3,7,1,6),\quad (5,0,8,2,9),\quad (5,1,9,3,8).$$

□

Lemma 4.10 *There exists an equitably 2-coloured 5-cycle decomposition of $K_{12} - F$.*

Proof. Let the vertex set of $K_{12}-F$ be $\mathbb{Z}_{12}$. Colour the vertices $0, 1, \ldots, 5$ black and colour the remaining vertices white. Let the edges in F be $\{0,1\}$, $\{2,3\}$, $\{4,5\}$, $\{6,7\}$, $\{8,9\}$ and $\{10,11\}$. A suitable decomposition of $K_{12}-F$ is given by:

$$(0,2,1,8,6),\quad (3,0,4,9,10),\quad (1,5,0,7,10),\quad (11,8,10,2,4),\quad (9,7,11,2,5),$$
$$(9,6,11,5,3),\quad (1,3,8,0,9),\quad (3,4,10,0,11),\quad (4,1,6,2,8),\quad (11,9,2,7,1),$$
$$(8,7,3,6,5),\quad (10,6,4,7,5).$$

□

Lemma 4.11 *There exists an equitably 2-coloured 5-cycle decomposition of $K_{20} - F$.*

Proof. Let the vertex set of $K_{20}-F$ be $\mathbb{Z}_{20}$. Colour the vertices $0, 1, \ldots,$ 11 black and colour the vertices $12, 13, \ldots, 19$ white. Let the edges in F be $\{0,1\}$, $\{2,3\}$, $\{4,5\}$, $\{6,7\}$, $\{8,9\}$, $\{10,11\}$, $\{12,13\}$, $\{14,15\}$, $\{16,17\}$ and $\{18,19\}$. A suitable decomposition of $K_{20} - F$ is given by:

$$(12,14,3,1,2),\quad (13,15,0,4,6),\quad (16,18,5,7,8),\quad (17,19,10,9,11),$$
$$(12,15,1,4,3),\quad (13,14,0,2,5),\quad (16,19,6,8,10),\quad (17,18,11,7,9),$$
$$(12,16,0,3,5),\quad (13,17,1,6,9),\quad (14,18,2,7,10),\quad (15,19,4,8,11),$$
$$(12,17,0,5,1),\quad (13,16,2,6,10),\quad (14,19,7,4,11),\quad (15,18,8,3,9),$$
$$(12,18,0,6,11),\quad (13,19,1,7,3),\quad (14,16,4,2,9),\quad (15,17,8,5,10),$$
$$(12,19,2,10,4),\quad (13,18,3,11,1),\quad (14,17,7,0,8),\quad (15,16,9,5,6),$$
$$(12,6,14,1,10),\quad (13,7,18,9,0),\quad (15,4,17,2,8),\quad (17,6,18,10,3),$$
$$(12,7,14,4,9),\quad (13,2,15,5,11),\quad (16,3,19,9,1),\quad (17,5,19,0,10),$$
$$(12,8,19,11,0),\quad (13,4,18,1,8),\quad (14,5,16,11,2),\quad (15,7,16,6,3).$$

□

Lemma 4.12 *There exists an equitably 2-coloured 5-cycle decomposition of $K_{22} - F$.*

Proof. Let the vertex set of $K_{22} - F$ be $\mathbb{Z}_{22}$. Colour the vertices $0, 1, \ldots,$ 11 black and colour the vertices $12, 13, \ldots, 21$ white. Let the edges in F be $\{0,1\}$, $\{2,3\}$, $\{4,5\}$, $\{6,7\}$, $\{8,9\}$, $\{10,11\}$, $\{12,13\}$, $\{14,15\}$, $\{16,17\}$, $\{18,19\}$ and $\{20,21\}$ A suitable decomposition of $K_{22} - F$ is given by:

$(12,14,13,0,2)$,	$(12,15,13,1,3)$,	$(12,16,13,2,1)$,	$(12,17,13,3,0)$,
$(12,18,13,4,6)$,	$(12,19,13,5,7)$,	$(12,21,13,7,4)$,	$(14,16,15,0,4)$,
$(14,17,15,1,5)$,	$(14,18,15,2,6)$,	$(14,19,15,3,7)$,	$(14,20,0,5,2)$,
$(14,21,0,6,1)$,	$(15,20,1,4,8)$,	$(15,21,1,7,9)$,	$(16,18,0,7,2)$,
$(16,19,0,8,1)$,	$(16,20,2,4,3)$,	$(16,21,2,8,5)$,	$(17,18,1,9,0)$,
$(17,19,1,10,2)$,	$(17,20,3,5,9)$,	$(17,21,3,6,8)$,	$(18,20,4,9,2)$,
$(18,21,4,10,3)$,	$(19,20,5,10,6)$,	$(19,21,5,11,2)$,	$(12,8,13,9,10)$,
$(12,9,14,0,11)$,	$(13,10,14,3,11)$,	$(14,8,16,4,11)$,	$(15,4,17,1,11)$,
$(15,5,17,7,10)$,	$(15,6,16,11,7)$,	$(16,7,18,8,10)$,	$(16,9,18,10,0)$,
$(17,6,20,8,3)$,	$(17,10,19,8,11)$,	$(18,4,19,9,11)$,	$(18,5,19,11,6)$,
$(19,7,21,9,3)$,	$(20,10,21,6,9)$,	$(20,11,21,8,7)$.	

□

Theorem 4.13 *There exist equitably 2-colourable 5-cycle decompositions of $K_v - F$ if and only if $v \equiv 0, 2 \pmod{10}$, $v \geq 10$.*

Proof. By Lemma 1.1, a 5-cycle decomposition of $K_v - F$ exists if and only if $v \equiv 0, 2 \pmod{10}$, $v \geq 10$. We need to consider these two cases separately.

Case 1: $v \equiv 0 \pmod{10}$.

Let $v = 10x$, $x \geq 1$. By Corollary 4.2, we can take either a GDD$[3, 1, 2; 2x]$ or a GDD$[3, 1, \{2, 4^*\}; 2x]$ and simultaneously construct $K_v - F$ and its equitably 2-coloured 5-cycle decomposition as follows. Replace each element of the design with five vertices, colouring three vertices black and two vertices white. Within each set of vertices arising from a group of the design of size 2, let there be three one-coloured black edges and two one-coloured white edges in F. Similarly, within any set of vertices arising from a group of size 4, let there be six one-coloured black edges and four one-coloured white edges in F.

By Lemma 4.6, we can place an equitably 2-coloured 5-cycle decomposition of $K_{3(5)}$ on each set of vertices arising from a block of the design. Furthermore, by Lemmas 4.9 and 4.11, we can place an equitably 2-coloured 5-cycle decomposition of $K_{10} - F$ or $K_{20} - F$ on g for each set of vertices g arising from a group of the design of size 2 or 4 respectively. It is not difficult to check that the result is an equitably 2-coloured 5-cycle decomposition of $K_v - F$.

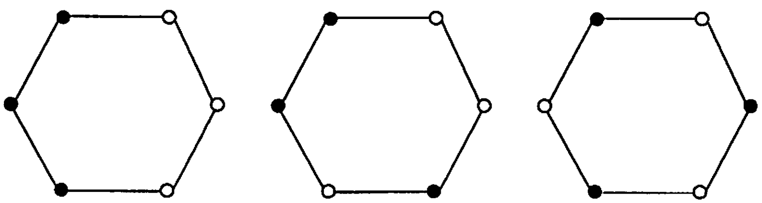
Case 2: $v \equiv 2 \pmod{10}$.

Let $v = 10x + 2$, $x \geq 1$. Proceeding as for Case 1, we replace each element of either a GDD$[3, 1, 2; 2x]$ or GDD$[3, 1, \{2, 4^*\}; 2x]$ with five vertices, colouring three vertices black and two vertices white. Let the edges in F be as described in Case 1. Adjoin two new independent vertices ∞_1 and ∞_2, both coloured white, such that ∞_1 and ∞_2 are adjacent to all other vertices except each other; that is, $\{\infty_1, \infty_2\}$ is an edge of F.

We can again place an equitably 2-coloured 5-cycle decomposition of $K_{3(5)}$ on each set of vertices arising from a block of the design. Furthermore, by Lemmas 4.10 and 4.12, we can place an equitably 2-coloured 5-cycle decomposition of $K_{12} - F$ or $K_{22} - F$ on $\{\infty_1, \infty_2\} \cup g$ for each set of vertices g arising from a group of the design of size 2 or 4 respectively. It is not difficult to check that the result is an equitably 2-coloured 5-cycle decomposition of $K_v - F$. □

5 Equitably 2-colourable 6-cycle decompositions

For 6-cycles, we proceed in much the same manner as for 4-cycles. In this case, the only possible equitable 2-colourings of a 6-cycle are shown in Figure 3.



Type 6.1 Type 6.2 Type 6.3

Figure 3: Possible equitable 2-colourings of 6-cycles.

We use the following existence results when proving Theorem 5.4.

Lemma 5.1 *There exists an equitably 2-coloured 6-cycle decomposition of $K_6 - F$.*

Proof. Let the vertex set of $K_6 - F$ be $\mathbb{Z}_6$. Colour the vertices 0, 1 and 2 black and colour the vertices 3, 4 and 5 white. Let the edges in F be

$\{0,4\}$, $\{1,5\}$ and $\{2,3\}$. A suitable decomposition of $K_6 - F$ is given by: $(0,1,2,5,4,3)$, $(0,2,4,1,3,5)$. □

Lemma 5.2 *There exists an equitably 2-coloured 6-cycle decomposition of $K_8 - F$.*

Proof. Let the vertex set of $K_8 - F$ be $\mathbb{Z}_8$. Colour the vertices 0, 1, 2 and 3 black and colour the vertices 4, 5, 6 and 7 white. Let the edges in F be $\{i, i+4\}$, for $i = 0,1,2,3$. A suitable decomposition of $K_8 - F$ is given by: $(0,1,3,4,5,7)$, $(1,2,3,5,6,7)$, $(2,0,6,1,4,7)$, $(3,0,5,2,4,6)$. □

Lemma 5.3 *There exists an equitably 2-coloured 6-cycle decomposition of $K_{6,6}$.*

Proof. Let the vertex set of $K_{6,6}$ be $\bigcup_{i=1,2} \{0_i, 1_i, \ldots, 5_i\}$, with the obvious vertex partition. Colour the vertices 0_i, 2_i and 4_i black, for $i = 1,2$, and colour the remaining vertices white. A suitable decomposition of $K_{6,6}$ is given by:

$$\begin{array}{lll} (0_1,0_2,4_1,5_2,1_1,3_2), & (2_1,2_2,0_1,1_2,3_1,5_2), & (4_1,4_2,2_1,3_2,5_1,1_2), \\ (4_1,2_2,1_1,0_2,3_1,3_2), & (0_1,4_2,3_1,2_2,5_1,5_2), & (2_1,0_2,5_1,4_2,1_1,1_2). \end{array}$$

□

We prove the following:

Theorem 5.4 *There exist equitably 2-colourable 6-cycle decompositions of $K_v - F$ if and only if $v \equiv 0,2 \pmod 6$, $v \geq 6$.*

Proof. By Lemma 1.1, a 6-cycle decomposition of $K_v - F$ exists if and only if $v \equiv 0,2 \pmod 6$, $v \geq 6$. We need to consider these two cases separately.

Case 1: $v \equiv 0 \pmod 6$.

Let $v = 6x$, $x \geq 1$. Let the vertex set of $K_v - F$ be $\bigcup_{i=1,\ldots,x} V_i$, where $V_i = \{0_i, 1_i, \ldots, 5_i\}$. Colour the vertices 0_i, 1_i and 2_i black for $i = 1,2,\ldots,x$, and colour the remaining vertices white. Let the edges in F be $\{0_i, 3_i\}$, $\{1_i, 4_i\}$ and $\{2_i, 5_i\}$, for $i = 1,2,\ldots,x$.

By Lemma 5.3 we can place an equitably 2-coloured 6-cycle decomposition of $K_{6,6}$ on $V_i \vee V_j$, for $1 \leq i < j \leq x$. By Lemma 5.1, we can place an equitably 2-coloured 6-cycle decomposition of $K_6 - F$ on V_i for $1 \leq i \leq x$.

Case 2: $v \equiv 2 \pmod 6$.

Let $v = 6x+2$, $x \geq 1$. Label and colour the vertices of a copy of $K_{6x} - F$ as for Case 1. Also, let the edges in F be as described for Case 1. Create the graph $K_{6x+2} - F$ by adjoining two new independent vertices, ∞_1, coloured black, and ∞_2, coloured white, such that ∞_1 and ∞_2 are adjacent to all other vertices except each other; that is, $\{\infty_1, \infty_2\}$ is an edge in F.

By Lemma 5.3 we can place an equitably 2-coloured 6-cycle decomposition of $K_{6,6}$ on $V_i \vee V_j$, for $1 \leq i < j \leq x$. By Lemma 5.2, we can place an equitably 2-coloured 6-cycle decomposition of $K_8 - F$ on $V_i \cup \{\infty_1, \infty_2\}$ for $1 \leq i \leq x$. □

6 Equitably 2-colourable 2-perfect cycle decompositions

In the previous sections, we have shown how to construct equitably 2-coloured m-cycle decompositions for m = 4, 5 and 6. In this section, we show that not *every* cycle decomposition which satisfies the obvious necessary conditions can be equitably 2-coloured. That is, we show that there exist cycle decompositions which are not equitably 2-colourable.

If C is a cycle of length m, the *distance i graph of C*, denoted $C(i)$, is formed by connecting the vertices in C that are at distance i. If $\mathcal{C}$ = $\{C_1, C_2, \ldots, C_n\}$ forms a cycle decomposition of G, and the collection of graphs $\{C_1(i), C_2(i), \ldots, C_n(i)\}$ also forms a decomposition of G, then we say that $\mathcal{C}$ is an *i-perfect cycle decomposition of G*. The spectrum problem for i-perfect m-cycle systems has been considered for a number of cycle lengths; see [1]. We proceed by briefly considering the existence of 2-perfect, equitably 2-coloured m-cycle systems for m = 5 and 7.

Theorem 6.1 [7] *There exists a 2-perfect 5-cycle decomposition of K_v if and only if*
$v \equiv 1, 5 \pmod{10}$, $v \geq 5$ *and* $v \neq 15$.

Theorem 6.2 *No 2-perfect 5-cycle system of order $v > 5$ can be equitably 2-coloured.*

Proof. Consider the four possible equitable 2-colourings of a 5-cycle using two colours as shown in Figure 2. Suppose that the vertices of K_v are coloured with a total of b black and w white vertices, and that there exists a 2-perfect 5-cycle decomposition of K_v which contains: α 5-cycles of Type 5.1; β of Type 5.2; γ of Type 5.3; and δ of Type 5.4. We now establish equations relating the number of vertices of each colour in K_v to the required number of 5-cycles of each type.

The number of one-coloured white edges in K_v is $\frac{1}{2}w(w-1)$. Since 5-cycles of Types 5.1 and 5.4 each contain one one-coloured white edge, those of Type 5.2 contain two such edges and those of Type 5.3 contain none, we deduce that:

$$\alpha + 2\beta + \delta = \frac{1}{2}w(w-1). \tag{1}$$

Since every vertex in K_v appears in $\frac{1}{2}(v-1)$ 5-cycles of the decomposition the cycles in the decomposition contain a total of $\frac{1}{2}(v-1)b$ black vertices and $\frac{1}{2}(v-1)w$ white vertices. Noting that our decomposition has $\frac{1}{10}v(v-1)$ 5-cycles, let x denote the number of 5-cycles with three white vertices (Types 5.2 and 5.4) and y denote the number with two white vertices (Types 5.1 and 5.3). Then

$$x + y = \frac{1}{10}v(v-1), \text{ and}$$
$$3x + 2y = \frac{1}{2}(v-1)w.$$

Solving the system for x and y, we find that

$$x = \frac{1}{10}(v-1)(5w-2v) = \beta + \delta, \text{ and} \tag{2}$$

$$y = \frac{1}{10}(v-1)(3v-5w) = \alpha + \gamma. \tag{3}$$

Taking the distance 2 graph of a 5-cycle we again obtain a 5-cycle. Clearly, the distance 2 graph of a cycle of Type 5.1 is a cycle of Type 5.3 and vice versa, and the distance 2 graph of a cycle of Type 5.2 is a cycle of Type 5.4 and vice versa. As the system is 2-perfect, the collection of distance 2 graphs must also be a 5-cycle decomposition of K_v, this time containing: γ 5-cycles of Type 5.1; δ of Type 5.2; α of Type 5.3; and β of Type 5.4. Based on the distance 2 decomposition, we can derive a new equation by following the reasoning used to derive Equation 1. We have

$$\gamma + 2\delta + \beta = \frac{1}{2}w(w-1). \tag{4}$$

Adding Equations (1) and (4), we obtain

$$w(w-1) = (\alpha + \gamma) + 3(\beta + \delta). \tag{5}$$

Substituting our expressions for $(\alpha+\gamma)$ and $(\beta+\delta)$ as given by Equations 2 and 3, we can manipulate equation Equation 5 to give

$$w^2 - vw + \frac{3}{10}v(v-1) = 0. \tag{6}$$

This is a quadratic equation in w. For a real solution to this quadratic to exist, it is required that $v(6-v) \geq 0$, which implies that $v \leq 6$. Combining this result with Theorem 4.8, we deduce that no 2-perfect 5-cycle system of order $v > 5$ can be equitably 2-coloured. □

Lemma 6.3 *There exists a 5-cycle decomposition of K_{15} which is not equitably 2-colourable.*

Proof. Let the vertex set of K_{15} be $\mathbb{Z}_{15}$. A computational search has shown that the following 5-cycle decomposition of K_{15} cannot be equitably 2-coloured.

(14, 4, 12, 13, 8),	(9, 2, 5, 4, 6),	(6, 5, 12, 7, 3),	(13, 9, 0, 4, 2),
(10, 8, 1, 6, 0),	(3, 4, 11, 8, 0),	(0, 1, 2, 3, 5),	(0, 2, 6, 7, 11),
(0, 7, 1, 3, 12),	(0, 13, 1, 5, 14),	(1, 4, 7, 2, 10),	(1, 9, 3, 8, 12),
(1, 11, 2, 12, 14),	(2, 8, 4, 9, 14),	(3, 10, 4, 13, 11),	(3, 13, 5, 7, 14),
(5, 8, 6, 10, 9),	(5, 10, 13, 6, 11),	(6, 12, 11, 10, 14),	(7, 8, 9, 12, 10),
(7, 9, 11, 14, 13).			

□

Theorem 6.4 *For all $v \equiv 1, 5 \pmod{10}$, $v > 5$, there exists a 5-cycle decomposition of K_v which is not equitably 2-colourable.*

Proof. This follows immediately from Theorem 6.2 and Lemma 6.3. □

However, Theorem 6.2 does not generalise to all 2-perfect m-cycle decompositions for $m > 5$; see the following example.

Example 6.5 There exists a 2-perfect 7-cycle decomposition of K_{15} which can be equitably 2-coloured.

Proof. Let the vertex set of K_{15} be $\mathbb{Z}_{15}$. Colour the vertices 0, 1, 3, 5, 7, 9, 11 and 13 black and colour the remaining vertices white. Then the starter cycle $(0, 1, 3, 10, 7, 2, 6)$ can be developed modulo 15 to produce a 2-perfect 7-cycle decomposition. It is easy to check that the decomposition is equitably 2-coloured. □

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