The crossing number of P(10,3) is six

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Abstract

We use a computer to show that the crossing number of generalized Petersen graph P(10,3) is six.

1 Introduction

Fiorini made the first substantial attempt to compute the crossing numbers of the generalized Petersen graphs $P(n,k)$ [2]. He identified $cr(P(3n,3))$ and $cr(P(3n+2,3))$ as being n and $n+2$, respectively. Left open was the crossing number $cr(P(3n+1,3))$. Correcting an error in [2], Richter and Salazar proved Fiorini's results and showed $cr(P(3n+1,3)) = n+3$ [4]. However, the induction required the base case $cr(P(10,3)) = 6$.

At the time, the claim had been made that this was established using a computer. No article has since appeared to support that claim.

The purpose of this note is to describe how we used a computer to establish $cr(P(10,3)) = 6$. Our method seems to be effective for graphs with 30 or so edges, a relatively large automorphism group, and crossing number at most 6. Without the symmetry, it seems that the method would take too long. It would be interesting to have a method that will be effective on asymmetric graphs with roughly these parameters.

We are extremely grateful to John Boyer for providing us with his planarity testing code. This efficient program made everything else possible. We also used Brendan McKay's "nauty" package to compute generators for the automorphism group of $P(10,3)$.

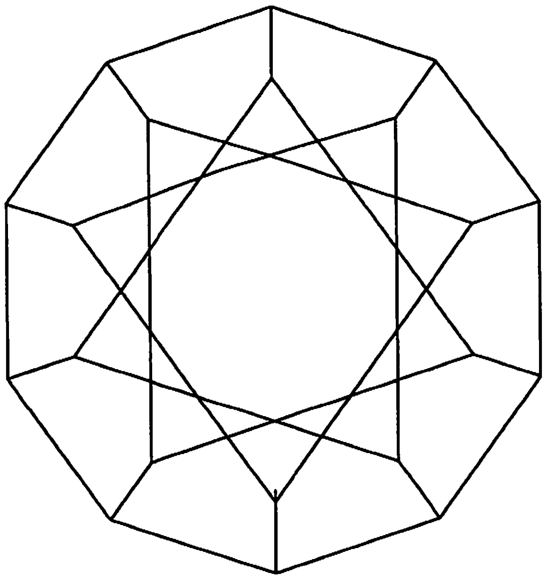

FIGURE 1. $P = P(10,3)$.

2 Method

In this section, we describe how we verified $cr(P(10,3)) = 6$. We begin with some simple concepts.

Definition 1 *The* skewness $\mu(G)$ *of a graph G the smallest number of edges whose removal from G results in a planar graph.*

It is a completely trivial matter to check, by removing all possible sets of three or four edges, that $\mu(P(10,3)) = 4$. We did not try to prove by hand that $\mu(P(10,3)) > 3$.

In studying crossing numbers, we are interested in "drawings" of graphs in the plane. It is well-known that it suffices to consider drawings in which:

1. vertices are mapped to distinct points of the plane;
2. each edge is represented by a piecewise linear arc joining the points representing the ends of the edge and otherwise disjoint from the vertices;
3. any two edges have at most one intersection, which is either a common end or a crossing; and
4. no three edges are concurrent at a point.

Definition 2 *For a drawing ϕ of a graph G, the* skewness $\mu(\phi)$ of ϕ *is the minimum number of edges that can be deleted from ϕ to make the drawing free of crossings.*

Henceforth, let P denote the generalised Petersen Graph $P(10,3)$. Since $\mu(P)=4$, the following is trivial.

Lemma 3 *For any drawing ϕ of P, $\mu(\phi)\geq 4$.*

The following was proved in [3].

Theorem 4 $cr(P)\geq 5$.

Figure 2 proves the upper bound.

Theorem 5 $cr(P)\leq 6$.

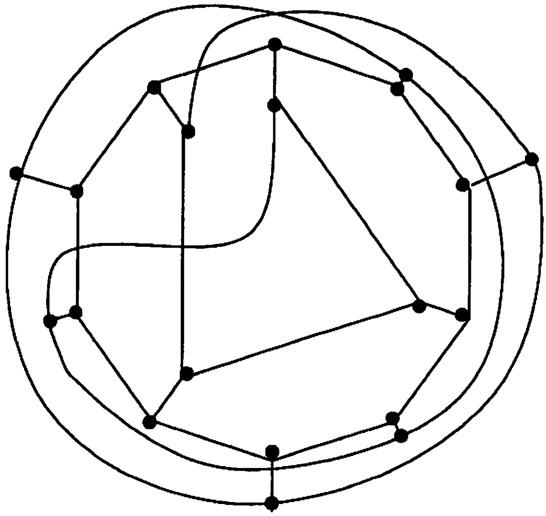

FIGURE 2. $cr(P)\leq 6$.

We use the computer to show that there is no drawing of P with five crossings. To do this, we assume that ϕ is a drawing of P with five crossings. Because $\mu(\phi)\geq 4$, there are only a handful of configurations that could contain all the crossings. For example, if some edge e crossed three or more others in ϕ, then deleting e and at most two other edges would result in a planar drawing of some subgraph of P, violating Lemma 3. Thus, every edge crosses at most two others.

A simple way to account for the possibilities is the following. For the drawing ϕ, we create an auxiliary graph A_ϕ, whose vertices are the edges of P and two vertices are adjacent in A_ϕ if the corresponding edges of P cross each other in ϕ. The assumption that ϕ has five crossings translates into A_ϕ has five edges. The value of $\mu(\phi)$ is the covering number $c(A_\phi)$ of A_ϕ, i.e., the minimum number of vertices whose deletion leaves an edgeless graph.

Thus, we are interested in all graphs on five edges with covering number at least four. It is easy to use nauty to produce all such graphs; it is also easy to produce them by hand. The earlier example becomes the assertion that, in such a graph, no vertex has degree three. We see equally easily that there is no path of length four or more, and no cycle of length four or more. There are only four possibilities; these are shown in Figures 3–6.

FIGURE 3. Configuration 1.

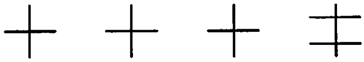

FIGURE 4. Configuration 2.

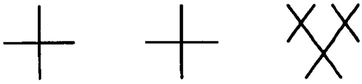

FIGURE 5. Configuration 3.

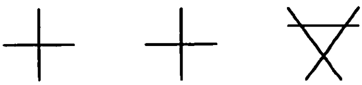

FIGURE 6. Configuration 4.

We note that crossing edges do not have common ends, but that other edges in these configurations may have common ends.

Our first goal was to find all possible sets, up to automorphisms, of three pairs of edges; these would be three pairs, each pair giving a different one of the five crossings. Thus, we want the two edges in each pair to have no common ends. We began by finding all different pairs; there are 5 of these. Then we extended each of these in all possible ways, up to automorphisms, to sets of two pairs; there are 309 of these. Finally, each of these was extended in all possible ways, up to automorphisms, to sets of three pairs; there are 24,321 of these.

So now we tried to find a drawing with five crossings for all of the configurations.

For the first configuration, if $\{\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}\}$ is one of the 24,321 sets of three pairs of edges, we found all pairs of additional edges e_7 and e_8 so that the two sets $\{e_1, e_3, e_5, e_7, e_8\}$ and $\{e_2, e_4, e_6, e_7, e_8\}$ have the property that deleting the edges from either of these sets results in a planar subgraph of P. (In principle, we could have checked that eight subgraphs – all ways of choosing one of the two from each pair from the original set of three pairs – are all planar; in fact, we checked only three. More checking here might reduce the number of possibilities, but this did not turn out to be important.)

This gives a list of pairs of edges. Picking two disjoint pairs, say e_7, e_8 and e_9, e_{10}, we obtain a candidate for Configuration 1 with the hypothesized crossing pairs $\{e_1, e_2\}$, $\{e_3, e_4\}$, $\{e_5, e_6\}$, $\{e_7, e_9\}$, and $\{e_8, e_{10}\}$. We also have to check the five pairs $\{e_1, e_2\}$, $\{e_3, e_4\}$, $\{e_5, e_6\}$, $\{e_7, e_{10}\}$, and $\{e_8, e_9\}$.

With the given hypothesized crossing pairs, we constructed a new graph by deleting the 10 edges, and, for each of the five pairs, introduced a new vertex adjacent to the four ends of the two edges in the pair. If there were a drawing of P with five crossings occurring as in Configuration 1, then, for one such selection, the new graph would be planar. Since, for none of the possibilities is the result planar, we deduce that there is no drawing of P having five crossings occurring as in Configuration 1.

We proceed similarly for the remaining three configurations. We do not explain them in such full detail.

For Configuration 2, again we began with one of the 24,321 sets of three pairs of edges, say $\{\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}\}$. In this case, we found both all the single edges e and all the pairs e_7, e_8 for which all of $P - \{e_1, e_3, e_5, e\}$, $P - \{e_2, e_4, e_6, e\}$, $P - \{e_1, e_3, e_5, e_7, e_8\}$, and $P - \{e_2, e_4, e_6, e_7, e_8\}$ are planar. Then we have the three single crossing pairs $\{e_1, e_2\}$, $\{e_3, e_4\}$, $\{e_5, e_6\}$, together with e crossing e_7 and e_8 (there are two possibilities here for the order of the crossings on e). Trying all the possibilities, we found that there is no drawing of P with five crossings occurring as in Configuration 2.

For the third and fourth configuration we considered each of the 309 sets of two pairs $\{\{e_1, e_2\}, \{e_3, e_4\}\}$ as the two single crossings. For each such pair, we found all sets of two edges whose deletion, once together with e_1, e_3 and once together with e_2, e_4, yield a planar subgraph. For Configuration 3, we take two pairs of these and make the new graph by insertion of vertices of degree 4 at the crossing points (there are four possibilities). For Configuration 3, we take each pair and any other edge to make the configuration (there are eight ways to do the ordering of crossings). In every possibility, the resulting graph is not planar, so we conclude that no drawing of P exists having five crossings occurring as in Configurations 3 or 4.

Since we eliminated all possibilities, there is no drawing of P having five crossings.

These deliberations and computations combine with Theorems 4 and 5 to prove the following.

Theorem 6 $cr(P) = 6$.

Our confidence in our program was boosted by the following two trials. When we ran our program looking for drawings having six crossings, in certain configurations we found many different examples of drawings with six crossings. Therefore, our program did not simply always produce a negative answer. Furthermore, it confirmed the result of [1] that a certain 5-regular graph on 10 vertices also has crossing number six. The algorithm in [1] was completely different from the one used here.

References

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