ADDING EVIDENCE TO THE ERDŐS-FABER-LOVÁSZ CONJECTURE

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Abstract. A hypergraph is linear if no two distinct edges intersect in more than one vertex. A long standing conjecture of Erdős, Faber and Lovász states that if a linear hypergraph has n edges, each of size n, then its vertices can be properly colored with n colors. We prove the correctness of the conjecture for a new, infinite class of linear hypergraphs.

1 INTRODUCTION

The aim of this paper is to add evidence to the long standing conjecture of Erdős, Faber and Lovász, originally stated in [4] as:

"*... if $|A_k| = n$, $1 \le k \le n$, and $|A_k \cap A_j| \le 1$, for $k < j \le n$, then one can color the elements of the union $\bigcup_{k=1}^{n} A_k$ by n colors so that every set has elements of all the colors ...*"

The correctness of this conjecture has been verified only in a handful of cases —for a recent survey see [8]—, motivating more than one pessimistic point of view [5]. However, here we will show that the conjecture is true in a new, unpublished case. Our discussion will be expressed in terms of hypergraphs [1].

A *hypergraph* $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ consists of a finite set $\mathcal{V}_\mathcal{H} \neq \emptyset$, the *vertices* of $\mathcal{H}$, and a finite family $\mathcal{E}_\mathcal{H}$ of non-empty subsets of $\mathcal{V}_\mathcal{H}$, the *edges* of $\mathcal{H}$; it is assumed that each vertex belongs to at least one edge. Thus, a *graph* is a hypergraph in which every edge has at most two elements. A hypergraph is *linear* if no two distinct edges intersect in more than one vertex, and it is *intersecting* if every two edges have a nonempty intersection.

The *degree* $\delta(v)$ of a vertex $v \in \mathcal{V}_\mathcal{H}$ is the number of edges containing v.

A (*proper*) *k-vertex coloring* of $\mathcal{H}$ is a surjective map of $\mathcal{V}_\mathcal{H}$ into a color set $\{1, \dots, k\}$, such that in every edge of $\mathcal{E}_\mathcal{H}$ all vertices have distinct color. The (vertex) *chromatic number* $\chi(\mathcal{H})$ is the smallest k such that there is a k-vertex coloring of $\mathcal{H}$.

Thus, in terms of hypergraphs, the original Erdős, Faber and Lovász conjecture can be stated as (see [8, 9]):

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ARS COMBINATORIA 85(2007), pp. 71-84

Conjecture 1 *If a linear hypergraph $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ has n edges, each of size n, then $\chi(\mathcal{H}) = n$.*

For our approach we find it convenient to coin the n-cluster concept. An *n-cluster* is a hypergraph consisting of n edges, each of size n, such that $|E \cap F| = 1$, for every two distinct edges E and F. In other words, an n-cluster is a linear, intersecting hypergraph with n edges, each of size n.

Conjecture 2 *If $\mathcal{H}$ is an n-cluster then $\chi(\mathcal{H}) = n$.*

Theorem 3 *Conjectures 1 and 2 are equivalent.*

Proof. Let $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ be a linear hypergraph with n edges, each of size n. Proceeding as follows, we prove that if Conjecture 1 is true for intersecting hypergraphs, then $\chi(\mathcal{H}) = n$.

1. *For any pair of disjoint edges $E, F \in \mathcal{E}_\mathcal{H}$ there are vertices $x_1, x_2 \in E$, and $y_1, y_2 \in F$ such that $\delta(x_i) = \delta(y_i) = 1$ for $i = 1, 2$.* This is because, as $\mathcal{H}$ has n edges of size n, edge E (as F) intersects at most $n-2$ of the remaining edges, leaving at least two vertices of degree one. In other words, *no edge in $\mathcal{E}_\mathcal{H} \backslash \{E, F\}$ contains a vertex from $\{x_1, x_2, y_1, y_2\}$.*

2. Define $F^* = F - y_1 + x_1$, and let $\mathcal{H}' = (\mathcal{V}_{\mathcal{H}'}, \mathcal{E}_{\mathcal{H}'})$, where $\mathcal{E}_{\mathcal{H}'} = \mathcal{E}_\mathcal{H} - F + F^*$ and $\mathcal{V}_{\mathcal{H}'} = \mathcal{V}_\mathcal{H} - y_1$.

3. *Any k-vertex coloring (φ) of $\mathcal{H}'$ can be extended to a k-vertex coloring of $\mathcal{H}$.* This can be done by defining $\psi(y_1) = \varphi(x_1)$ and $\psi(x) = \varphi(x)$ for $x \in \mathcal{V}_\mathcal{H} - y_1$. By (1) above, if φ is proper then ψ is also proper. Thus ψ is a (proper) k-vertex coloring of $\mathcal{H}$.

4. Construct the sequence of hypergraphs $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_r$ so as:
   a. $\mathcal{H}_0 = \mathcal{H}$.
   b. For $i = 1, \ldots$ construct $\mathcal{H}_i$ by applying the operation defined in (2) above to $\mathcal{H}_{i-1}$, until an intersecting hypergraph $\mathcal{H}_r$ is obtained. Namely, replace edge F_i in $\mathcal{H}_{i-1}$ with a new edge $F_i^* = F_i - y_i + x_i$ to obtain $\mathcal{H}_i$, where vertices y_i and x_i have, each, degree one in $\mathcal{H}_{i-1}$.

5. *The vertices $x_1, \ldots, x_r$ all are distinct.* On the contrary assume $x_i = x_j$ for $i \neq j$. If $i < j$ then in $\mathcal{H}_{j-1}$ vertex x_j has degree greater than one, in contradiction with (4) above.

6. Now apply (3) above to the sequence $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_r$, proceeding backwards from $\mathcal{H}_r$. Hypergraph $\mathcal{H}_r$ is intersecting and by assumption Conjecture 1 is true for intersecting hypergraphs, thus $\mathcal{H}_r$ can be colored with n colors. Finally, by (3) above we obtain $\chi(\mathcal{H}_0) = \chi(\mathcal{H}_r)$. This finishes the proof. ∎

Our approach in Section 3 to the Erdős-Faber-Lovász conjecture through n-clusters is justified by Theorem 3.

2 PREVIOUSLY SOLVED CASES

Using computer search Hindman [5] verified that Conjecture 1 holds true for 'small' hypergraphs, those having at most ten edges.

The approach of Colbourn and Colbourn [3] was through Steiner systems. A *Steiner system* $S(2,k,n)$ can be defined as a linear hypergraph with n vertices, whose edges all have the same cardinality k. A Steiner system $S(2,k,n)$ is *cyclic* if its vertices can be labelled $0,1,\ldots,n-1$, such that the mapping $i \rightarrow i+1 \bmod n$ is an automorphism.

The *dual* of a hypergraph $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$, with $\mathcal{V}_\mathcal{H} = \{v_1,\ldots,v_n\}$ and $\mathcal{E}_\mathcal{H} = \{E_1,\ldots,E_m\}$, is the hypergraph $\mathcal{H}^* := (\mathcal{V}_{\mathcal{H}^*}, \mathcal{E}_{\mathcal{H}^*})$ whose vertices $\mathcal{V}_{\mathcal{H}^*} = \{e_1,\ldots,e_m\}$ are the edges of $\mathcal{H}$, and whose n edges are defined by the sets: $V_i = \{e_j : v_i \in E_j \in \mathcal{H}\}$. It follows that $(\mathcal{H}^*)^*$ is isomorphic to $\mathcal{H}$. Thus, Colbourn and Colbourn [3] established what comes to:

Theorem 4 *Let $\mathcal{H}$ be an instance of Conjecture 1, and let $\mathcal{H}'$ be the hypergraph obtained from $\mathcal{H}$ by deleting the vertices of degree one. If $\mathcal{H}'$ is the dual hypergraph of a cyclic Steiner system $S(2,k,n)$, then $\chi(\mathcal{H}) = n$.*

Let $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ be a hypergraph, and let $\mathcal{U} \subseteq \mathcal{V}_\mathcal{H}$. The set $\mathcal{U}$ is *independent* if it intersects each edge of $\mathcal{E}_\mathcal{H}$ in at most one vertex. Berge and Hilton [2] proved a result that includes

Theorem 5 *Let $\mathcal{H}$ be an instance of Conjecture 1, in which each vertex has degree at most three. If the set of vertices of degree greater than two is independent, then $\chi(\mathcal{H}) = n$.*

The *partial hypergraph* of $\mathcal{H}$ induced by $\mathcal{U}$ is the hypergraph $\mathcal{H}' = (\mathcal{U}, \mathcal{E}_{\mathcal{H}'})$, where $E \in \mathcal{E}_{\mathcal{H}'}$ if and only if $E = \mathcal{U} \cap F \neq \emptyset$, for some $F \in \mathcal{E}_\mathcal{H}$. Jackson, Sethuraman and Whitehead [6] obtained

Theorem 6 *Let $\mathcal{H}$ be an instance of Conjecture 1, and consider $\mathcal{H}' = (\mathcal{V}_{\mathcal{H}'}, \mathcal{E}_{\mathcal{H}'})$ as the partial hypergraph of $\mathcal{H}$ induced by the vertices of degree at least three. If $|E| \leq 3$ for each $E \in \mathcal{E}_{\mathcal{H}'}$, and $\mathcal{H}'$ can be colored with three colors, then $\chi(\mathcal{H}) = n$.*

Call a hypergraph $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ *dense* if $\min\{\delta(v) \mid v \in \mathcal{V}_\mathcal{H}\} \geq \sqrt{|\mathcal{E}_\mathcal{H}|}$. Among the possible examples of dense linear hypergraphs we have all finite projective planes. Quite recently, Sánchez-Arroyo [9] proved

Theorem 7 *Let $\mathcal{H}$ be an instance of Conjecture 1, and let $\mathcal{H}'$ be the hypergraph obtained from $\mathcal{H}$ by deleting the vertices of degree one. If $\mathcal{H}'$ is dense, then $\chi(\mathcal{H}) = n$.*

To our knowledge, no other result exists stating the validity of Conjecture 1.

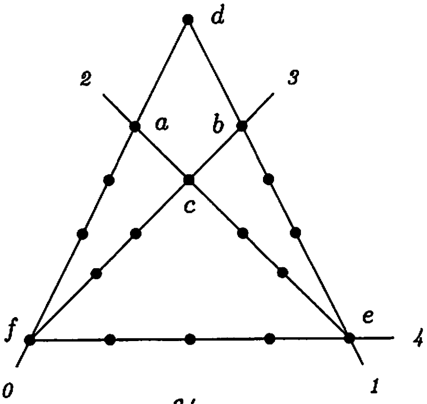
3 MAIN RESULT

Let $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ be an n-cluster, with edges numbered from 0 to $n-1$. Let $M(\mathcal{H})$ be an n-by-n matrix whose rows (and columns) are numbered from 0 to $n-1$, and correspond to the edges of $\mathcal{H}$. Denote by $M'(\mathcal{H})$ the set of cells (i,j) of $M(\mathcal{H})$ such that $i \neq j$. Finally, let $\tilde{\mathcal{V}}_\mathcal{H} \subset \mathcal{V}_\mathcal{H}$ be the set of vertices with degree greater than or equal to two. To each $v \in \tilde{\mathcal{V}}_\mathcal{H}$ we associate the "block" $C_v = \{(i,j) \mid v \in i \cap j \text{ and } (i,j) \in M'(\mathcal{H})\}$.

Remark 8 *$C_{v_1} \cap C_{v_2} = \emptyset$ for any distinct vertices $v_1, v_2 \in \tilde{\mathcal{V}}_\mathcal{H}$, because $\mathcal{H}$ is linear.*

Remark 9 *$\bigcup_{v \in \tilde{\mathcal{V}}_\mathcal{H}} C_v = M'(\mathcal{H})$ because $\mathcal{H}$ is intersecting.*

For example, consider the 5-cluster $\mathcal{H}_1$ schematically depicted in Figure 1, where edges are shown as line segments numbered from *0* to *4*, and vertices in $\tilde{\mathcal{V}}_{\mathcal{H}_1}$ are labelled $a, \ldots, f$. At right, in matrix $M(\mathcal{H}_1)$ a cell (i,j) has label $x \in \{a,b,c,d,e,f\}$ if $(i,j) \in C_x$.



$\mathcal{H}_1$

	0	1	2	3	4
0	\	d	a	f	f
1	d	\	e	b	e
2	a	e	\	c	e
3	f	b	c	\	f
4	f	e	e	f	\

$M(\mathcal{H}_1)$

FIGURE 1. A 5-cluster $\mathcal{H}_1$ and matrix $M(\mathcal{H}_1)$.

A coloring of the cells of $M'(\mathcal{H})$ —a coloring of $M(\mathcal{H})$, for short— is *proper* when for any distinct cells $r, s \in M'(\mathcal{H})$:

α. If r and s belong to the same block then they share color.

β. If r and s do not belong to the same block but share row or column, then they have distinct color.

Remark 10 *Every proper coloring of $M(\mathcal{H})$ is symmetrical with respect to its main diagonal.*

Conjecture 11 *Let $\mathcal{H}$ be an n-cluster. Then there is a proper coloring of $M(\mathcal{H})$ with at most n colors.*

Lemma 12 *Conjectures 2 and 11 are equivalent.*

Proof. Assume first that Conjecture 11 is correct, and let $\mathcal{H}$ be an n-cluster. A proper coloring of $M(\mathcal{H})$ with $k \leq n$ colors yields a coloring of the vertices in $\tilde{\mathcal{V}}_{\mathcal{H}}$, which in turn might be trivially extended to a proper n-vertex coloring of $\mathcal{H}$.

Assume now that Conjecture 2 is correct. From a proper n-vertex coloring of $\mathcal{H}$, and considering only the vertices in $\tilde{\mathcal{V}}_{\mathcal{H}}$, one can obtain in a straightforward manner a proper coloring of $M(\mathcal{H})$ with at most n colors. ∎

In view of Lemma 12, the challenging Erdős-Faber-Lovász conjecture can be approached by searching for a proper coloring of $M(\mathcal{H})$ with at most n colors. Although this route seems as formidable, it allowed us to find out a new class of hypergraphs for which the conjecture is correct.

Conformal labelling

A nonempty set W of nonnegative integers is *compact*, if either $|W| = 1$, or there is an order $(a_1, \ldots, a_{|W|})$ on W, such that $a_{i+1} = a_i + 1$, for $i = 1, \ldots, |W| - 1$.

Let $\mathcal{H} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}_{\mathcal{H}})$ be a hypergraph with n edges. $\mathcal{H}$ is *edge conformable*, if there is a bijection $\varphi : \mathcal{E}_{\mathcal{H}} \to \{0, \ldots, n-1\}$ —that we call a *conformal labelling*— such that for each vertex $v \in \tilde{\mathcal{V}}_{\mathcal{H}}$, the set $F(v) = \{\varphi(e) \mid v \in e \in \mathcal{E}_{\mathcal{H}}\}$ can be partitioned in two compact sets. See, for example, a conformal labelling of the 5-cluster $\mathcal{H}_1$ in Figure 1.

Remark 13 *A hypergraph whose vertices have degree at most two is edge conformable.*

We are ready for our main result.

Theorem 14 *Let $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ be an n-cluster. If $\mathcal{H}$ is edge conformable then $\chi(\mathcal{H}) = n$.*

Proof. Assume that an n-cluster $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ is edge conformable, and let the edges of $\mathcal{H}$ be indexed by a conformal labelling $\varphi : \mathcal{E}_\mathcal{H} \to \{0, \dots, n-1\}$. In view of Remark 10 we find it convenient to consider in $M(\mathcal{H})$ the set $M^\circ(\mathcal{H}) = \{(i,j) \mid 0 \le i < j \le n-1\}$, where we will look for a cell coloring —that we call proper— which by symmetry could be extended to a proper coloring of $M(\mathcal{H})$. For clarity's sake, $M(\mathcal{H})$ and $M^\circ(\mathcal{H})$ will be denoted in this proof as M and M°, respectively.

A vertex $v \in \tilde{\mathcal{V}}_\mathcal{H}$ and the cells of $\hat{C}_v = C_v \cap M^\circ$ will be called *free* or *tied*, if $\delta(v) = 2$ or $\delta(v) \ge 3$, respectively. If v is free then $|\hat{C}_v| = 1$; on the other hand, if v is tied then $|\hat{C}_v| \ge 3$ and the cells of $\hat{C}_v$ display in M° one of the four patterns in Figure 2, which we now proceed to explain.

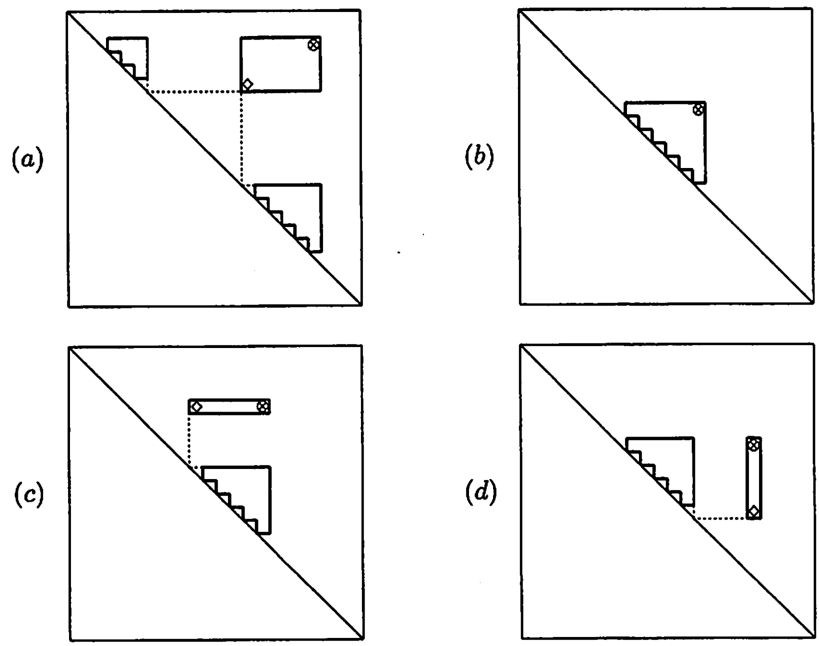


FIGURE 2. The possible shapes of $\hat{C}_v$ when $\delta(v) \ge 3$.
Heads and tails are indicated by ⊗ and ◇, respectively.

Let $(x_1, \dots, x_{\delta(v)})$ be an order on $F(v)$ satisfying $x_1 < x_2 < \cdots < x_{\delta(v)}$, and call the cell $(x_1, x_{\delta(v)})$ the *head* (of $\hat{C}_v$); further, let $t_v \in \{1, \dots, \delta(v)-1\}$ be an index such that both $\{x_1, \dots, x_{t_v}\}$ and $\{x_{t_v+1}, \dots, x_{\delta(v)}\}$ are compact. If $x_{t_v} + 1 < x_{t_v+1}$ then the cell (x_{t_v}, x_{t_v+1}) will be called the *tail*. Two cases arise:

- When $x_{t_v}+1 < x_{t_v+1}$ the set $\hat{C}_v$ looks as in Figure 2 (a), (c), or (d), if $2 \le t_v \le \delta(v)-2$, $t_v = 1$, or $t_v = \delta(v)-1$, respectively.
- When $x_{t_v}+1 = x_{t_v+1}$ the set $\hat{C}_v$ has the staircase shape of Figure 2(b).

For example, observe in Figure 1 that $\mathcal{H}_1$ has two tied vertices, e and f. Moreover, $\hat{C}_e$ has shape (d) and $\hat{C}_f$ has shape (c). Cells $(0,4)$ and $(1,4)$ are heads; cells $(0,3)$ and $(2,4)$ are tails.

For our purposes we find it convenient to enlarge M° by adding one column and one row to it, namely, the set of cells $P = \{(-1,2),(-1,3),\dots,(-1,n),(0,n),(1,n),\dots,(n-3,n)\}$. In the sequel the cells of P will also be considered as free. Let $r=(i,j)$ and $s=(k,\ell)$ be distinct cells of the enlarged M°. Call cell s the *immediate successor* of r whenever one of the following four conditions is satisfied.

I. r is a head, and there is a vertex $v \in \mathcal{V}_\mathcal{H}$ such that $r, s \in \hat{C}_v$.

II. r is a tail or free, s is free or a head, $k = i+1$, and $\ell = j-1$.

III. r is a tail or free, $(i+1,j),(i+1,j-1) \in \hat{C}_v$ for some $v \in \mathcal{V}_\mathcal{H}$, $s \notin \hat{C}_v$, $k = x_{t_v}+1$, and $\ell = j-1$.

IV. r is a tail or free, $(i,j-1),(i+1,j-1) \in \hat{C}_v$ for some $v \in \mathcal{V}_\mathcal{H}$, $s \notin \hat{C}_v$, $k = i+1$, and $\ell = x_{t_v+1}-1$.

Remark 15 *Each head has two or more immediate successors. Each tail or free cell has at most one immediate successor. The remaining cells have no immediate successors.*

If there is a sequence of cells $r = c_1,\dots,c_\alpha = s$, such that c_{i+1} is an immediate successor of c_i, for $i = 1,\dots,\alpha-1$, then cell s is a *successor* of cell r, and r is a *predecessor* of s. Clearly, from Remark 8 and Remark 9, any cell of M° is a successor of at least one cell of the enlarged M°, but it is a successor of a unique cell of P.

Remark 16 *If two distinct successors of a given cell share row or column, then they belong to the same block.*

Remark 17 *If cell (i,j) is a successor of cell (k,ℓ), then $i \ge k$ and $\ell \ge j$.*

Remark 18 *If two cells share a predecessor, and neither is a successor of the other, then at least one of them is tied.*

These concepts are exemplified with the (not enlarged) matrices of Figure 3, where thick lines serve to indicate blocks. The matrix above corresponds to a 17-cluster with only one tied vertex, where edges 2, 3, 4, 10, 11, 12, and

13, intersect; the immediate successor of cell y^x is indicated by y^{x+1}; bullets (•) indicate the other immediate successors of e^2 (besides e^3). Below four blocks can be distinguished in a 34-by-34 matrix; all successors of cell × are marked with bullets.

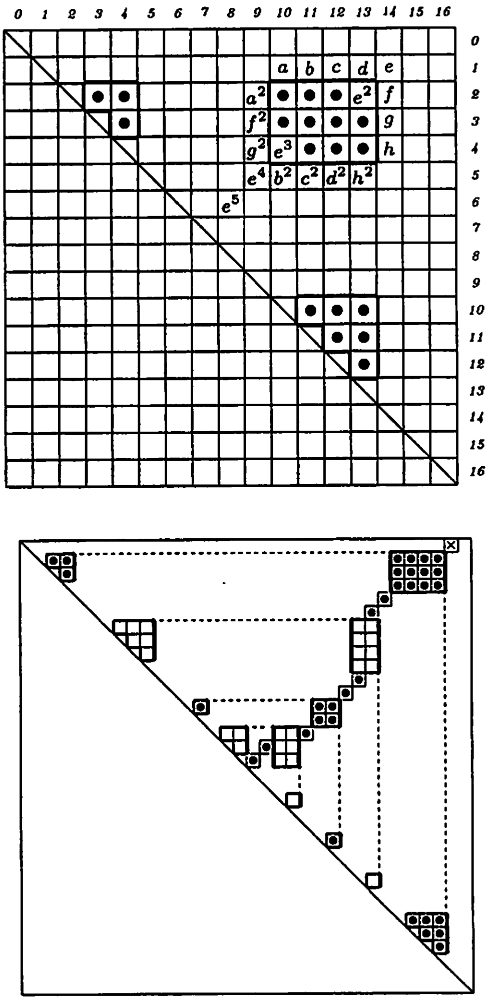


FIGURE 3. Top: the immediate successor of y^x is y^{x+1}.
Bottom: cells marked • are the only successors of cell ×.

As φ is a conformal labelling, for each tied vertex $v \in \mathcal{V}_{\mathcal{H}}$ the cells of $\hat{C}_v$ display in M° one of the patterns in Figure 2. With this in mind consider

Algorithm Λ

STEP 1. Assign color $(i + j) \bmod n$ to every cell $(i, j) \in P$.

STEP 2. For every cell of P, assign its color to each of its successors.

STEP 3. By symmetry, extend the coloring of M° to M.

We proceed now to prove that Algorithm Λ —a couple of examples are given in the appendix to help understanding it— produces a coloring of M that

- uses at most n colors, and
- is proper, namely, it satisfies conditions α and β above.

When coloring a cell of M, Algorithm Λ either explicitly assigns an element of $\{0, 1, \dots, n-1\}$ in Step 1, or it replicates an already existing color in Step 2 or Step 3; hence the total number of distinct assigned colors never exceeds n.

Additionally, by condition **I** above every tied cell that is not a head inherits the color of its head in Step 2. This coloring is extended in Step 3, and condition α —which concerns solely tied cells— is satisfied.

The remaining lines are devoted to proving that condition β is indeed satisfied. Reasoning by contradiction assume that cells r and s of M share row —or column, which by symmetry is the same—, belong to distinct blocks, and receive color w. Let then $r = (i, j)$ and $s = (i, \ell)$.

In regard to color w, note that in Step 1 of Algorithm Λ, if $w \in \{1, \dots, n-3\}$ this color is assigned to cells $(-1, w+1)$ and (w, n) of P, otherwise it is assigned to only one cell of P. Consider $w \in \{1, \dots, n-3\}$, as the case $w \in \{0, n-1, n-2\}$ could be dealt with similarly.

Assume that one of r and s is in M°, say $s \in M^\circ$. Then $i < \ell$. If $r \in M^\circ$ also, then $i < j$ and by Remark 16, one of r and s is a successor of $(-1, w)$ and the other is a successor of (w, n). But using Remark 17, both these possibilities lead to a contradiction. We conclude $\bar{r} = (j, i) \in M^\circ$ and hence $j < i$, giving $j < i < \ell$. Again the possibility that s and $\bar{r}$ are the successors of different cells in P leads to a contradiction by Remark 17. We can therefore assume they are (**1**) both successors of $(-1, w+1)$ or (**2**) both successors of (w, n).

Case **1** gives $0 \leq j < i < \ell \leq w$, with cells $\bar{r}$ and s sharing color w. As by Remark 17 neither is a successor of the other, then by Remark 18 one of them is tied. Assume $\bar{r}$ is tied, for the proof also applies when s is tied.

Let Γ be the set of <u>head</u> cells that are common predecessors of $\bar{r}$ and s (clearly, $|\Gamma| \neq \emptyset$), and let $(p, q) \in \Gamma$ be such that no element in Γ is a successor of (p, q). Consequently, $p \leq j < i < \ell \leq q$.

Let v be a vertex such that (p, q) is head in $\hat{C}_v$. Recall that $F(v) = \{\varphi(e) \mid v \in e \in \mathcal{E}_\mathcal{H}\}$, and let $(x_1, \dots, x_{\delta(v)})$ be an order on $F(v)$ satisfying $p = x_1 < x_2 < \cdots < x_{\delta(v)} = q$. As, by assumption, $\bar{r}$ and s belong to distinct blocks, then $\hat{C}_v$ does not have the shape as in Figure 2(b). Assume $\hat{C}_v$ with shape as in Figure 2(a). Now, $\bar{r}$ is tied and it should belong to $\hat{C}_v$, hence $p \leq j < i \leq x_{t_v}$. On the other hand $x_{t_v} < i < \ell < x_{t_v+1}$, because cell s is a successor of the tail in $\hat{C}_v$. These two last expressions contradict each other. Thus case 1 follows.

Case 2 gives $w+1 \leq j < i < \ell \leq n-1$, and could be dealt with analogously; therefore condition β is satisfied.

Thus, we have proved that a proper coloring of M with at most n colors can be found. Therefore, $\mathcal{H}$ can be properly vertex colored with n colors, and hence the theorem. ∎

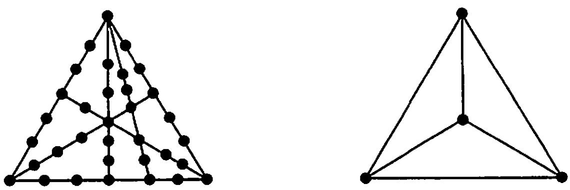
4 SKELETONS

Let $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ be a hypergraph. A vertex $v \in \mathcal{V}_\mathcal{H}$ will be called *tied* whenever $\delta(v) \geq 3$. An edge of cardinality one is a *singleton*. Clearly, Conjecture 2 is correct for n-clusters without tied vertices.

The *skeleton* of a hypergraph $\mathcal{H}$ is the hypergraph $\mathcal{H}^1$ obtained by first removing from $\mathcal{H}$ its vertices that are not tied, and then the singletons, if any. Note that if $\mathcal{H}$ is an n-cluster, then $\mathcal{H}^1$ is linear or the null graph.

The skeleton of $\mathcal{H}^1$ will in turn be denoted $\mathcal{H}^2$, and in general, for $k \geq 1$, $\mathcal{H}^{k+1}$ denotes the skeleton of $\mathcal{H}^k$. Assume that the skeleton of the null graph is still the null graph, and define $\mathcal{H}^0 = \mathcal{H}$. Clearly, if $\mathcal{H}^k = \mathcal{H}^{k+1}$ for some k, then $\mathcal{H}^t = \mathcal{H}^{t+1}$ for $t > k$. Hence, let $w(\mathcal{H})$ denote the smallest integer k such that $\mathcal{H}^k = \mathcal{H}^{k+1}$.

Some examples might clarify these concepts. First, a 7-cluster $\mathcal{G}$ and its skeleton $\mathcal{G}^1$ —which happens to be a graph— are depicted in Figure 4, with edges shown as line segments. In turn, $\mathcal{G}^2$ is identical to $\mathcal{G}^1$, thus $\mathcal{G}^0 \neq \mathcal{G}^1 = \mathcal{G}^2$ and we get $w(\mathcal{G}) = 1$.



(a) (b)

FIGURE 4. A 7-cluster $\mathcal{G}$ (a) and its skeleton $\mathcal{G}^1$ (b).

Consider now the 10-cluster $\mathcal{K}$ depicted in Figure 5(a), with edges shown as line segments. Its skeleton $\mathcal{K}^1$ is clearly composed of a single vertex and no edges; furthermore, as $\mathcal{K}^2$ is the null graph we get $w(\mathcal{K}) = 2$. A final example comes from the 17-cluster $\mathcal{L}$ sketched in Figure 5(c); its 17 edges are depicted either as circles or as line segments, and only tied vertices are shown. By removing from $\mathcal{L}$ its vertices that are not tied and then the singletons we get the graph $\mathcal{L}^1$, namely, a pentagon plus one chord. In turn, $\mathcal{L}^2$ is the complete graph on two vertices. Finally, $\mathcal{L}^3$ is the null graph and hence $w(\mathcal{L}) = 3$.

Proposition 19 *Let $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ be a hypergraph. If its skeleton $\mathcal{H}^1 \neq \mathcal{H}$ is either an edge conformable hypergraph or the null graph, then $\mathcal{H}$ is edge conformable.*

Proof. Consider first that $\mathcal{H}^1$ is an edge conformable hypergraph, with a conformal labelling ℓ of $\mathcal{H}^1$ at hand. Let F be the set of edges removed from $\mathcal{H}$ to obtain its skeleton $\mathcal{H}^1$. Assume $|F| \geq 1$ as otherwise we get a triviality. To derive from ℓ a conformal labelling of $\mathcal{H}$, first assign to every $e \in \mathcal{E}_\mathcal{H} \backslash F$ the label of the corresponding edge in $\mathcal{H}^1$, then repeat the following steps until every edge of $\mathcal{H}$ has been labelled.

1. Find an unlabelled edge $e \in \mathcal{E}_\mathcal{H}$. Let V_e denote the set of tied vertices in e. Clearly, $|V_e| \leq 1$.
2. If $|V_e| = 0$ then make $\ell(e) = z + 1$, where z is the so far maximum assigned label.
3. Otherwise $V_e = \{v\}$. If there are at least three labelled edges of $\mathcal{H}$ containing v then there are at least two labelled with consecutive numbers, say $a - 1$ and a; add one to the label of every edge of $\mathcal{H}$ with label greater than o equal to a, and make $\ell(e) = a$. On the other hand, if there are exactly two labelled edges of $\mathcal{H}$ containing v, say x and y, labelled as $\ell(x) = a < b = \ell(y)$, then add one to the label of every edge of $\mathcal{H}$ with label greater than o equal to a, and make $\ell(e) = a$. Finally, if there is one or zero labelled edges of $\mathcal{H}$ containing v, then make $\ell(e) = z + 1$, where z is the so far maximum assigned label.

When $\mathcal{H}^1$ is the null graph, either $\mathcal{H}$ does not have tied vertices and by Remark 13 is edge conformable, or no edge of $\mathcal{H}$ contains two or more tied vertices. In the latter case a conformal labelling of $\mathcal{H}$ is obtained in two steps. First consider the tied vertices of $\mathcal{H}$, one at a time, and for each assign consecutive numbers to the edges that contain it. In the second step arbitrarily label the remaining edges (if any) with distinct numbers in the range $[1, |\mathcal{E}_\mathcal{H}|]$.

We have shown that, in each case, it is possible to get a conformal labelling, and hence $\mathcal{H}$ is edge conformable. ∎

As a consequence of Proposition 19, finding out if a given n-cluster $\mathcal{H}$ is edge conformable might be somewhat simplified by looking at $\mathcal{H}^{w(\mathcal{H})}$. Note that if $\mathcal{H}^{w(\mathcal{H})}$ is not the null graph then each of its vertices is tied. We leave the characterization of edge conformable hypergraphs as an open problem, and close this section with two minor results.

Proposition 20 *If a hypergraph $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ is edge conformable, and $\mathcal{E}' \subset \mathcal{E}_\mathcal{H}$ then $\mathcal{H}' = (\mathcal{V}_\mathcal{H}, \mathcal{E}')$ is also edge conformable.*

Proof. It is enough to prove that if $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ is edge conformable then $\mathcal{H}^* = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H} \backslash e)$ is also edge conformable, for any edge $e \in \mathcal{E}_\mathcal{H}$.

Let $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ be an edge conformable hypergraph, and let ℓ be a conformal labelling of $\mathcal{E}_\mathcal{H}$. Remove any edge $e \in \mathcal{E}_\mathcal{H}$, and let every edge $f \in \mathcal{E}_\mathcal{H} \backslash e$ (if any) with $\ell(f) > \ell(e)$ be re-labelled as $\ell(f) = \ell(f) - 1$. It can be seen that now ℓ is a conformal labelling of $\mathcal{H}^* = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H} \backslash e)$, and hence $\mathcal{H}^*$ is edge conformable. ∎

Corollary 21 *If a hypergraph $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ is not edge conformable, and $\mathcal{E}_\mathcal{H} \subset \mathcal{E}'$ then $\mathcal{H}' = (\mathcal{V}_\mathcal{H}, \mathcal{E}')$ is not edge conformable.*

5 CONCLUSION

In Section 3 we introduced an infinite class of hypergraphs for which the Conjecture of Erdős-Faber-Lovász holds. This class is not contained in any of the previously solved cases mentioned in Section 2. To see this consider the 7-cluster $\mathcal{G}$ of Figure 4(a); although it is edge conformable, it does not satisfy the conditions asked for in theorems 4, 5, 6, and 7. Furthermore, we show now that the case of Berge and Hilton [2] is strictly contained in the class of edge conformable hypergraphs.

Let hypergraph $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ be an instance of Conjecture 1, in which each vertex has degree at most three, and the set $U = \{u_1, \ldots, u_{|U|}\} \subseteq \mathcal{V}_\mathcal{H}$ of vertices of degree three is independent. Denote by $D(u) \subseteq \mathcal{E}_\mathcal{H}$ the set of edges containing $u \in U$, then $|D(u)| = 3$ and $D(u) \cap D(v) = \emptyset$ for any distinct $u, v \in U$. Thus, for $j = 1, \ldots, |U|$, arbitrarily label the edges of $D(u_j)$ with $3j-3$, $3j-2$, and $3j-1$, then arbitrarily label the $n - 3|U|$ remaining edges of $\mathcal{E}_\mathcal{H}$ with numbers $3|U|, \ldots, n-1$. It is clear that this is a conformal labelling of $\mathcal{E}_\mathcal{H}$, and hence $\mathcal{H}$ is edge conformable. Thus, Theorem 5 is contained in Theorem 14. The hypergraphs $\mathcal{K}$ and $\mathcal{L}$ in Figure 5 are simple examples at hand to see that the contention is strict.

6 APPENDIX. EXAMPLES

We present here two examples of the application of Algorithm Λ in Section 4. Consider first the 10-cluster $\mathcal{K}$ in Figure 5(a), which has only one tied vertex. A conformal labelling from *0* to *9* has been given to its edges. Also, a proper coloring with colors 0 to 9 is shown for vertices in $\tilde{\mathcal{V}}_{\mathcal{K}}$. This vertex coloring derives from the (proper) coloring of $M^\circ(\mathcal{K})$ in Figure 5(b) —obtained with Algorithm Λ— trivially extended to vertices with degree one in $\mathcal{K}$.

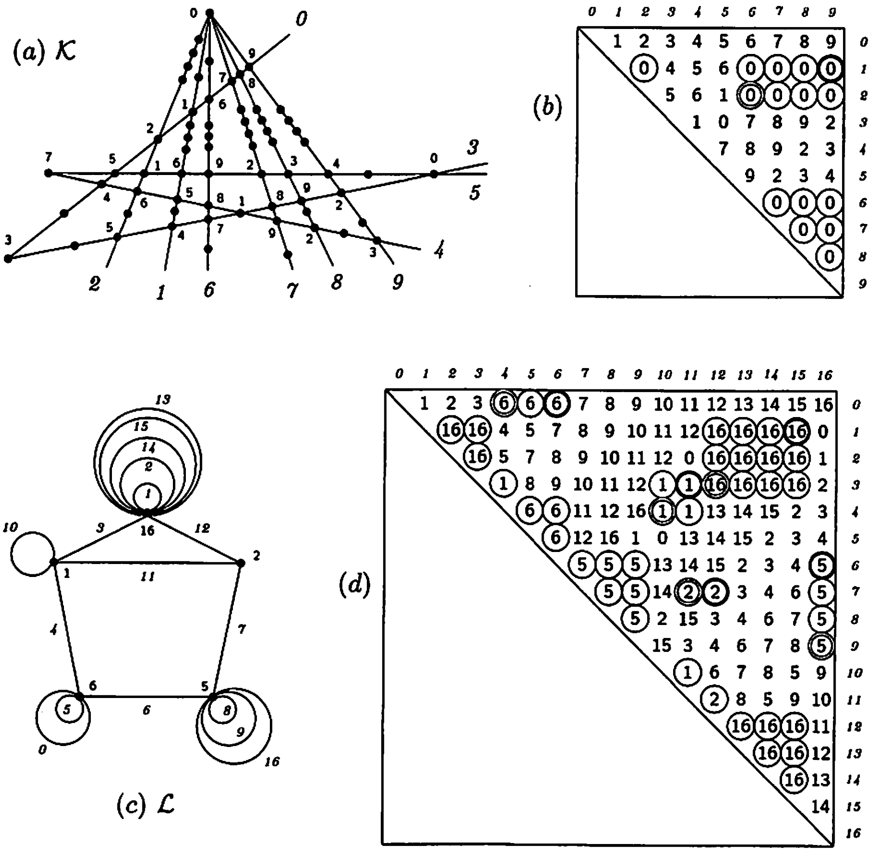


FIGURE 5. (a) and (c): Sketches of $\mathcal{K}$ and $\mathcal{L}$, respectively. (b) and (d): Proper coloring of $M^\circ(\mathcal{K})$ and $M^\circ(\mathcal{L})$, respectively. Tied cells indicated with circles (heads = thick, tails = double).

Now, consider the 17-cluster $\mathcal{L}$ sketched in Figure 5(c), whose edges — circles and line segments— have been assigned a conformal labelling. Only tied vertices are shown; their color (1, 2, 5, 6, and 16) comes directly from the coloring of matrix $M^\circ(\mathcal{L})$ once Algorithm Λ has been performed, see Figure 5(d).

Acknowledgements. This work was supported in part by CONACyT project number U40004-F. We specially thank the referee for his careful reading of the paper and useful comments.

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