An extension of Seller's theorem on partitions and its application

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Abstract: Let $\mathcal{K} = (K_{ij})$ be an infinite lower triangular matrix of non-negative integers such that $K_{i0} = 1$ and $K_{ii} \geq 1$ for $i \geq 0$. Define a sequence $\{V_m(\mathcal{K})\}_{m\geq 0}$ by the recurrence $V_{m+1}(\mathcal{K}) = \sum_{j=0}^{m} K_{mj}V_j(\mathcal{K})$ with $V_0(\mathcal{K}) = 1$. Let $P(n;\mathcal{K})$ be the number of partitions of n of the form $n = p_1 + p_2 + p_3 + p_4 + \cdots$ such that $p_j \geq \sum_{i\geq j} K_{ij}p_{i+1}$ for $j \geq 1$ and let $\mathrm{P}(n; V(\mathcal{K}))$ denote the number of partitions of n into summands in the set $V(\mathcal{K}) = \{V_1(\mathcal{K}), V_2(\mathcal{K}), \dots\}$. Based on the technique of MacMahon's partitions analysis, we prove that $P(n;\mathcal{K}) = \mathrm{P}(n; V(\mathcal{K}))$ which generalizes a recent result of Sellers'. We also give several applications of this result to many classical sequences such as Bell numbers, Fibonacci numbers, Lucas numbers and Pell numbers.

1. Introduction

A *partition* of n is of the form $n = p_1 + p_2 + p_3 + p_4 + \cdots$ with $p_1 \geq p_2 \geq p_3 \geq p_4 \geq \cdots \geq 0$. There are many interesting partition identities in the theory of partitions [3]. The most celebrated Euler's identity states as follows:

Theorem 1.1 *Let $o(n)$ be the number of partitions of n into odd parts and $d(n)$ be the number of partitions of n into distinct parts. Then, for all $n \geq 0$, $o(n) = d(n)$.*

Recently, Santos [14] proved bijectively that $o(n)$ also equals the number of partitions of n of the form $p_1 + p_2 + p_3 + p_4 + \cdots$ such that $p_1 \geq p_2 \geq p_3 \geq p_4 \geq \cdots \geq 0$ and $p_1 \geq 2p_2 + p_3 + p_4 + \cdots$. Later, by the technique

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of MacMahon's partitions analysis [13], Sellers [15, 16] obtained a more general result, namely,

Theorem 1.2 *Let $A = (a_2, a_3, a_4, \cdots)$ be an infinite vector of nonnegative integers with $a_2 \geq 1$. Define $p(n; A)$ as the number of partitions of n of the form $p_1 + p_2 + p_3 + p_4 + \cdots$ with $p_1 \geq p_2 \geq p_3 \geq p_4 \cdots \geq 0$ and $p_1 \geq a_2p_2 + a_3p_3 + a_4p_4 + \cdots$. Then, for all $n \geq 0$, $p(n; A)$ equals the number of partitions of n whose parts must be 1's or of the form $\left(\sum_{i=2}^{m} a_i\right) + (m-1)$ for some integer $m \geq 2$.*

More recently, a refinement of Sellers' result has been obtained by [11]. Our main theorem reads as follows:

Theorem 1.3 *Let $\mathcal{K} = (K_{ij})$ be an infinite lower triangular matrix of nonnegative integers such that $K_{i0} = 1$ and $K_{ii} \geq 1$ for $i \geq 0$. Define a sequence $\{V_m(\mathcal{K})\}_{m \geq 0}$ by the recurrence $V_{m+1}(\mathcal{K}) = \sum_{j=0}^{m} K_{mj} V_j(\mathcal{K})$ with $V_0(\mathcal{K}) = 1$. Let $P(n; \mathcal{K})$ be the number of partitions of n of the form $n = p_1 + p_2 + p_3 + p_4 + \cdots$ such that $p_j \geq \sum_{i \geq j} K_{ij} p_{i+1}$ for $j \geq 1$ and let $\mathrm{P}(n; V(\mathcal{K}))$ denote the number of partitions of n into summands in the set $V(\mathcal{K}) = \{V_1(\mathcal{K}), V_2(\mathcal{K}), \ldots\}$. Then*

$$\sum_{n \geq 0} P(n; \mathcal{K}) q^n = \prod_{m \geq 1} \frac{1}{1 - q^{V_m(\mathcal{K})}} = \sum_{n \geq 0} \mathrm{P}(n; V(\mathcal{K})) q^n.$$

In this paper, based on the technique of MacMahon's partitions analysis, we prove this result in the next Section. And in Section 3, we give several applications of this result to many classical sequences such as Bell numbers, Fibonacci numbers, Lucas numbers and Pell numbers.

2. The Proof of Main Theorem

Before proving our main theorem, we briefly recall the technique of MacMahon's partitions analysis, which has been exploited further by G. Andrews, P. Paule, A. Riese and other authors [1, 2, 4, 5, 6, 7, 8, 9, 10].

Definition 2.1 *The Omega operator $\underset{\geq}{\Omega}$ is defined by*

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_j=-\infty}^{\infty} T_{s_1,\ldots,s_j} \lambda_1^{s_1} \cdots \lambda_j^{s_j} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_j=0}^{\infty} T_{s_1,\ldots,s_j},$$

where the domain of the $T_{s_1,\ldots,s_j}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to annuli of the form $1 - \varepsilon < |\lambda_i| < 1 + \varepsilon$.

According to this definition, it is easy to show the following lemma which plays an important role in the paper.

Lemma 2.2

$$\Omega \frac{1}{\left(1-\lambda x\right)\left(1-\frac{y}{\lambda}\right)} = \frac{1}{(1-x)(1-xy)}.$$

Now we give the proof of our result.

Proof. By the definition of the Omega operator, applying Lemma 2.2 multiple times to λ_i for $i \geq 1$, we obtain that

$$\begin{aligned}
\sum_{n=0}^{\infty} P(n;\mathcal{K})q^n &= \sum_{\substack{p_1 \geq p_2 \geq \cdots \geq p_j \geq \cdots \geq 0 \\ p_j \geq \sum_{i\geq j} K_{ij} p_{i+1}}} q^{p_1+p_2+p_3+\cdots} \\
&= \Omega \sum_{p_1, p_2, p_3, \cdots \geq 0} q^{\sum_{i\geq 1} p_i} \Big(\prod_{i\geq 1} \lambda_i^{p_i - p_{i+1}}\Big)\Big(\prod_{j\geq 1} \mu_j^{p_j - \sum_{i\geq j} K_{ij} p_{i+1}}\Big) \\
&= \Omega \frac{\prod_{i\geq 3}\left(1-\frac{q\lambda_{i+1}\mu_{i+1}}{\lambda_i \mu_1^{K_{i1}} \mu_2^{K_{i2}} \cdots \mu_i^{K_{ii}}}\right)^{-1}}{\left(1-q\lambda_1\mu_1\right)\left(1-\frac{q\lambda_2\mu_2}{\lambda_1\mu_1^{K_{11}}}\right)\left(1-\frac{q\lambda_3\mu_3}{\lambda_2\mu_1^{K_{21}}\mu_2^{K_{22}}}\right)} \\
&= \Omega \frac{\prod_{i\geq 3}\left(1-\frac{q\lambda_{i+1}\mu_{i+1}}{\lambda_i \mu_1^{K_{i1}} \mu_2^{K_{i2}} \cdots \mu_i^{K_{ii}}}\right)^{-1}}{\left(1-q\mu_1\right)\left(1-\frac{q^2\lambda_2\mu_2}{\mu_1^{K_{11}-1}}\right)\left(1-\frac{q\lambda_3\mu_3}{\lambda_2\mu_1^{K_{21}}\mu_2^{K_{22}}}\right)} \\
&= \Omega \frac{\prod_{i\geq 3}\left(1-\frac{q\lambda_{i+1}\mu_{i+1}}{\lambda_i \mu_1^{K_{i1}} \mu_2^{K_{i2}} \cdots \mu_i^{K_{ii}}}\right)^{-1}}{\left(1-q\mu_1\right)\left(1-\frac{q^2\mu_2}{\mu_1^{K_{11}-1}}\right)\left(1-\frac{q^3\lambda_3\mu_3}{\mu_1^{K_{11}+k_{21}-1}\mu_2^{K_{22}-1}}\right)}
\end{aligned}$$

Continuing to apply Lemma 2.2 to annihilate all parameters λ_i, we have

$$\sum_{n\geq 0} P(n;\mathcal{K})q^n = \Omega \frac{\left(\frac{1}{1-q\mu_1}\right)\left(\frac{1}{1-\frac{q^2\mu_2}{\mu_1^{K_{11}-1}}}\right)}{\prod_{j\geq 2}\left(1-\frac{q^{j+1}\mu_{j+1}}{\prod_{i=1}^{j}\mu_i^{-1+\sum_{k\geq i}^{j} K_{ki}}}\right)} \tag{2.1}$$

Now expanding (2.1) in terms of geometric series and utilizing the Omega operator to eliminate μ_i for $i \geq 1$, we attain,

$$
\begin{aligned}
\sum_{n\geq 0} P(n;\mathcal{K})q^n = \Omega \sum_{t_1\geq 0}(q\mu_1)^{t_1} \sum_{t_2\geq 0}(q^2\mu_1^{-K_{11}+1}\mu_2)^{t_2} \\
\cdot\prod_{j\geq 2}\sum_{t_{j+1}\geq 0}\left(q^{j+1}\mu_{j+1}\prod_{i=1}^{j}\mu_i^{1-\sum_{k\geq i}^{j}K_{ki}}\right)^{t_{j+1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \Omega \sum_{t_1,t_2,t_3,\cdots\geq 0} q^{t_1+2t_2+3t_3+\cdots}\prod_{j\geq 1}\mu_j^{t_j+\sum_{i\geq j}(1-\sum_{k\geq j}^{i}K_{kj})t_{i+1}} \\
&= \Omega \sum_{t_2,t_3,\cdots\geq 0} q^{2t_2+3t_3+\cdots}\prod_{j\geq 2}\mu_j^{t_j+\sum_{i\geq j}(1-\sum_{k\geq j}^{i}K_{kj})t_{i+1}} \\
&\qquad \cdot\sum_{t_1\geq 0} q^{t_1}\mu_1^{t_1+\sum_{i\geq 1}(1-\sum_{k\geq 1}^{i}K_{k1})t_{i+1}} \\
&= \Omega \sum_{t_2,t_3,\cdots\geq 0} q^{2t_2+3t_3+\cdots}\prod_{j\geq 2}\mu_j^{t_j+\sum_{i\geq j}(1-\sum_{k\geq j}^{i}K_{kj})t_{i+1}} \\
&\qquad \cdot \sum_{\substack{t_1\geq 0\\ t_1\geq \sum_{i\geq 1}(-1+\sum_{k\geq 1}^{i}K_{k1})t_{i+1}}} q^{t_1} \\
&= \Omega \sum_{t_2,t_3,\cdots\geq 0} q^{2t_2+3t_3+\cdots}\prod_{j\geq 2}\mu_j^{t_j+\sum_{i\geq j}(1-\sum_{k\geq j}^{i}K_{kj})t_{i+1}} \\
&\qquad \cdot\frac{q^{\sum_{i\geq 1}(-1+\sum_{k\geq 1}^{i}K_{k1})t_{i+1}}}{1-q} \\
&= \frac{1}{(1-q)(1-q^{1+K_{11}})(1-q^{1+K_{21}+(1+K_{11})K_{22}})\dots} \\
&= \prod_{m\geq 1}\frac{1}{1-q^{V_m(\mathcal{K})}} = \sum_{n\geq 0}\mathrm{P}(n;V(\mathcal{K}))q^n,
\end{aligned}
$$

where $V_{m+1}(\mathcal{K}) = \sum_{j=0}^{m} K_{mj}V_j(\mathcal{K})$ with $V_0(\mathcal{K}) = 1$.

Now we take the step of eliminating μ_{m+1} into detailed consideration,

$$\Omega \sum_{t_{m+1}\geq 0} q^{(m+1)t_{m+1}} \prod_{j\geq 1}^{m} q^{V_j(\mathcal{K})(-1+\sum_{k\geq j}^{m} K_{kj})t_{m+1}}$$

$$\cdot \mu_{m+1}^{t_{m+1}+\sum_{i\geq m+1}(1-\sum_{k\geq m+1}^{i} K_{k(m+1)})t_{i+1}}$$

$$= \Omega \sum_{t_{m+1}\geq 0} q^{\left((m+1)+\sum_{j\geq 1}^{m} V_j(\mathcal{K})(-1+\sum_{k\geq j}^{m} K_{kj})\right)t_{m+1}}$$

$$\cdot \mu_{m+1}^{t_{m+1}+\sum_{i\geq m+1}(1-\sum_{k\geq m+1}^{i} K_{k(m+1)})t_{i+1}}$$

$$= \frac{q^{\left((m+1)+\sum_{j\geq 1}^{m} V_j(\mathcal{K})(-1+\sum_{k\geq j}^{m} K_{kj})\right)\left(\sum_{i\geq m+1}(-1+\sum_{k\geq m+1}^{i} K_{k(m+1)})t_{i+1}\right)}}{1-q^{\left((m+1)+\sum_{j\geq 1}^{m} V_j(\mathcal{K})(-1+\sum_{k\geq j}^{m} K_{kj})\right)}}.$$

Then

$$\begin{aligned}
V_{m+1}(\mathcal{K}) &= (m+1)+\sum_{j\geq 1}^{m} V_j(\mathcal{K})(-1+\sum_{k\geq j}^{m} K_{kj}) \\
&= 1+\left(m+\sum_{j\geq 1}^{m-1} V_j(\mathcal{K})(-1+\sum_{k\geq j}^{m-1} K_{kj})\right) \\
&\quad +\sum_{j=1}^{m-1} V_j(\mathcal{K})K_{mj}+(K_{mm}-1)V_m(\mathcal{K}) \\
&= 1+V_m(\mathcal{K})+\sum_{j=1}^{m-1} V_j(\mathcal{K})K_{mj}+(K_{mm}-1)V_m(\mathcal{K}) \\
&= \sum_{j=0}^{m} K_{mj}V_j(\mathcal{K})
\end{aligned}$$

Hence the result holds. □

3. Applications

In this section, we give several applications of our main theorem to many classical sequences such as Bell numbers, Fibonacci numbers, Lucas numbers and Pell numbers.

First, define an infinite lower triangular matrix $\mathcal{K} = (K_{ij})$ with $K_{11} \geq 1$

and

$$K_{ij} = \begin{cases} 1 & \text{if } j = 0, \text{ or } i = j \geq 2, \\ K_{i1} & \text{if } j = 1 \text{ and } i \geq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

Using the recurrence relation

$$V_{m+1}(\mathcal{K}) = \sum_{j=0}^{m} K_{mj} V_j(\mathcal{K}), \qquad V_0(\mathcal{K}) = 1, \tag{3.2}$$

we get

$$V_{m+1}(\mathcal{K}) = m + \delta_{0m} + \sum_{i=1}^{m} K_{i1}, \quad (m \geq 0),$$

where δ_{0m} is the Kronecker symbol, which is Sellers' result. Clearly, in addition, setting $K_{11} = 2, K_{i1} = 1$ for $i \geq 2$ in (3.1), we obtain Santos' result. Other special cases have been given in [15], including an interesting case related to graphic forest partitions [12], at this point.

Next, we list nine interesting corollaries of Theorem 1.3 in the following table according to different $\mathcal{K}$.

$\mathcal{K} = (K_{ij})$	$V_m(\mathcal{K})$ $(m \geq 1)$
$\binom{i}{j}$	B_m
$\begin{cases} 1 & \text{if } j=0, \\ u-1 & \text{if } i \geq j \geq 1, \\ 0 & \text{otherwise.} \end{cases}$	u^{m-1}
$\begin{cases} 1 & \text{if } j=0, \\ j & \text{if } i \geq j \geq 1, \\ 0 & \text{otherwise.} \end{cases}$	$m!$
$\begin{cases} 2 & \text{if } i=j \geq 1, \\ 0 & \text{if } i<j, \\ 1 & \text{otherwise.} \end{cases}$	F_{2m-1}
$\begin{cases} 2 & \text{if } i=j \geq 2, \\ 0 & \text{if } i<j, \text{ or } i \geq 2 \text{ and } j=1, \\ 1 & \text{otherwise.} \end{cases}$	F_{2m-2}
$\begin{cases} 1 & \text{if } j=0, \text{ or } i-j \equiv 0 \pmod 2 \text{ and } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$	F_m
$\begin{cases} 1 & \text{if } j=0, \\ u-1 & \text{if } 1 \leq j \leq k, \\ i-j+1 & \text{if } i \geq j \geq k+1, \\ 0 & \text{otherwise.} \end{cases}$	$\begin{cases} u^k F_{2m-2k-2} & \text{if } m \geq k+1 \\ u^{m-1} & \text{if } 1 \leq m \leq k \end{cases}$
$\begin{cases} 1 & \text{if } j=0, \text{ or } i=j, \\ 0 & \text{if } i<j, \\ 2 & \text{otherwise.} \end{cases}$	P_m
$\begin{cases} 1 & \text{if } j=0, \text{ or } i-j \equiv 0 \pmod 2 \text{ and } i \geq j \geq 2, \\ 2 & \text{if } i-j \equiv 0 \pmod 2 \text{ and } i \geq j=1, \\ 0 & \text{otherwise.} \end{cases}$	L_m

where B_m is the mth Bell number defined by $B_{m+1} = \sum_{k=0}^{m} \binom{m}{k} B_k$ with $B_0 = 1$; F_m is the mth Fibonacci number defined by $F_{m+1} = F_m + F_{m-1}$ with $F_0 = F_1 = 1$; P_m is the mth Pell number defined by $P_{m+1} = 2P_m + P_{m-1}$ with $P_1 = 1, P_1 = 2$; L_m is the mth Lucas number defined by $L_{m+1} = L_m + L_{m-1}$ with $L_1 = 1, L_2 = 3$.

It seems that some entries of $\mathcal{K}$ could be negative integers. For example, define another infinite lower triangular matrix $\mathcal{K}^{\varsigma} = (K^{\varsigma}_{ij})$ with

$$K^{\varsigma}_{ij} = \begin{cases} 1 & \text{if } j = 0, \\ 2 & \text{if } i = j \geq 1, \\ -1 & \text{if } i - j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By (3.2), we obtain $V_m(\mathcal{K}^{\varsigma}) = \binom{m+1}{2}$ for $m \geq 1$. For this case, Andrews [2] has proved the following:

Theorem 3.1 *Let $P(n; \mathcal{K}^{\varsigma})$ be the number of partitions of n of the form $n = p_1 + p_2 + p_3 + p_4 + \cdots$ such that $p_j - 2p_{j+1} + p_{j+2} \geq 0$ for $j \geq 1$. Then*

$$\sum_{n \geq 0} P(n; \mathcal{K}^{\varsigma}) q^n = \prod_{m \geq 1} \frac{1}{1 - q^{\binom{m+1}{2}}}.$$

Namely, $P(n; \mathcal{K}^{\varsigma})$ also equals the number of partitions of n into triangular numbers.

Our goal in future work is to consider a general principle so that K can be extended to include negative integer entries.

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