Super Vertex-magic Total Labelings of $W_{3,n}$

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Abstract. A graph G is called super vertex-magic total labelings if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(v) + \sum f(vu) = C$ where the sum is over all vertices u adjacent to v and $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$, $f(E(G)) = \{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)|\}$. *The Knödel graphs* $W_{\Delta,n}$ have even $n \geq 2$ vertices and degree Δ, $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$. The vertices of $W_{\Delta,n}$ are the pairs (i, j) with $i = 1, 2$ and $0 \leq j \leq n/2 - 1$. For every j, $0 \leq j \leq n/2 - 1$, there is an edge between vertex $(1, j)$ and every vertex $(2, (j + 2^k - 1) \bmod (n/2))$, for $k = 0, ..., \Delta - 1$. In this paper, we show that $W_{3,n}$ is super vertex-magic for $n \equiv 0 \bmod 4$.

Keywords. *Knödel graphs*, super vertex-magic total labeling, vertex labeling, edge labeling

1 Introduction

Let $G = (V, E)$ be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $p = |V(G)|$, $q = |E(G)|$ be the number of vertices and edges of G, respectively. A connected graph $G = (V, E)$ is said to be an vertex-magic labeling if there exist a constant C and a bijection $f : E \to \{p + 1, p + 2, \ldots, p + q\}$ such that the induced mapping

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$g_f : V \to N$, defined by $g_f(v) = C - \sum f(uv)$, $uv \in E(G)$, is injective and $g_f(V) = \{1, 2, \ldots, p\}$. In this case f is called an vertex-magic labeling of G.

MacDougall, Miller, Slamin and Wallis [10] introduced the notion of a vertex-magic total labeling in 1999. Miller, Bača and MacDougall [12] have proved that the generalized Petersen graphs $P(n, k)$ are vertex-magic total when n is even and $k \leq n/2 - 1$. They conjecture that all $P(n, k)$ are vertex-magic total when $k \leq (n-1)/2$. Bača, Miller and Slamin [1] proved the conjecture.

MacDougall, Miller and Sugeng [11] show: C_n has s super vertex-magic total labeling if and only if n is odd, and no wheel, ladder, fan, friendship graph, complete bipartite graph or graph with a vertex of degree 1 has a super vertex-magic total labeling. They conjecture that no tree has a super vertex-magic total labeling and that K_{4n} has a super vertex-magic labeling when $n > 1$. In [6], Gómez proves the conjecture: If $n \equiv 0 \bmod 4$, $n > 4$, then K_n has a super vertex-magic total labeling. Moreover, there are some works that present several methods to obtain super VMTL of graphs from graphs that admit super VMTLs. For instance, P. Kovář presented a method for constructing super vertex-magic total labelings of graphs at IWOGL held in Herlany 2005 [9]; hitherto unpulished. To be more precise, Kovář proved: Let $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, $G = (V, E)$ be graphs such that $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E$. If G_1 admits a super VMTL and G_2 is a regular graph of even degree, then G admits a super VMTL. In [7], two methods to obtain super VMTL of graphs, obtained from graphs that admit a super VMTL, are presented. For the literature on super edge-magic graphs we refer to [5] and the relevant references given in it.

The *Knödel graphs* $W_{\Delta,n}$, introduced in 1975 by Knödel[8] and formally defined[2] in 2001, have even $n \geq 2$ vertices and degree Δ, $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$. The vertices of $W_{\Delta,n}$ are the pairs (i, j) with $i = 1, 2$ and $0 \leq j \leq n/2 - 1$. For every j, $0 \leq j \leq n/2 - 1$, there is an edge between vertex $(1, j)$ and every vertex $(2, (j + 2^k - 1) \bmod (n/2))$, for $k = 0, \ldots, \Delta - 1$.

For $W_{\Delta,n}$, let v_j represent vertex $(1, j)$ and u_j represent vertex $(2, j)$. In this paper, the vertex labels are read modulo $n/2$ unless specified otherwise. From the definition of the Knödel graph, for $\Delta = 3$ and even $n \geq 8$, we have

$$
\begin{aligned}
V(W_{3,n}) &= \{v_0, v_1, \cdots, v_{n/2-1}, u_0, u_1, \cdots, u_{n/2-1}\}, \\
E(W_{3,n}) &= \bigcup_{i=0}^{n/2-1} \{v_i u_i, v_i u_{i+1}, v_i u_{i+3}\}.
\end{aligned}
$$

Figure 1.1 shows Knödel graphs $W_{3,8}$ and $W_{3,14}$.

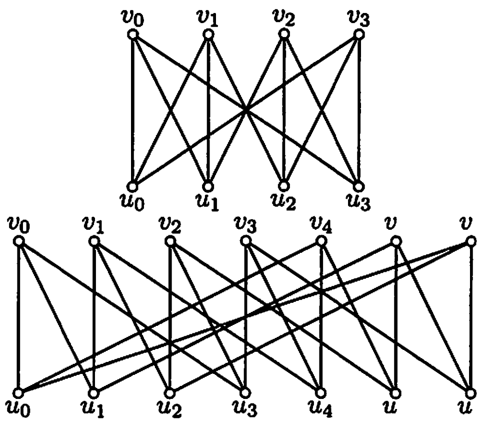

Figure 1.1 The Knödel graphs $W_{3,8}$ and $W_{3,14}$

The *Knödel graphs* $W_{\Delta,n}$ are regular graphs of even order n and degree $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$ that have been widely studied[3][4]. Since $W_{3,n}$ is a 3-regular graph with n vertices and $3n/2$ edges, according to the definition of super vertex-magic total labeling graph, $W_{3,n}$ is super vertex-magic only if $n \equiv 0 \bmod 4$, and the magic constant C is $23n/4 + 2$. In this paper, we show that $W_{3,n}$ is super vertex-magic for $n \equiv 0 \bmod 4$.

2 Main Result

Theorem 2.1 $W_{3,n}$ is super vertex-magic for $n \equiv 0 \bmod 4$.

Proof. We give a supervertex-magic total labeling of $W_{3,8}$ shown in Figure 2.1.

We define the edge labeling f of $W_{3,n}$ for $n \equiv 0 \bmod 4$ as follows:

$$
f(v_i u_i) = \begin{cases} n + 1 + i/2, & 0 \leq i \leq n/2 - 2 \wedge i \bmod 2 = 0, \\ (3n - 1 + i)/2, & 1 \leq i \leq n/2 - 1 \wedge i \bmod 2 = 1. \end{cases}
$$

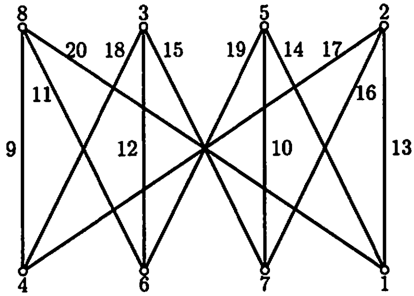

Figure 2.1 A super vertex-magic labeling for $W_{3,8}$

$$f(v_iu_{i+1}) = \begin{cases} 5n/4+1+i/2, & 0 \le i \le n/2-4 \wedge i \bmod 2 = 0, \\ 2n-(1+i)/2, & 1 \le i \le n/2-3 \wedge i \bmod 2 = 1, \\ 7n/4, & i = n/2-2, \\ 2n+1, & i = n/2-1. \end{cases}$$

$$f(v_iu_{i+3}) = \begin{cases} (5n-i)/2, & 0 \le i \le n/2-2 \wedge i \bmod 2 = 0, \\ 2n+(3+i)/2, & 1 \le i \le n/2-3 \wedge i \bmod 2 = 1, \\ 2n, & i = n/2-1. \end{cases}$$

Now we verify that f is a bijection from the edge set $E(W_{3,n})$ onto $\{n+1, n+2, \ldots, 5n/2\}$.

Denote by

$$S_1 = \{f(v_iu_i) | 0 \le i \le n/2-1\},$$
$$S_2 = \{f(v_iu_{i+1}) | 0 \le i \le n/2-1\},$$
$$S_3 = \{f(v_iu_{i+3}) | 0 \le i \le n/2-1\}.$$

Then

$$\begin{aligned} S_1 &= S_{11} \cup S_{12}, \\ S_{11} &= \{f(v_iu_i) | 0 \le i \le n/2-2 \wedge i \bmod 2 = 0\} \\ &= \{n+1+i/2 | 0 \le i \le n/2-2 \wedge i \bmod 2 = 0\} = \{n+1, n+2, \ldots, 5n/4\}, \\ S_{12} &= \{f(v_iu_i) | 1 \le i \le n/2-1 \wedge i \bmod 2 = 1\} \\ &= \{(3n-1+i)/2 | 1 \le i \le n/2-1 \wedge i \bmod 2 = 1\} \\ &= \{3n/2, 3n/2+1, \ldots, 7n/4-1\}, \\ S_2 &= S_{21} \cup S_{22} \cup S_{23} \cup S_{24}, \\ S_{21} &= \{f(v_iu_{i+1}) | 0 \le i \le n/2-4 \wedge i \bmod 2 = 0\} \\ &= \{5n/4+1+i/2 | 0 \le i \le n/2-4 \wedge i \bmod 2 = 0\} \\ &= \{5n/4+1, 5n/4+2, \ldots, 3n/2-1\}, \\ S_{22} &= \{f(v_iu_{i+1}) | 1 \le i \le n/2-3 \wedge i \bmod 2 = 1\} \\ &= \{2n-(1+i)/2 | 1 \le i \le n/2-3 \wedge i \bmod 2 = 1\} \\ &= \{2n-1, 2n-2, \ldots, 7n/4+1\} \\ &= \{7n/4+1, 7n/4+2, \ldots, 2n-1\}, \end{aligned}$$

$$
\begin{aligned}
S_{23} &= \{f(v_iu_{i+1})|i = n/2-2\} = \{7n/4\},\\
S_{24} &= \{f(v_iu_{i+1})|i = n/2-1\} = \{2n+1\},\\
S_3 &= S_{31} \cup S_{32} \cup S_{33},\\
S_{31} &= \{f(v_iu_{i+3})|0 \le i \le n/2-2 \wedge i \bmod 2 = 0\}\\
&= \{(5n-i)/2|0 \le i \le n/2-2 \wedge i \bmod 2 = 0\}\\
&= \{5n/2, 5n/2-1, \dots, 9n/4+1\}\\
&= \{9n/4+1, 9n/4+2, \dots, 5n/2\},\\
S_{32} &= \{f(v_iu_{i+3})|1 \le i \le n/2-3 \wedge i \bmod 2 = 1\}\\
&= \{2n+(3+i)/2|1 \le i \le n/2-3 \wedge i \bmod 2 = 1\}\\
&= \{2n+2, 2n+3, \dots, 9n/4\},\\
S_{33} &= \{f(v_iu_{i+3})|i = n/2-1\} = \{2n\}.
\end{aligned}
$$

Hence, $S_1 \cup S_2 \cup S_3$ is the set of labels of all edges, and

$$
\begin{aligned}
S = {} & S_1 \cup S_2 \cup S_3\\
= {} & S_{11} \cup S_{21} \cup S_{12} \cup S_{23} \cup S_{22} \cup S_{33} \cup S_{24} \cup S_{32} \cup S_{31}\\
= {} & \{n+1, n+2, \dots, 5n/4\} \cup \{5n/4+1, 5n/4+2, \dots, 3n/2-1\} \cup \{3n/2, 3n/2+1,\\
& \dots, 7n/4-1\} \cup \{7n/4\} \cup \{7n/4+1, 7n/4+2, \dots, 2n-1\} \cup \{2n\} \cup \{2n+1\}\\
& \cup \{2n+2, 2n+3, \dots, 9n/4\} \cup \{9n/4+1, 9n/4+2, \dots, 5n/2\}\\
= {} & \{n+1, n+2, \dots, 5n/2\}.
\end{aligned}
$$

Therefore we conclude that f is a bijection from $E(G)$ onto $\{n+1, n+2, \dots, 5n/2\}$. Denote by

$$
\begin{aligned}
g_f(v) &= C - \sum f(vu),\ vu \in E(G),\\
W &= \{g_f(v)|v \in V(G)\}.
\end{aligned}
$$

Now, we show that g_f is a bijective mapping from $V(G)$ onto W. Let us denote the sets of the weights (under an edge labeling f) of vertices v_i and u_i of $W_{3,n}$ by

$$
\begin{aligned}
W_1 &= \{g_f(v_i)|0 \le i \le n/2-1\}\\
&= \{C - (f(v_iu_i) + f(v_iu_{i+1}) + f(v_iu_{i+3}))|0 \le i \le n/2-1\},\\
W_2 &= \{g_f(u_i)|0 \le i \le n/2-1\}\\
&= \{C - (f(u_iv_i) + f(u_iv_{i-1}) + f(u_iv_{i-3}))|0 \le i \le n/2-1\}.
\end{aligned}
$$

Where

$$
\begin{aligned}
W_1 &= W_{11} \cup W_{12} \cup W_{13} \cup W_{14},\\
W_{11} &= \{C - (f(v_iu_i) + f(v_iu_{i+1}) + f(v_iu_{i+3}))|0 \le i \le n/2-4 \wedge i \bmod 2 = 0\}\\
&= \{23n/4+2-(19n/4+2+i/2)|0 \le i \le n/2-4 \wedge i \bmod 2 = 0\}\\
&= \{n, n-1, \dots, 3n/4+2\} = \{3n/4+2, 3n/4+3, \dots, n\},\\
W_{12} &= \{C - (f(v_iu_i) + f(v_iu_{i+1}) + f(v_iu_{i+3}))|1 \le i \le n/2-3 \wedge i \bmod 2 = 1\}\\
&= \{23n/4+2-(11n+1+i)/2|1 \le i \le n/2-3 \wedge i \bmod 2 = 1\}\\
&= \{n/4+1, n/4, \dots, 3\} = \{3, 4, \dots, n/4+1\},\\
W_{13} &= \{C - (f(v_iu_i) + f(v_iu_{i+1}) + f(v_iu_{i+3}))|i = n/2-2\} = \{n/2+1\},\\
W_{14} &= \{C - (f(v_iu_i) + f(v_iu_{i+1}) + f(v_iu_{i+3}))|i = n/2-1\} = \{2\},\\
W_2 &= W_{21} \cup W_{22} \cup W_{23} \cup W_{24} \cup W_{25} \cup W_{26},\\
W_{21} &= \{C - (f(u_iv_i) + f(u_iv_{i-1}) + f(u_iv_{i-3}))|i = 0\} = \{n/2\},\\
W_{22} &= \{C - (f(u_iv_i) + f(u_iv_{i-1}) + f(u_iv_{i-3}))|i = 1\} = \{3n/4\},\\
W_{23} &= \{C - (f(u_iv_i) + f(u_iv_{i-1}) + f(u_iv_{i-3}))|i = 2\} = \{3n/4+1\},
\end{aligned}
$$

$$\begin{aligned}
W_{24} &= \{C-(f(u_iv_i)+f(u_iv_{i-1})+f(u_iv_{i-3}))|3\le i\le n/2-3\wedge i \bmod 2=1\}\\
&= \{23n/4+2-(21n+6+2i)/4|3\le i\le n/2-3\wedge i \bmod 2=1\}\\
&= \{n/2-1,n/2-2,\ldots,n/4+2\}=\{n/4+2,n/4+3,\ldots,n/2-1\},\\
W_{25} &= \{C-(f(u_iv_i)+f(u_iv_{i-1})+f(u_iv_{i-3}))|4\le i\le n/2-2\wedge i \bmod 2=0\}\\
&= \{23n/4+2-(5n+1+i/2)|4\le i\le n/2-2\wedge i \bmod 2=0\}\\
&= \{3n/4-1,3n/4-2,\ldots,n/2+2\}=\{n/2+2,n/2+3,\ldots,3n/4-1\},\\
W_{26} &= \{C-(f(u_iv_i)+f(u_iv_{i-1})+f(u_iv_{i-3}))|i=n/2-1\}=\{1\}.
\end{aligned}$$

Hence, $W=W_1\cup W_2$ is the set of the weights of all vertices, and

$$\begin{aligned}
W= {} & W_1\cup W_2\\
= {} & W_{11}\cup W_{12}\cup W_{13}\cup W_{14}\cup W_{21}\cup W_{22}\cup W_{23}\cup W_{24}\cup W_{25}\cup W_{26}\\
= {} & W_{26}\cup W_{14}\cup W_{12}\cup W_{24}\cup W_{21}\cup W_{13}\cup W_{25}\cup W_{22}\cup W_{23}\cup W_{11}\\
= {} & \{1\}\cup\{2\}\cup\{3,4,\ldots,n/4+1\}\cup\{n/4+2,n/4+3,\ldots,n/2-1\}\cup\{n/2\}\\
& \cup\{n/2+1\}\cup\{n/2+2,n/2+3,\ldots,3n/4-1\}\cup\{3n/4\}\cup\{3n/4+1\}\\
& \cup\{3n/4+2,3n/4+3,\ldots,n\}\\
= {} & \{\ 1,\ 2,\ldots,\ n-1,\ n\}.
\end{aligned}$$

It is clear that the labels of each edge are distinct, and the edge labels are $\{1,2,\ldots,n\}$. According to the definition of super vertex-magic total labeling, we thus conclude that the graph $W_{3,n}$ is super vertex-magic for $n\equiv 0 \bmod 4$. □

In Figure 2.2, we show our super vertex-magic total labeling for $W_{3,24}$.

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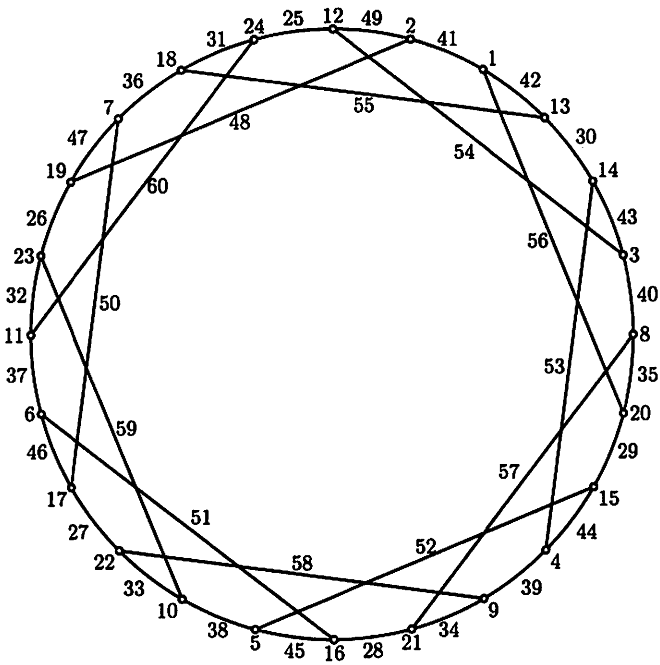

Figure 2.2 A super vertex-magic total labeling for $W_{3,24}$