On the order of close to regular graphs without a matching of given size

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Abstract

A graph G is a $(d, d+k)$-graph, if the degree of each vertex of G is between d and $d+k$. Let $p \geq 0$ and $d, k \geq 2$ be integers. If G is a $(d, d+k)$-graph of order n with at most p odd components and without a matching M of size $2|M| = n - p$, then we show in this paper that
(i) $n \geq 2d + p + 2$ when $p \leq k - 2$,
(ii) $n \geq 2\lceil (d(p+2))/k \rceil + p + 2$ when $p \geq k - 1$.

Corresponding results for $0 \leq p \leq 1$ and $0 \leq k \leq 1$ were given by Wallis [6], Zhao [8], and Volkmann [5].

Examples will show that the given bounds (i) and (ii) are best possible.

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We shall assume that the reader is familiar with standard terminology on graphs (see, e.g., Chartrand and Lesniak [2]). In this paper, all graphs are finite and simple. The vertex set of a graph G is denoted by $V(G)$. The *neighborhood* $N_G(x) = N(x)$ of a vertex x is the set of vertices adjacent with x, and the number $d_G(x) = d(x) = |N(x)|$ is the *degree* of x in the graph G. If $d \leq d_G(x) \leq d + k$ for each vertex x in a graph G, then we speak of a *close to regular graph* or more precisely of a $(d, d+k)$-*graph*. If M is a matching in a graph G with the property that every vertex (with exactly one exception) is incident with an edge of M, then M is a *perfect* *matching* (an *almost perfect matching*). We denote by $K_{r,s}$ the complete bipartite graph with partite sets A and B, where $|A| = r$ and $|B| = s$. If G is a graph and $A \subseteq V(G)$, then we denote by $q(G-A)$ the number of odd components in the subgraph $G - A$.

As a generalization of a result by Wallis [6] (see also [7]), Zhao [8] proved in 1991 the following theorem.

Theorem 1 (Zhao [8] 1991) Let $d \geq 2$ be an integer. If a $(d, d+1)$-graph G has no odd component and no perfect matching, then

$$|V(G)| \geq 3d+4.$$

Theorem 1 follows easily from the next result by Volkmann [5].

Theorem 2 (Volkmann [5] 2004) Let $d \geq 2$ be an integer, and let G be a $(d, d+1)$-graph with exactly one odd component and without any almost perfect matching. Then

1) $|V(G)| \geq 4(d+1)+1$,

2) $|V(G)| \geq 4(d+1)+3$ when $d \geq 3$ is odd or $d = 2$ and G is connected,

3) $|V(G)| \geq 4(d+1)+5$ when $d \geq 3$ is odd and G is connected.

In [5] one can also find the following corresponding result for $(d, d+2)$-graphs.

Theorem 3 (Volkmann [5] 2004) If G is a $(d, d+2)$-graph with exactly one odd component and without any almost perfect matching, then

$$|V(G)| \geq 3d+3.$$

Instead of $(d, d+1)$-graphs or $(d, d+2)$-graphs, we investigate in this paper the general case of $(d, d+k)$-graphs for $k \geq 2$. Our main theorem (Theorem 4) is a supplement to Theorems 1 and 2 and an extension of Theorem 3. The proof of our main theorem is based on the following generalization of Tutte's famous 1-factor theorem [3] by Berge [1] in 1958, and we call it the Theorem of Tutte-Berge (for a proof see e.g., [4]).

Theorem of Tutte-Berge (Berge [1] 1958) Let G be a graph of order n. If M is a maximum matching of G, then

$$n - 2|M| = \max_{A \subseteq V(G)} \{q(G-A) - |A|\}.$$

Theorem 4 Let $p \geq 0$ and $d, k \geq 2$ be integers. If G is a $(d, d+k)$-graph of order n with at most p odd components and without any matching M of size $2|M| = n - p$, then

(i) $n \geq 2d + p + 2$ when $p \leq k - 2$,

(ii) $n \geq 2\lceil (d(p+2))/k \rceil + p + 2$ when $p \geq k - 1$.

Proof In view of the hypotheses, we observe that n and p are of the same parity. Suppose to the contrary that there exists a $(d, d+k)$-graph G with at most p odd components and without any matching M of size $2|M| = n - p$ such that

a) $n \leq 2d + p + 1$ when $p \leq k - 2$,

b) $n \leq 2\lceil (d(p+2))/k \rceil + p + 1$ when $p \geq k - 1$.

By the hypotheses and the Theorem of Tutte-Berge, there exists a non-empty set $A \subseteq V(G)$ such that $q(G-A) \geq |A| + p + 1$. However, since n and p are of the same parity, it is straightforward to verify that this even leads to the better bound $q(G - A) \geq |A| + p + 2$. We call an odd component of $G - A$ large if it has more than d vertices and small otherwise. If we denote by α and β the number of large and small components, respectively, then we deduce that

$$\alpha + \beta = q(G - A) \geq |A| + p + 2, \tag{1}$$

$$n \geq |A| + \beta + \alpha(d+1), \tag{2}$$

$$n \geq |A| + \beta + \alpha(d+2) \text{ when } d \geq 3 \text{ is odd.} \tag{3}$$

Firstly, we show that $\alpha \leq p + 1$. In the case that $p \leq k - 2$, it follows from assumption a) and inequality (2) that

$$2d + p + 1 \geq n \geq |A| + \beta + \alpha(d+1) \geq 1 + \alpha(d+1).$$

This leads to $(2 - \alpha)(d+1) + p - 2 \geq 0$ and thus $\alpha \leq p + 1$. In the other case that $p \geq k - 1$, we conclude from assumption b) and (2) that

$$2\left\lceil \frac{d(p+2)}{k} \right\rceil + p + 1 \geq n \geq |A| + \beta + \alpha(d+1) \geq 1 + \alpha(d+1).$$

This inequality chain yields

$$2\left(\frac{d(p+2)}{k} + \frac{k-1}{k} \right) + p - \alpha(d+1) \geq 0.$$

Because of $k \geq 2$, it is a simple matter to verify that this inequality implies $\alpha \leq p + 1$. Applying (1), we arrive at

$$\beta \geq |A| + 1. \tag{4}$$

Since G is a $(d, d+k)$-graph, it is easy to show that there are at least d edges of G joining each small component of $G - A$ with A. Therefore it follows from the hypothesis that G has at most p odd components that

$$\alpha - p + d\beta \leq |A|(d+k), \tag{5}$$

$$d\beta \leq |A|(d+k) \text{ when } \alpha \leq p. \tag{6}$$

Case 1. Assume that $p \leq k-2$.

If $|A| \geq d$, then inequalities (1) and (2) lead to the following contradiction to assumption a):

$$\begin{aligned} n &\geq |A| + \beta + \alpha(d+1) \\ &\geq |A| + |A| + p + 2 - \alpha + \alpha(d+1) \\ &\geq 2d + p + 2 + \alpha d \\ &\geq 2d + p + 2 \end{aligned}$$

Let U be a small component of $G-A$. Since $N(x) \subseteq V(U) \cup A$ for $x \in V(U)$, we observe that $|A| + |V(U)| \geq d+1$. If $|A| < d$, say $|A| = d - t$ with $1 \leq t \leq d-1$, then we deduce that each small component U contains at least $t+1$ vertices. Thus (1) implies that

$$\begin{aligned} n &\geq |A| + \beta(t+1) + \alpha(d+1) \\ &\geq d - t + (|A| + p + 2 - \alpha)(t+1) + \alpha(d+1) \\ &= 2d + p + 2 + (d - t + p - \alpha)t + \alpha d. \end{aligned}$$

Because of $\alpha \leq p+1$ and $t \leq d-1$, this leads to $n \geq 2d+p+2$, a contradiction to assumption a).

Case 2. Assume that $p \geq k-1$ and $\alpha = 0$.

From inequalities (6) and (1), we deduce that

$$|A|(d+k) \geq d\beta \geq d(|A| + p + 2).$$

This yields $|A|k \geq d(p+2)$ and thus $|A| \geq \lceil (d(p+2))/k \rceil$. Combining this with (1) and (2), we arrive at the following contradiction to assumption b):

$$\begin{aligned} n &\geq |A| + \beta \geq |A| + |A| + p + 2 \\ &\geq 2\left\lceil \frac{d(p+2)}{k} \right\rceil + p + 2 \end{aligned}$$

Case 3. Assume that $p \geq k-1$ and $\alpha \geq 1$.

We note that inequality (5) is equivalent to

$$\beta \leq |A| + \frac{p + k\beta - \alpha}{d+k}. \tag{7}$$

Subcase 3.1. Assume that $p + k\beta - \alpha \le d + k - 1$. It follows from (7) that $\beta \le |A|$, a contradiction to (4).

Subcase 3.2. Assume that $p + k\beta - \alpha \ge d + k$. This implies that $k\beta \ge d + k - p + \alpha$. Combining this with (5), we obtain

$$|A| \ge \frac{\alpha - p + d\beta}{d+k} \ge \frac{\alpha - p + \frac{d}{k}(d + k - p + \alpha)}{d+k}.$$

For $\alpha = p + 1$, this yields $|A| \ge (d+1)/k$ and hence (2) leads to

$$\begin{aligned} n &\ge |A| + \beta + \alpha(d+1) \\ &\ge \frac{d+1}{k} + \frac{d+k+1}{k} + (p+1)(d+1) \\ &= \frac{2d+2}{k} + d(p+1) + p + 2. \end{aligned} \tag{8}$$

Since $p \ge 0$ and $d, k \ge 2$, we observe that $k(pd + d - 2) \ge 2(pd + d - 2)$, and this is equivalent to

$$\frac{2d+2}{k} + d(p+1) \ge 2\frac{d(p+2) + (k-1)}{k}.$$

Combining this inequality with (8), we arrive at a contradiction to assumption b) as follows:

$$\begin{aligned} n &\ge \frac{2d+2}{k} + d(p+1) + p + 2 \\ &\ge 2\frac{d(p+2) + (k-1)}{k} + p + 2 \\ &\ge 2\left\lceil \frac{d(p+2)}{k} \right\rceil + p + 2 \end{aligned}$$

Assume next that $1 \le \alpha \le p$, and let $\alpha = p - s$ with $0 \le s \le p - 1$. We deduce from (1) that $\beta \ge |A| + s + 2$, and this yields together with (6) the inequality

$$|A| \ge \left\lceil \frac{d(s+2)}{k} \right\rceil \tag{9}$$

Subcase 3.2.1. Assume that $k = 2$ and that $d \ge 2$ is even. In this case (2) and (9) lead to the following contradiction to b):

$$\begin{aligned} n &\ge |A| + \beta + \alpha(d+1) \\ &\ge 2|A| + s + 2 + (p-s)(d+1) \\ &\ge 2\frac{d(s+2)}{2} + s + 2 + (p-s)(d+1) \\ &= d(p+2) + p + 2 \\ &= 2\left\lceil \frac{d(p+2)}{2} \right\rceil + p + 2 \end{aligned}$$

Subcase 3.2.2. Assume that $k = 2$ and that $d \geq 3$ and s are odd. Since d is odd, (3) and (9) yield the following contradiction to b):

$$\begin{aligned} n &\geq |A| + \beta + \alpha(d+2) \\ &\geq 2|A| + s + 2 + (p-s)(d+2) \\ &\geq 2\frac{d(s+2)+1}{2} + s + 2 + (p-s)(d+2) \\ &= d(p+2) + p + 3 + p - s \\ &\geq d(p+2) + p + 4 \\ &\geq 2\left\lceil \frac{d(p+2)}{2} \right\rceil + p + 2 \end{aligned}$$

Subcase 3.2.3. Assume that $k = 2$ and that $d \geq 3$ is odd and that s is even. Combining (3) and (9), we arrive at the following contradiction to b):

$$\begin{aligned} n &\geq |A| + \beta + \alpha(d+2) \\ &\geq 2|A| + s + 2 + (p-s)(d+2) \\ &\geq d(s+2) + s + 2 + (p-s)(d+2) \\ &= d(p+2) + 2p + 2 - s \\ &\geq d(p+2) + p + 3 \\ &= 2\frac{d(p+2)+1}{2} + p + 2 \\ &\geq 2\left\lceil \frac{d(p+2)}{2} \right\rceil + p + 2 \end{aligned}$$

Subcase 3.2.4. Assume that $k \geq 3$. It follows from (2) and (9) that

$$\begin{aligned} n &\geq |A| + \beta + \alpha(d+1) \\ &\geq 2|A| + s + 2 + (p-s)(d+1) \\ &\geq 2\left\lceil \frac{d(s+2)}{k} \right\rceil + (p-s)d + p + 2. \end{aligned} \tag{10}$$

To receive a contradiction to assumption b), it thus remains to show that

$$2\frac{d(s+2)}{k} + (p-s)d \geq 2\frac{d(p+2)+k-1}{k},$$

and this is equivalent to

$$d \geq \frac{2(k-1)}{(p-s)(k-2)}. \tag{11}$$

If $s \le p-2$, then $k \ge 3$ implies

$$\frac{2(k-1)}{(p-s)(k-2)} \le \frac{k-1}{k-2} \le 2 \le d$$

and (11) is valid. If $s = p-1$ and $d \ge 4$, then it is easy to see that (11) is also true. In the remaining case that $k \ge 3$, $s = p-1$, and $2 \le d \le 3$, we deduce that

$$\begin{aligned}
2\left\lceil \frac{d(p+2)}{k} \right\rceil &= 2\left\lceil \frac{d(p+1)}{k} + \frac{d}{k} \right\rceil \\
&\le 2\left\lceil \frac{d(p+1)}{k} \right\rceil + 2\left\lceil \frac{d}{k} \right\rceil \\
&= 2\left\lceil \frac{d(p+1)}{k} \right\rceil + 2 \\
&\le 2\left\lceil \frac{d(p+1)}{k} \right\rceil + d \\
&= 2\left\lceil \frac{d(s+2)}{k} \right\rceil + (p-s)d.
\end{aligned}$$

Combining this inequality chain with (10), we finally obtain a contradiction to b).

Since we have discussed all possible cases, the proof of Theorem 4 is complete. □

The following examples show that the bounds in Theorem 4 are best possible.

Example 5 *Case 1.* Assume that $p+2 \le k$. In this case, the complete bipartite graph $K_{d,d+p+2}$ is a $(d, d+k)$-graph of order $n = 2d+p+2$ without a matching M of size $2|M| = n-p$. Consequently, Condition (i) is best possible.

Case 2. Assume that $p+2 \ge k+1$. Let H be a d-regular bipartite graph with the partite sets $X = \{x_1, x_2, \ldots, x_{\lceil \frac{d(p+2)}{k} \rceil}\}$ and $Y = \{y_1, y_2, \ldots, y_{\lceil \frac{d(p+2)}{k} \rceil}\}$. Now let G consists of H and $p+2$ additional vertices $u_1, u_2, \ldots, u_{p+2}$, which are connected with X by $d(p+2)$ edges such that $d_G(u_i) = d$ for $i = 1, 2, \ldots, p+2$ and $|d_G(x_i) - d_G(x_j)| \le 1$ for $1 \le i, j \le \lceil \frac{d(p+2)}{k} \rceil$. Now G is a $(d, d+k)$-graph of order $2\lceil \frac{d(p+2)}{k} \rceil + p + 2$ without a matching M of size $2|M| = n-p$. This example shows that Condition (ii) is also best possible.

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