1-FACTORIZATIONS OF CAYLEY GRAPHS

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ABSTRACT. In this note we prove that all connected Cayley graphs of every finite group $Q \times H$ are 1-factorizable, where Q is any non-trivial group of 2-power order and H is any group of odd order.

1. Introduction and Results

Let G be a non-trivial group, $S \subseteq G \backslash \{1\}$ and $S^{-1} = \{s^{-1} : s \in S\}$. The Cayley graph $\Gamma(S : G)$ of the group G with respect to the set S has the vertex set G and the edge set $\{\{g, sg\} : g \in G, s \in S \cup S^{-1}\}$.

A j-factor of a graph is a spanning subgraph which is regular of valence j. In particular, a 1-factor of a graph is a collection of edges such that each vertex is incident with exactly one edge. A 1-factorization of a regular graph is a partition of the edge set of the graph into disjoint 1-factors. A 1-factorization of a regular graph of valence v is equivalent to a coloring of the edges in v colors (coloring each 1-factor a different color). This enables us to use a very helpful result: Any simple, regular graph of valence v can be edge-colored in either v or $v+1$ colors. This is a specific case of Vizing's theorem (see [2, pp. 245-248]).

We study the conjecture that says all Cayley graphs $\Gamma(S : G)$ of groups G of even order are 1-factorizable whenever $G = \langle S \rangle$. There are some partial results on this conjecture obtained by Stong [1]. Here we prove

Theorem. *Let H be a finite group of odd order and let Q be a finite group of order 2^k $(k > 0)$. Then the Cayley graph $\Gamma(S : Q \times H)$ is 1-factorizable for all generating sets S of $Q \times H$.*

As a corollary we prove that all connected Cayley graphs of every finite nilpotent group of even order are 1-factorizable which has been proved by

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Stong in [1, Corollary 2.4.1] only for Cayley graphs on minimal generating sets.

2. Proof of the Theorem

We need the following lemma whose proof is more or less as Lemma 2.1 of [1] with some modifications.

Lemma 2.1. *Let H be a finite group of odd order. Then the Cayley graph $\Gamma(S : \mathbb{Z}_2 \times H)$ is 1-factorizable, for any generating set S of $\mathbb{Z}_2 \times H$ containing exactly one element of even order.*

Proof. Let a be the only element of S of even order. Then $a = zh$, where $z \in \mathbb{Z}_2$ and $h \in H$ and z of order 2. If $a^2 = 1$, then $h = 1$ and $S \backslash \{a\} \subseteq H$ and so $axa^{-1} = x$ for all $x \in S \cap H$. Thus, in this case, Theorem 2.3 of [1] completes the proof. Therefore we may assume that $a^2 \neq 1$. Let $\Gamma' = \Gamma(S\backslash\{a\} : \mathbb{Z}_2 \times H)$ and Γ_1 and Γ_2 be the induced subgraphs of Γ' on the sets H and zH, respectively. It can be easily seen that the map $x \mapsto zx$ is an graph isomorphism from Γ_1 to Γ_2. By Vizing's theorem the edges in both Γ_1 and Γ_2 can be edges-colored in the same manner in $|S\backslash\{a\}| + 1$ colors (by "the same manner" we mean that the edge $\{h_1, h_2\}$ in Γ_1 has "the same" color as $\{zh_1, zh_2\}$ in Γ_2, and vice versa). Then all that remains to be done is to color the edges from H to zH, that is the following two 'disjoint' 1-factors of $\Gamma(S : \mathbb{Z}_2 \times H)$ (here we use $a^2 \neq 1$):

$$\big\{\{x, ax\} \mid x \in H\big\} \text{ and } \big\{\{x, a^{-1}x\} \mid x \in H\big\}. \qquad (*)$$

(note that the edges of $\Gamma(S : \mathbb{Z}_2 \times H)$ are exactly the edges of Γ_1, Γ_2 and those in the above 1-factors). Now since both $x \in H$ and $zx \in zH$ have edges (in Γ_1 and Γ_2, respectively) of the same $|S\backslash\{a\}|$ colors to them, there are 'two' colors (note that here we again use $a^2 \neq 1$) that can be used to color 1-factors in $(*)$. This completes the proof. □

Proof of the Theorem. Let $G = Q \times H$ and S be any generating set of G. We argue by induction on $|S|$. If $|S| = 1$, then G is a cyclic group of even order and Corollary 2.3.1 of [1] completes the proof. Now assume that $|S| > 1$ and for any non-trivial group Q_1 of 2-power order and subgroup H_1 of H the Cayley graph $\Gamma(S_1 : Q_1 \times H_1)$ is 1-factorizable for any generating set S_1 of $Q_1 \times H_1$ with $|S_1| < |S|$. Since the set of elements of odd order in G is the subgroup H and $G = \langle S \rangle$, S has at least one element a of even order. First assume that S has another element distinct from a of even order. Consider the subgroup G_1 generated by $S\backslash\{a\}$ of G. Then $G_1 = Q_1 \times H_1$ for some subgroups $Q_1 \leq Q$ and $H_1 \leq H$ such that $Q_1 \neq 1$. Therefore the induction hypothesis implies that $\Gamma(S\backslash\{a\} : G_1)$ has a 1-factorization. Since $\Gamma(S\backslash\{a\}, G)$ consists of disjoint copies of $\Gamma(S\backslash\{a\} : G_1)$ which are 1-factorizable, $\Gamma(S\backslash\{a\}, G)$ has a 1-factorization. Now since the only element

of $S\backslash(S\backslash\{a\})$ has even order, Lemma 2.2 of [1] shows that $\Gamma(S:G)$ is 1-factorizable.
Hence we may assume that a is the only element of S of even order. Since $a = a_1a_2$ for some $a_1 \in Q$ and $a_2 \in H$, we have

$$G = \langle S \rangle = \langle S\backslash\{a\}, a_1a_2 \rangle = \langle a_1 \rangle \times \langle S\backslash\{a\}, a_2 \rangle .$$

It follows that $Q = \langle a_1 \rangle$. Consider the subgroup $N = \langle a_1^2 \rangle$. Then N is a normal subgroup of G such that $N \cap S = \varnothing$. It is easy to see that when $s, t \in S$ with $s \neq t^{\pm 1}$, neither st nor st^{-1} belongs to N. Now by Lemma 2.4 of [1], it is enough to show that $\Gamma(\frac{SN}{N} : \frac{G}{N})$ is 1-factorizable. Since $\frac{G}{N} \cong \mathbb{Z}_2 \times H$, it follows from Lemma 2.1 that $\Gamma(\frac{SN}{N} : \frac{G}{N})$ is 1-factorizable. This completes the proof. □

Corollary 2.2. *If G is a finite nilpotent group of even order, then $\Gamma(S:G)$ is 1-factorizable for all generating sets S of G.*

Proof. It follows from the Theorem and the fact that every finite nilpotent group is the direct product of its Sylow subgroups. □

References

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