CORDIALNESS OF CYCLES WITH PARALLEL P_k - CHORDS AND MULTIPLE SUBDIVISION GRAPHS

A. ELUMALAI*and G. SETHURAMAN†

Department of Mathematics

Crescent Engineering College, Chennai - 600 048

Email : s_elu@yahoo.com

Abstract

In this paper we prove that the cycle C_n with parallel chords and the cycle C_n with parallel P_k-chords are cordial for any odd positive integer $k \geq 3$ and for all $n \geq 4$ except for $n = 4r + 2$, $r \geq 1$. Further, we show that every even-multiple subdivision of any graph G is cordial and we show that every graph is a subgraph of a cordial graph.

Key words: Graph labeling, Cordial labeling, Cycle with parallel chords, Cycle with parallel P_k-chords.

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*Corresponding author.

†Currently working in the Department of Mathematics, Anna University, Chennai - 600 025.

1 Introduction

Let f be a function from the vertices of G to $\{0,1\}$ and for each edge xy assign the label $|f(x)-f(y)|$. The function f is called a *cordial labeling* of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by atmost 1 and the number of edges labeled 0 and the number of edges labeled 1 differ by atmost 1.

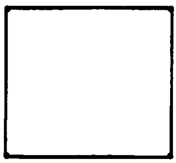
The notation of a cordial labeling was first introduced by Cahit [1] as a weaker version of graceful labeling.

A *chord* of a cycle is an edge joining two non adjacent vertices of the cycle.

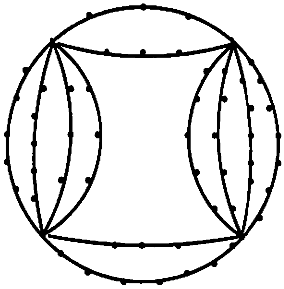
A graph G is called a *cycle with parallel chords* if G is obtained from the cycle $C_n : v_0v_1\cdots v_{n-1}v_0$ $(n \geq 4)$ by adding the chords v_1v_{n-1}, v_2v_{n-2}, $\cdots$, $v_{(\frac{n-2}{2})}v_{(\frac{n+2}{2})}$ or the chords $v_2v_{n-1}, v_3v_{n-2}, \cdots, v_{(\frac{n-1}{2})}v_{(\frac{n+3}{2})}$ depending on the parity of n.

A graph G is called a *cycle with parallel P_k-chords* if G is obtained from the cycle C_n of order n : $v_0v_1\cdots v_{n-1}v_0$ $(n \geq 4)$ by adding disjoint paths P_k's$(k \geq 3)$ between the pair of vertices (v_1, v_{n-1}), (v_2, v_{n-2}), $\cdots$, (v_i, v_{n-i}), $\cdots$, $(v_{\lfloor\frac{n}{2}\rfloor-1}, v_\beta)$, where $\beta = \lfloor\frac{n}{2}\rfloor + 2$, if n is odd and $\beta = \frac{n}{2}+1$, if n is even.

A grap H is said to be *even-multiple subdivision* of a graph G if H is obtained from G by replacing every edge e of G by a set of pairs of paths $(P,Q)'$s, with lengths of each of both the paths P and Q either of the form $0 \equiv (mod4)$ or $2 \equiv (mod4)$ by merging the origin and terminus of all the pairs of paths $(P,Q)'$s with the ends of the edge e.



Cycle C_4.



Multiple subdivision graph H of C_4.

In this paper we prove that the cycle C_n with parallel chords and the cycle C_n with parallel P_k-chords are cordial for all $n \geq 4$, except for $n = 4r+2$, $r \geq 1$ and for any odd positive integer $k \geq 3$. Further, we show that every even-multiple subdivision graph of any graph is cordial and we prove that every graph is a subgraph of a cordial graph.

2 Cordialness of cycles with parallel P_k-chords

In this section we prove that the cycle C_n with parallel P_k-chords is cordial, for all $n \geq 4$, except for $n = 4r+2$, for some $r \geq 1$ and for any odd positive integers $k \geq 3$.

Theorem 1 : *Cycle C_n with parallel chords is cordial for all $n \geq 4$ except for $n = 4r+2$, for some $r \geq 1$.*

Proof : Let G be a cycle C_n with parallel chords, where $n \geq 4$ and $n = 4r+k$; $k \in \{0,1,3\}$ and $r \geq 1$. Label the vertices of G (i.e., C_n) as $v_0v_1\cdots v_{n-1}(n \geq 4)$ such that $v_1v_{n-1}, v_2v_{n-2}, \cdots, v_{\frac{n-2}{2}}v_{\frac{n+2}{2}}$, are chords when n is even, while n is odd $v_2v_{n-1}, v_3v_{n-2}, \cdots, v_{\frac{n-1}{2}}v_{\frac{n+3}{2}}$, are the chords. Observe that G has n vertices and $\frac{3n-\rho}{2}$ edges, where $\rho = 3$, if n is odd or $\rho = 2$, if n is even. Arrange the vertices of G as a sequence $v_0v_1\cdots v_{n-1}$.

A particular 0-1 sequence of length n for the corresponding termwise matching with the above sequence of vertices of G is given in Table 1.

Let V_0 and V_1 respectively, denote the set of vertices (matched with) assigned the label 0 and the set of vertices (matched with) assigned the label 1.

From the Table 1, it is clear that when the number of vertices of G is of the form $4r$, the number of vertices assigned the label 0 and the number of vertices assigned the label 1 are equal, while the number of vertices of G is of the form $4r+1$ or $4r+3$, the number of vertices assigned the label 0 and the number of vertices assigned the label 1 differ by atmost 1.

Let A denote the set of edges $v_0v_1, v_1v_2, \cdots, v_{n-1}v_0$ (the cycle edges) of G and B denote the set of chords $\{v_1v_{n-1}, v_2v_{n-2}, \cdots, v_{(\frac{n-2}{2})}v_{(\frac{n+2}{2})}\}$, or the set of chords $\{v_2v_{n-1}, v_3v_{n-2} \cdots v_{(\frac{n-1}{2})}v_{(\frac{n+3}{2})}\}$, depends on n is even or odd.

The edges in the sets A and B get the edge values 0 or 1, the edge values of the edges in the set A and in the set B are arranged as a 0-1 sequences and they are given in the Table 1.

Let E_0 and E_1 respectively, denote the set of all edges of G getting the label 0 and the set of all edges of G getting the label 1. From Table 1, it is clear that G is cordial.

Table 1. Vertex and edge labeling of cycle with parallel chords.

Nature of the number of vertices n of G	The 0-1 sequence for termwise matching with the sequence of vertices of G	Relation between $\|V_0\|$ and $\|V_1\|$	The edge labels sequence for the set A	B	Relation between $\|E_0\|$ and $\|E_1\|$
$n = 4r$, $r \geq 1$	$(0011)^{\frac{n}{4}}$	$\|V_0\| = \|V_1\|$	$(01)^{\frac{n}{2}}$	$(10)^{\frac{n-4}{4}}1$	$\|E_0\| + 1 = \|E_1\|$
$n = 4r+1$, $r \geq 1$	$(0011)^{\frac{n-1}{4}}\ 0$	$\|V_0\| + 1 = \|V_1\|$	$(01)^{\frac{n-1}{2}}\ 0$	$(10)^{\frac{n-5}{4}}1$	$\|E_0\| = \|E_1\|$
$n = 4r+3$, $r \geq 1$	$(0011)^{\frac{n-3}{4}}\ 001$	$\|V_0\| + 1 = \|V_1\|$	$(01)^{\frac{n-1}{2}}1$	$(01)^{\frac{n-3}{4}}$	$\|E_0\| + 1 = \|E_1\|$

Theorem 2 : *Cycle C_n with parallel P_k-chords is cordial for all $n \geq 4$, except for $n = 4r + 2$, $r \geq 1$ and for any odd positive integer $k \geq 3$.*

Proof : Let G be a cycle C_n with parallel P_k-chords, where $n \geq 4$ and $n = 4r + \ell$; $\ell \in \{0, 1, 3\}$, for $r \geq 1$ and where $k \geq 3$ is any odd positive integer. By definition, G is obtained from the cycle C_n of order $n : v_0 v_1 \cdots v_{n-1} v_0$ by adding disjoint paths $P_k' s$ between the pair of vertices $(v_1, v_{n-1}), (v_2, v_{n-2}), \cdots, (v_i, v_{n-i}), \cdots, (v_{\lfloor \frac{n}{2} \rfloor - 1}, v_\beta)$, where $\beta = \lfloor \frac{n}{2} \rfloor + 2$, if n is odd or $\beta = \lfloor \frac{n}{2} \rfloor + 1$, if n is even. Observe that G has $N = \frac{nk - \alpha(k-2)}{2}$ vertices, where $\alpha = 3$, if n is odd or $\alpha = 2$, if n is even and $M = \frac{n(k+1) - \rho(k+1)}{2}$ edges, where $\rho = 3$, if n is odd or $\rho = 2$, if n is even.

Let G be a cycle $C_n : u_0 u_1 \cdots u_{n-1} u_0$ with parallel P_k-chords. We call the P_k-chords joined between the pair (u_i, u_{n-i}) of C_n in G, the i^{th} P_k-chords for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$. Observe that G has a hamiltonian path, starting with u_0 and ending up with u_α of the cycle C_n of G, where $\alpha = \lceil \frac{n}{2} \rceil$, if n is odd or $\alpha = \frac{n}{2}$, if n is even.

Let $v_0 v_1 \cdots v_{N-1}$, where $N = |V(G)|$ be a hamiltonian path in G starting with u_0 of C_n in G and ending up with u_α of C_n in G.

Arrange the vertices of G as a sequence $v_0 v_1 \cdots v_{N-1}$.

A particular 0-1 sequence of length N for the corresponding termwise matching with the above sequence of vertices of G is given in Table 2.

Let V_0 and V_1 respectively, denote the set of vertices assigned the label 0 and the set of vertices assigned the label 1.

Table 2. Vertex and edge labeling of cycle with parallel P_k chords, where k is odd.

Nature of the number of vertices n of G	The 0-1 seq. for termwise matching with the seq. of vertices of G: For $k = 4t-1$, $t \geq 1$	For $k = 4t+1$, $t \geq 1$	Relation between $\lvert V_0\rvert$ and $\lvert V_1\rvert$	The edge labels seq. for the set: For $k = 4t-1, t \geq 1$: A	For $k = 4t-1, t \geq 1$: B	For $k = 4t+1, t \geq 1$: A	For $k = 4t+1, t \geq 1$: B	Relation between $\lvert E_0\rvert$ and $\lvert E_1\rvert$
$n = 4r, r \geq 1$ for r odd	$(0011)^{\frac{N-1}{4}}0$	$(0011)^{\frac{N-3}{4}}0$	$\lvert V_0\rvert + 1 = \lvert V_1\rvert$	$(01)^{\frac{N-1}{2}}$	$11(01)^{\frac{n-8}{4}}00$	$(01)^{\frac{N-1}{2}}$	$(01)^{\frac{n}{4}}$	$\lvert E_0\rvert = \lvert E_1\rvert$
for r even	$(0011)^{\frac{N-3}{4}}001$	$(0011)^{\frac{N-1}{4}}001$						
$n = 4r+1, r \geq 1$ for r odd	$(0011)^{\frac{N-2}{4}}10$	$(0011)^{\frac{N}{4}}$	$\lvert V_0\rvert = \lvert V_1\rvert$	$(01)^{\frac{N-4}{2}}001$	$11(01)^{\frac{n-9}{4}}00$	$(01)^{\frac{N-2}{2}}0$	$(01)^{\frac{n-1}{4}}$	$\lvert E_0\rvert = \lvert E_1\rvert + 1$
for r even	$(0011)^{\frac{N}{4}}$	$(0011)^{\frac{N-2}{4}}10$		$(01)^{\frac{N-2}{2}}0$		$(01)^{\frac{N-4}{2}}001$		
$n = 4r+3, r \geq 1$ for r odd	$(0011)^{\frac{N-1}{4}}0$	$(0011)^{\frac{N-1}{4}}0$	$\lvert V_0\rvert + 1 = \lvert V_1\rvert$	$(01)^{\frac{N-1}{2}}$	$11(01)^{\frac{n-7}{4}}0$	$(01)^{\frac{N-1}{2}}$	$(01)^{\frac{n-3}{4}}1$	$\lvert E_1\rvert = \lvert E_0\rvert + 1$
for r even	$(0011)^{\frac{N-3}{4}}001$	$(0011)^{\frac{N-3}{4}}001$						

From the Table 2, it is clear that when the number of vertices of G is of the form $4r$ or $4r+3$, the number of vertices assigned the label 0 and the number of vertices assigned the label 1 differ by atmost 1, while when the number of vertices of G is of the from $4r+1$, the number of vertices assigned the label 0 and the number of vertices assigned the label 1 are equal.

Let A denote the set of edges $\{v_0v_1, v_1v_2, \cdots, v_iv_{i+1}, \cdots, v_{N-2}v_{N-1}\}$ of the hamiltonian path of G and let $B = E(G) - A$ of G. The edges in the sets A, B get the edge values 0 or 1, the edge values of the edges of the sets A and B are arranged as a 0-1 sequence and they are given in the Table 2.

Let E_0 and E_1 respectively, denote the set of all edges of G getting the label 0 and the set of all edges of G getting the label 1.

From Table 2, we observe that G is cordial. □

3 Cordialness of even-multiple subdivision graph

In this section, we prove that every even-multiple subdivision of any graph G is cordial.

Let H be an even-multiple subdivision graph of a graph G. Observe that $V(H)$ can be partitioned into two sets U and W, where U is the set of vertices of H which are originally belong to the graph G and W is the set of vertices of H which are internal vertices of the sets of pairs of paths replacing the edges of G in obtaining H. We call the vertices of the set U *the base vertices of* H and the vertices of the set W *the non base vertices of* H.

Note: From each graph G we can construct a family of even-multiple subdivision graphs H's of G.

Theorem 3 : *For every nontrivial graph G, each even-multiple subdivision graph H of G is cordial.*

Proof : Let G be any nontrivial graph and let H be any even-multiple subdivision graph of G. Let U be the set of base vertices of H (Note that U is precisely the set of vertices of G) and let W be the set of internal vertices of all the paths replacing the edges of G. Partition the set U into $U_0 \cup U_1$ such that $\big||U_0| - |U_1|\big| \leq 1$.

Give label 0 to all the vertices of U_0 and give the label 1 to all the vertices of U_1 and the vertices of W are assigned the label 0 or 1 as described below.

Let $x\ v_1\ v_2 \cdots v_{2k_1-1}y$ and $x\ v_1'\ v_2' \cdots v_{2k_2-1}'y$ be respectively a pair of paths (P_e, P_e') in the set of pairs of paths replacing an edge e of G in obtaining H, then x and y are base vertices of H which have already been assigned label either 0 or 1. For the sequence of internal vertices $v_1v_2 \cdots v_{2k_1-1}$ and $v_1'v_2' \cdots v_{2k_2-1}'$ we match termwise with a particular sequence of 0-1.

When k_1 and k_2 are odd, say $k_1 = 2r_1 + 1$ and $k_2 = 2r_2 + 1$, for some $r_1, r_2 \geq 1$, we match the vertices sequence $v_1v_2 \cdots v_{4r_1+1}$ termwise with $(0011)^{r_1}0$ and match the vertices sequence $v_1'v_2' \cdots v_{4r_2+1}'$ termwise with $(1100)^{r_2}1$. Then the edge values of the edges xv_1, v_iv_{i+1}, for $1 \leq i \leq 4r_1$, $v_{4r_1+1}y$ of the path P_e and the edges $x\ v_1', v_i'v_{i+1}'$, for $1 \leq i \leq 4r_2, v_{4r_2+1}'y$ of the path P_e' are given in the Table 3, as a 0-1 sequence with respect to the above order of the edges.

When k_1 and k_2 are even, say $k_1 = 2r_1$ and $k_2 = 2r_2$, for some $r_1, r_2 \geq 1$, we match the vertices sequence $v_1v_2 \cdots v_{4r_1-1}$ termwise with $(0011)^{r_1-1}001$ and match the vertices sequence $v_1'v_2' \cdots v_{4r_2-1}'$ termwise with $(1100)^{r_2-1}110$.

Then the edge values of the edges xv_1, v_iv_{i+1}, for $1 \leq i \leq 4r_1-2, v_{4r_1-1}y$ of the path P_e and the edges $xv_1', v_i'v_{i+1}'$, for $1 \leq i \leq 4r_2 - 2, v_{4r_2-1}'y$ of the path P_e' are given in the Table 3 as a 0-1 sequence with respect to the above order of the edges.

Let $V_0(P_e \cup P_e')$ denotes the set of internal vertices of the paths P_e and P_e' get the label 0 and $V_1(P_e \cup P_e')$ denotes the set of internal vertices of the paths P_e and P_e' get the label 1. Then $|V_0(P_e \cup P_e')| = |V_1(P_e \cup P_e')|$.

Let $E_0(P_e \cup P_e')$ and $E_1(P_e \cup P_e')$ respectively denote the set of edges of $P_e \cup P_e'$ getting the edge value 0 and the set of edges of $P_e \cup P_e'$ getting the edge value 1.

Suppose an edge e has been replaced by t pairs of paths $(P_e^{(1)}, P_e^{(1')})$, $(P_e^{(2)}, P_e^{(2')}), \cdots, (P_e^{(t)}, P_e^{(t')})$ in obtaining H. Let S_e denote the set of internal vertices of the pairs of paths $(P_e^{(1)}, P_e^{(1')}), (P_e^{(2)}, P_e^{(2')}) \cdots, (P_e^{(t)}, P_e^{(t')})$ and let F_e denote the set of all edges of the pairs of paths $(P_e^{(1)}, P_e^{(1')})$, $(P_e^{(2)}, P_e^{(2')}), \cdots, (P_e^{(t)}, P_e^{(t')})$. From the above argument, $|V_0(S_e)| = |V_1(S_e)|$ and $|E_0(F_e)| = |E_1(F_e)|$.

If G has m edges, say $e_1, e_2, \cdots, e_m$ in H, then we have $|V_0(S_{e_i})| = |V_1(S_{e_i})|$ and $|E_0(F_{e_i})| = |E_1(F_{e_i})|$, for $1 \leq i \leq m$.

Therefore,

$$|V_0(H)| = \sum_{i=1}^{m} |V_0(S_{e_i})| + U_0 \quad \text{and} \quad |V_1(H)| = \sum_{i=1}^{m} |V_1(S_{e_i})| + U_1.$$

Then $|V_0(H)|$ and $|V_1(H)|$ differ by atmost 1 as U_0 and U_1 differ by atmost 1.

Further,

$$|E_0(H)| = \sum_{i=1}^{m} |E_0(F_{e_i})| = \sum_{i=1}^{m} |E_1(F_{e_i})| = |E_1(H)|.$$

Hence, H is cordial. □

Table 3. Vertex and edge labeling of even-multiple subdivision graph.

The vertex label of x and y	When k_1 and k_2 are odd		When k_1 and k_2 are even		Relation between $\lvert E_0(P_e \cup P_e')\rvert$ and $\lvert E_1(P_e \cup P_e')\rvert$
	Edge value sequence of the edge of P_e	Edge value sequence of the edge of P_e'	Edge value sequence of the edge of P_e	Edge value sequence of the edge of P_e'	
0 and 0	$0\,(01)^{2r_1}\,0$	$1\,(01)^{2r_2}\,1$	$0\,(01)^{2r_1-1}\,1$	$1\,(01)^{2r_2-1}\,0$	$\lvert E_0(P_e \cup P_e')\rvert = \lvert E_1(P_e \cup P_e')\rvert$
0 and 1	$0\,(01)^{2r_1}\,1$	$1\,(01)^{2r_2}\,0$	$0\,(01)^{2r_1-1}\,0$	$1\,(01)^{2r_2-1}\,1$	$\lvert E_0(P_e \cup P_e')\rvert = \lvert E_1(P_e \cup P_e')\rvert$
1 and 0	$1\,(01)^{2r_1}\,0$	$0\,(01)^{2r_2}\,1$	$1\,(01)^{2r_1-1}\,1$	$0\,(01)^{2r_2-1}\,0$	$\lvert E_0(P_e \cup P_e')\rvert = \lvert E_1(P_e \cup P_e')\rvert$
1 and 1	$1\,(01)^{2r_1}\,1$	$0\,(01)^{2r_2}\,0$	$1\,(01)^{2r_1-1}\,0$	$0\,(01)^{2r_2-1}\,1$	$\lvert E_0(P_e \cup P_e')\rvert = \lvert E_1(P_e \cup P_e')\rvert$

4 Some general results on cordial graphs

In this section, we introduce some general results on cordial graphs.

Let K_n^* denote the graphs obtained from the complete graph K_n, by taking a new vertex v and joining v with $\lfloor \frac{n}{2} \rfloor$ vertices of K_n.

Theorem 4 : *The graph K_n^* is cordial.*

Proof : Assign label 1 to v and all of its $\lfloor \frac{n}{2} \rfloor$ adjacent vertices and label 0 to all the other remaining vertices of K_n^*. Let V_0 and V_1 respectively, denote the set of vertices of K_n^* assigned the label 0 and the set of vertices of K_n^* assigned the label 1 and let E_0 and E_1 respectively, denote the set of all edges of K_n^* getting the label 0 and the set of all edges of K_n^* getting the label 1.

Observe that when n is even, say $n = 2r$, then $|V_0| = r$ and $|V_1| = r + 1$ and $|E_0| = |E_1| = r^2$, while when n is odd, say $n = 2r + 1$, then $|V_0| = |V_1| = r + 1$ and $|E_0| = |E_1| = r^2 + r$. Hence K_n^* is cordial. □

Corollary 1 : *Every graph is a subgraph of a cordial graph.*

A cordial graph H is said to be *even cordial* if $|V_0(H)| = |V_1(H)|$ and $|E_0(H)| = |E_1(H)|$, where $V_0(H), V_1(H)$ are respectively the set of all vertices of H with label 0 and 1 and $E_0(H), E_1(H)$ are respectively the set of all edges of H with label 0 and 1.

Let H_1 and H_2 be two copies of an even cordial graph H with cordial labeling. *Same label joining of H_1 and H_2* is addition of $|V(H)|$ independent edges between the vertices of $V_0(H_1)$ and $V_0(H_2)$ and between the vertices of $V_1(H_1)$ and $V_1(H_2)$.

Similarly, *opposite joining of H_1 and H_2* is addition of $|V(H)|$ independent edges between the vertices $V_0(H_1)$ and $V_1(H_2)$ and between the vertices of $V_1(H_1)$ and $V_0(H_2)$.

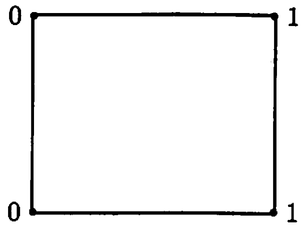


Figure 1. A graph H.

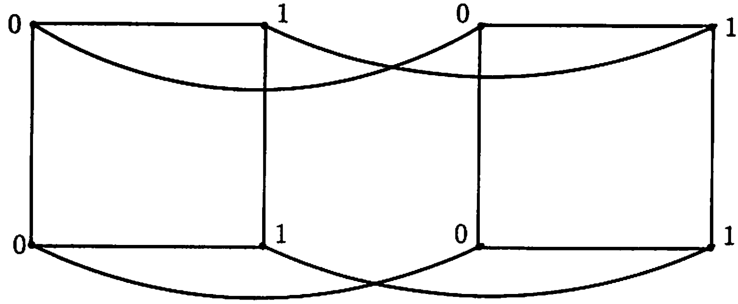


Figure 2. Same joining of H_1 and H_2.

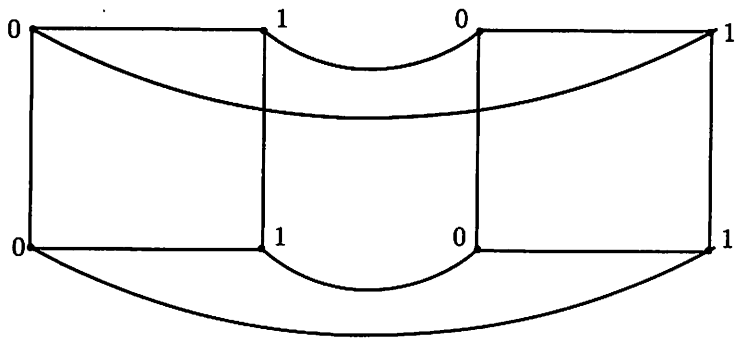


Figure 3. Opposite joining of H_1 and H_2.

Let H be an even-cordial graph and let G be any graph with even number of edges. Let $V(G) = \{v_1, v_2, \cdots, v_{|v(G)|}\}$ and $E(G) = \{e_1, e_2, \cdots, e_{2r}\}$ and let G^* be a graph obtained by taking $|V(G)|$ copies of H, each copy of H corresponds to a distinct vertex of G. Join of two copies of H by same join or opposite join depends on whether the edge joining the corresponding vertices of the copies in G is even suffixed or odd suffixed. Then G^* is cordial.

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