On the Local Colorings of Graphs

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Abstract

A local coloring of a graph G is a function $c : V(G) \longrightarrow \mathbb{N}$ having the property that for each set $S \subseteq V(G)$ with $2 \leq |S| \leq 3$, there exist vertices $u, v \in S$ such that $|c(u) - c(v)| \geq m_S$, where m_S is the size of the induced subgraph $\langle S \rangle$. The maximum color assigned by a local coloring c to a vertex of G is called the value of c and is denoted by $\chi_\ell(c)$. The local chromatic number of G is $\chi_\ell(G) = \min\{\chi_\ell(c)\}$, where the minimum is taken over all local colorings c of G. If $\chi_\ell(c) = \chi_\ell(G)$, then c is called a minimum local coloring of G. The local coloring of graphs introduced by Chartrand et. al. in 2003. In this paper, following the study of this concept, first an upper bound for $\chi_\ell(G)$ where G is not complete graphs K_4 and K_5, is provided in terms of maximum degree $\Delta(G)$. Then the exact value of $\chi_\ell(G)$ for some special graphs G such as the cartesian product of cycles, paths and complete graphs is determined.

Key Words: local coloring, local chromatic number.

1 Introduction

A *standard coloring* or simply a (*vertex*) coloring of a graph G is a function $c : V(G) \longrightarrow \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers, having the property that $c(u) \neq c(v)$ for every pairs u, v of adjacent vertices of G. The *chromatic number* $\chi(G)$ is defined as the minimum number of colors

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used in any coloring of G. A *k-coloring* of G uses k colors. Define the *value* of a coloring c of G by $\chi(c) = \max\{c(v) : v \in V(G)\}$. Then $\chi(G) = \min\{\chi(c) : c$ is a coloring of $G\}$. In each k-coloring of G, the vertex set $V(G)$ is partitioned into subsets $V_1, V_2, \ldots, V_k$, where each set V_i, $1 \le i \le k$, is referred to as a *color class* with each vertex in V_i being assigned the color i, in fact each set V_i, $1 \le i \le k$, is an independent set.

Variations and generalizations of graph coloring have been studied by many authors and in many ways. The idea of defining the coloring of graphs by means of conditions placed on color classes was discussed in [4] and [5].

The standard definition of coloring can also be modified so that the local requirement that adjacent vertices must be assigned distinct colors is replaced by a more global requirement.

For a graph G and a nonempty subset $S \subseteq V(G)$, let m_S denote the size of the induced subgraph $\langle S \rangle$. A standard coloring of a graph G can be considered as a function $c : V(G) \longrightarrow \mathbb{N}$ with the property that for every 2-element set $S = \{u, v\}$ of vertices of G, $|c(u) - c(v)| \ge m_S$.

Defining the standard coloring of a graph in this way suggested the extension of this concept introduced in [2] and [3].

Let G be a graph of order $n \ge 2$, and let k be a fixed integer with $2 \le k \le n$. A *k-local coloring* of a graph G is a function $c : V(G) \longrightarrow \mathbb{N}$ having the property that for each set $S \subseteq V(G)$ with $2 \le |S| \le k$, there exist vertices $u, v \in S$ such that $|c(u) - c(v)| \ge m_S$, where m_S is the size of the induced subgraph $\langle S \rangle$. The maximum color assigned by a k-local coloring c to a vertex of G is called the *value* of c and is denoted by $lc_k(c)$. The *k-local chromatic number* of G is $lc_k(G) = \min\{lc_k(c)\}$, where the minimum is taken over all k-local coloring c of G. For every integer $2 \le k \le n$, it follows that $\chi(G) = lc_2(G) \le lc_3(G) \le \ldots \le lc_k(G)$.

The k-local coloring of graphs for $k = 3$ was discussed in [2] and [3]. A 3-local coloring c of a graph G is referred to as a *local coloring* of G and $lc_3(G)$ is denoted by $\chi_\ell(G)$ which is also referred to as *local chromatic number* of G. If $\chi_\ell(c) = \chi_\ell(G)$, then c is called a *minimum local coloring* of G.

Therefore, the local chromatic number of G is slightly more global than the chromatic number of G since the conditions on colors that can be assigned to the vertices of G depend on subgraphs of order 2 and 3 in G rather than only on subgraphs of order 2.

Just as with standard coloring, where $\chi(H) \le \chi(G)$ for any subgraph H of a graph G, it follows that $\chi_\ell(H) \le \chi_\ell(G)$ as well.

It is often useful to observe that if c is a local coloring of a graph G whose value is s, then the *complementary coloring* $\bar{c}$ of c defined by $\bar{c}(v) = s + 1 - c(v)$ for all $v \in V(G)$ is a local coloring of G as well.

In [2] and [3] among other facts the following results are established.

Theorem A. *For every graph G of order at least 3,*

$$\chi(G) \leq \chi_\ell(G) \leq 2\chi(G) - 1.$$

Theorem B. *If G is a connected graph with maximum degree $\Delta(G)$ that is not a triangle, then*

$$\chi_\ell(G) \leq 2\Delta(G) - 1.$$

Theorem C. *If G is a connected bipartite graph of order at least 3, then $\chi_\ell(G) = 3$.*

Theorem D. *Let $G = K_{n_1, n_2, \ldots, n_{r+s}}$ be a complete multipartite graph, where r of the integers n_i are at least 2, the remaining s integers n_i are 1, and $r + s \geq 2$. Then*

$$\chi_\ell(G) = 2r + \left\lfloor \frac{3s-1}{2} \right\rfloor.$$

In particular,

$$\chi_\ell(K_n) = \left\lfloor \frac{3n-1}{2} \right\rfloor$$

for every positive integer n.

The local chromatic number of all paths and cycles are also known.

Theorem E. *For $n \geq 4$ and $m \geq 3$, $\chi_\ell(C_n) = \chi_\ell(P_m) = 3$.*

By Theorem C, if G is a 3-regular bipartite graph, then $\chi_\ell(G) = 3$. Furthermore, $\chi_\ell(K_4) = 5$. The following conjecture is stated in [3].

Conjecture 1. *If G is a connected 3-regular graph that is neither bipartite nor complete, then $\chi_\ell(G) = 4$.*

In the next section we prove that the above conjecture is true.

2 Upper Bound for Local Chromatic Number

In this section first we provide an upper bound for the local chromatic number of a connected graph G except K_4 and K_5 in terms of maximum degree $\Delta(G)$. Then we conclude that if G is a connected 3-regular graph that is neither bipartite nor complete, then $\chi_\ell(G) = 4$, which proves that Conjecture 1 is true.

The following theorem is due to Brooks [1].

Theorem F. *If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.*

Theorem 1. *For every connected graph G with maximum degree $\Delta(G)$ greater than 2, except K_4 and K_5; we have*

$$\chi_\ell(G) \leq 2\Delta(G) - 2.$$

Proof. For $G = K_n$, $n \geq 6$, by Theorem D,

$$\chi_\ell(K_n) = \left\lfloor \frac{3n-1}{2} \right\rfloor \leq 2\Delta(G) - 2 = 2n - 4.$$

If G is not a complete graph, since $\Delta(G) \geq 3$, G is not a cycle, therefore by Theorem F, we have $\chi(G) \leq \Delta(G)$. If $\chi(G) \leq \Delta(G) - 1$ then by Theorem A, $\chi_\ell(G) \leq 2\chi(G) - 1 \leq 2\Delta(G) - 2$ and we are done. Now let $\chi(G) = \Delta(G) = \Delta$. For every Δ-coloring c of graph G, let $\{A_1^c, A_2^c, \ldots, A_\Delta^c\}$ be a partition of $V(G)$ to Δ color classes. Define the family $\mathcal{F}$ as follows

$$\mathcal{F} = \{P = \{|A_1^c|, |A_2^c|, ..., |A_\Delta^c|\} \mid c \text{ is a } \Delta\text{-coloring of } G\}.$$

Let $\alpha := \min\{\min P \mid P \in \mathcal{F}\}$; in fact α is the size of smallest color calss among all of the Δ-coloring of graph G. Consider the partition $P_\alpha = \{A_1, ..., A_\Delta\}$ where $|A_\Delta| = \alpha$ and $A_i = \{a_1^i, \ldots, a_{n_i}^i\}$, $1 \leq i \leq \Delta$, is a color class of color i in a Δ-coloring of graph G. We define a local coloring $c : V(G) \longrightarrow \mathbb{N}$ by

$$c(v) = \begin{cases} 2i-1 & \text{If } v \in A_i, 1 \leq i \leq \Delta - 2, \\ 2\Delta - 2 & \text{If } v \in A_{\Delta-1}, \\ 2\Delta - 4 & \text{If } v \in A_\Delta, \text{ and } |N(v) \cap A_{\Delta-1}| = 2, \\ 2\Delta - 3 & \text{If } v \in A_\Delta, \text{ and } |N(v) \cap A_{\Delta-1}| = 1, \end{cases}.$$

where $N(v)$ is the set of vertices adjacent to v.

Since A_Δ is the smallest color class among all partitions of $V(G)$ to Δ color classes, each vertex $v \in A_\Delta$ has at least one neighbor in each A_i, $i = 1, \ldots, \Delta - 1$. Hence each vertex $v \in A_\Delta$ has at most two neighbors in $A_{\Delta-1}$, so $|N(v) \cap A_{\Delta-1}| = 1$ or 2. Therefore the assignment c is well defined.

To see that c is a local coloring of G, let S be a subset of $V(G)$ with $2 \leq |S| \leq 3$, we show that there exist vertices u and v in S such that $|c(u) - c(v)| \geq m_S$, where m_S is the size of the induced subgraph $\langle S \rangle$.

Clearly c is a vertex coloring of G, so when $|S| = 2$ or $m_S = 1$ we are done. Now assume $|S| = 3$ and $m_S \geq 2$. Let $A := \bigcup_{i=1}^{\Delta-2} A_i$, we consider the following cases.

(a) $S = \{a_r^i, a_s^j, a_t^k\}$ where $1 \leq i < j < k \leq \Delta$.

In this case $|c(a_t^i) - c(a_r^k)| \geq m_S$ for $i < \Delta - 2$, and $|c(a_r^i) - c(a_s^j)| \geq m_S$ for $i = \Delta - 2$.

Not that $m_S = 3$ is possible only in case (a), so in the following cases we have $m_S = 2$.

(b) $S = \{a_r^i, a_s^j, a_t^k\}$ where $1 \leq i \leq j \leq k \leq \Delta - 1$.

It is obvious that there exist vertices u and v in S where $|c(u) - c(v)| \geq 2$.

(c) $S = \{u, v, a_t^\Delta\}$ where $u, v \in A, a_t^\Delta \in A_\Delta$.

If $c(a_t^\Delta) = 2\Delta - 3$ then $|c(a_t^\Delta) - c(v)| \geq 2$, because $1 \leq c(v) \leq 2\Delta - 5$. If $c(a_t^\Delta) = 2\Delta - 4$, then a_t^Δ has one neighbor in each A_i, $1 \leq i \leq \Delta - 2$. Since $m_S = 2$, we must have $u \in A_i, v \in A_j$ and $1 \leq i \neq j \leq \Delta - 2$, hence $|c(u) - c(v)| \geq 2$.

(d) $S = \{u, v, a_t^\Delta\}$ where $u, v \in A_{\Delta-1}$.

Since $m_S = 2$, we must have $u, v \in N(a_t^\Delta)$. Hence $c(a_t^\Delta) = 2\Delta - 4$ and $|c(a_\Delta^t) - c(v)| \geq 2$.

(e) $S = \{u, v, a_t^{\Delta-1}\}$ where $u, v \in A_\Delta$.

In this case if $c(u)$ or $c(v)$ is $2\Delta - 4$, we are done. Otherwise $c(u) = c(v) = 2\Delta - 3$. Since $m_S = 2$, $a_t^{\Delta-1}$ is the only neighbor of u and v in $A_{\Delta-1}$. If $a_t^{\Delta-1}$ has no any other neighbor in A_Δ, we can put vertices u and v in $A_{\Delta-1}$ and put vertex $a_t^{\Delta-1}$ in A_Δ. Therefore we obtain a color class of size smaller than α, which is contradiction.

If $a_t^{\Delta-1}$ has neighbors in A_Δ except u and v, since $\deg(a_t^{\Delta-1}) \leq \Delta$, there exists A_i, $1 \le i \le \Delta - 2$, which $a_t^{\Delta-1}$ has no neighbor in A_i. In this case we can put $a_t^{\Delta-1}$ in A_i and put vertices u and v in $A_{\Delta-1}$. Therefore we obtain a color class of size smaller than α, which is contradiction.

(f) $S = \{u, v, w\}$ where $u, v \in A_\Delta$ and $w \in A$.

If one of the vertices u and v has color $2\Delta - 3$, since $c(w) \le 2\Delta - 5$, then we are done. Otherwise $c(u) = c(v) = 2\Delta - 4$, hence each vertex u and v has two neighbors in $A_{\Delta-1}$. Since $m_S = 2$, w is the only neighbor of vertices u and v in some A_j, $1 \le j \le \Delta - 2$. Now if w has neighbors in A_Δ except u and v, since $\deg(w) \le \Delta$, there exists A_i, $1 \le i \le \Delta - 1$, which w has no neighbor in A_i. In this case we can put vertex w in A_i and put vertices u and v in A_j. Therefore we obtain a color class of size smaller than α, which is contradiction. If w has no any other neighbor in A_Δ, we can put vertex w in A_Δ and put vertices u and v in A_j. Hence we obtain a color class of size smaller than α, which is contradiction. □

Proposition 1. *If G is not a bipartite graph and $\delta(G) \ge 3$, then $\chi_\ell(G) \ge 4$.*

Proof. Since G is not bipartite graph, G contains an odd cycle C_{2k+1}. Hence $\chi_\ell(G) \ge \chi(C_{2k+1}) \ge 3$. If the local chromatic number of G is 3, then for any local coloring c of G of value 3, there exists a vertex $v \in V(C_{2k+1})$ such that $c(v) = 2$. The vertex v has at least three neighbors, at least two of them have colors either 1 or 3. Each case contradicts that c is a local coloring. Hence $\chi_\ell(G) \ge 4$. □

The following corollary proves that Conjecture 1 is true.

Corollary 1. *If G is a connected 3-regular graph that is neither bipartite nor complete, then $\chi_\ell(G) = 4$.*

Proof. By Theorem 1, $\chi_\ell(G) \le 2\Delta(G) - 2 = 4$. Also by Proposition 1, $\chi_\ell(G) \ge 4$. Hence $\chi_\ell(G) = 4$. □

3 Local Chromatic Number of Some Graphs

In this section we study the local chromatic number of the graphs W_n, $C_m \times C_n$, $C_m \times P_n$, $P_m \times P_n$ and $K_m \times K_n$.

Given two graphs G and H, the *join* of G and H, denoted by $G \vee H$ is a graph with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$.

Theorem 2. *For every two graphs G and H, we have*

$$\chi_\ell(G \vee H) \leq \chi_\ell(G) + \chi_\ell(H) + 1.$$

Proof. Let c_1 and c_2 be local colorings of graphs G and H of values s_1 and s_2 , respectively. We define a local coloring $c : V(G) \longrightarrow \mathbb{N}$ by

$$c(v) = \begin{cases} c_1(v) & \text{If } v \in V(G), \\ c_2(v) + s_1 + 1 & \text{If } v \in V(H). \end{cases}$$

It is easy to see that c is a local coloring of graph $G \vee H$ of value $s_1 + s_2 + 1$. Therefore $\chi_\ell(G \vee H) \leq \chi_\ell(G) + \chi_\ell(H) + 1$. □

Theorem 3. *Let $n \geq 3$ and $W_n = K_1 \vee C_n$. Then $\chi_\ell(W_n) = 5$.*

Proof. We know that $W_n = K_1 \vee C_n$. Therefore by Theorem 2, $\chi_\ell(W_n) \leq \chi_\ell(C_n) + 2 = 5$. For $n = 3$, $W_n = K_4$ and by Theorem D, $\chi_\ell(K_4) = 5$. Since for $n \geq 4$, C_3 is a subgraph of W_n, we have $\chi_\ell(W_n) \geq 4$. Now let c be a local coloring of W_n of value 4 and $V(K_1) = \{v\}$, there are two cases to be considered.

Case 1. $c(v) \in \{1, 4\}$.

If $c(v) = 1$ then the vertices of cycle C_n are colored 2, 3, and 4. Since v and every two adjacent vertices in C_n induced a cycle C_3, then the vertices of C_n must have color 4 alternatively. Therefore there is a vertex with color 3 in C_n, with two neighbors colored either 2 or 4, which both cases contradict that c is local coloring. The case $c(v) = 4$ is also failed by considering the complementary coloring c.

Case 2. $c(v) \in \{2, 3\}$.

If $c(v) = 2$ then the vertices of C_n must have colors 1 and 4, alternatively, because v and every two adjacent vertices in C_n induced a cycle C_3, . But two vertices of color 1 in C_n and v of color 2 induced a path P_3 which contradicts that c is a local coloring. The case $c(v) = 3$ is also failed by considering the complementary coloring c.

Therefore $\chi_\ell(W_n) = 5$. □

By Theorem C, $\chi_\ell(C_{2k} \times C_{2p}) = 3$, $\chi_\ell(P_m \times P_n) = 3$, $m + n \geq 4$ and $\chi_\ell(C_{2k} \times P_n) = 3$. Hence we consider graphs $C_m \times C_n$ and $C_m \times P_n$, when m is odd.

Theorem 4. *For positive integer $n \geq 3$, we have*

$$\chi_\ell(C_3 \times C_n) = \begin{cases} 4 & \text{If } n \text{ is even,} \\ 5 & \text{If } n \text{ is odd.} \end{cases}$$

Proof. Since $C_3 \times C_n$ contains C_3 as a subgraph, $\chi_\ell(C_3 \times C_n) \geq \chi_\ell(C_3) = 4$. For n even, graph $C_3 \times C_n$ contains three copies of a bipartite graph C_n. We denote the vertices of $C_3 \times C_n$ by $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3)$ which (X_i, Y_i) is a bipartition of C_n, such that there is no edge between vertices in X_i and Y_j, $i \neq j$. We define a local coloring $c : V(C_3 \times C_n) \longrightarrow \mathbb{N}$ by

$$c(v) = \begin{cases} 1 & \text{If } v \in X_1 \cup Y_3, \\ 4 & \text{If } v \in X_3 \cup Y_2, \\ 2 & \text{If } v \in X_2, \\ 3 & \text{If } v \in Y_1. \end{cases}$$

It can easily be checked that the above assignment is a local coloring .

For $n = 2k+1$ odd, in each local coloring of graph $C_3 \times C_{2k+1}$ of value 4, every copy of C_3 has two vertices with colors 1 and 4 and a vertex with color 2 or 3. If the third vertex in different consecutive copies of C_3 have the same color, say 3, then one of the vertices with color 3 has two neighbors with color 4 which is contradiction. So the third vertex in different consecutive copies of C_3 have different colors. Graph $C_3 \times C_{2k+1}$ has $2k+1$ copies of C_3 and if $\chi_\ell(C_3 \times C_{2k+1}) = 4$ then the colors of vertices in the first and the last copies of C_3 must have different coloring, which is impossible. Therefore by Theorem A, we have $5 \leq \chi_\ell(C_3 \times C_{2k+1}) \leq 2\chi(C_3 \times C_{2k+1}) - 1 = 5$. Hence $\chi_\ell(C_3 \times C_{2k+1}) = 5$. □

Theorem 5. *For every positive integers $k \geq 2$ and n, $\chi_\ell(C_{2k+1} \times C_n) = 4$.*

Proof. By Proposition 1, $\chi_\ell(C_{2k+1} \times C_n) \geq 4$. If n is even, then graph $C_{2k+1} \times C_n$ contains $2k + 1$ copies of bipartite graph $C_n = (X, Y)$. We denote the vertices of each copy by (X_i, Y_i), $i = 1, \ldots, 2k + 1$, and define the following local coloring c of $C_{2k+1} \times C_n$ (n is even) of value 4. For each vertex $v \in V(C_{2k+1} \times C_n)$, define

$$c(v) = \begin{cases} 2 & \text{If } v \in X_{2k+1}, \\ 4 & \text{If } v \in Y_{2k+1}, \\ 1 & \text{If } v \in X_i \cup Y_{i+1}, i \equiv 1 \pmod 2,\ 1 \leq i \leq 2k-1, \\ 3 & \text{If } v \in X_{i+1} \cup Y_i, i \equiv 1 \pmod 2,\ 1 \leq i \leq 2k-1. \end{cases}$$

It is easy to see that c is local coloring of $C_{2k+1} \times C_n$ of value 4, when n is even.

If n is odd, then graph $C_{2k+1} \times C_n$ contains $2k+1$ copies of three partied graph $C_n = (X, Y, \{v\})$. We denote the vertices in each copy by $(X_i, Y_i, \{v_i\})$, $i = 1, \ldots, 2k+1$, and define the following local coloring c of $C_{2k+1} \times C_n$ (n is odd) of value 4. For each vertex $v \in V(C_{2k+1} \times C_n)$, define.

$$c(v) = \begin{cases} 2 & \text{If } v \in X_{2k+1}, \\ 4 & \text{If } v \in Y_{2k+1}, \\ 1 & \text{If } v = v_{2k+1}, \\ 2 & \text{If } v = v_i, i \equiv 1 \pmod 2,\ 1 \leq i \leq 2k, \\ 4 & \text{If } v = v_i, i \equiv 0 \pmod 2,\ 1 \leq i \leq 2k, \\ 1 & \text{If } v \in X_i \cup Y_{i+1}, i \equiv 1 \pmod 2,\ 1 \leq i \leq 2k-1, \\ 3 & \text{If } v \in X_{i+1} \cup Y_i, i \equiv 1 \pmod 2,\ 1 \leq i \leq 2k-1. \end{cases}$$

It is easy to see that c is a local coloring of $C_{2k+1} \times C_n$ of value 4, when n is odd. □

Theorem 6. *For every positive integers $k \geq 2$ and n, $\chi_\ell(C_{2k+1} \times P_n) = 4$.*

Proof. By Proposition 1, $\chi_\ell(C_{2k+1} \times P_n) \geq 4$. On the other hand graph $C_{2k+1} \times P_n$ is a subgraph of $C_{2k+1} \times C_{2n}$. Therefore by Theorem 5, $\chi_\ell(C_{2k+1} \times P_n) \leq \chi_\ell(C_{2k+1} \times C_{2n}) = 4$. Hence $\chi_\ell(C_{2k+1} \times P_n) = 4$. □

Let $S_1, \ldots, S_n$ be sets. A *system of distinct representative* (SDR) for these sets is an n-tuple $(x_1, \ldots, x_n)$ of elements with the properties that $x_i \in S_i$ for $i = 1, \ldots, n$ and $x_i \neq x_j$ for $i \neq j$. It is well known theorem that if $|S_i| = r$ and each element in $\cup_{i=1}^n S_i$ is contained in exactly r of the sets $S_1, \ldots, S_n$, then the family $(S_1, \ldots, S_n)$ has an SDR [6].

Theorem 7. *For any two positive integers with $r \leq 2s$, we have*

$$\chi_\ell(K_r \times K_{2s}) = \chi_\ell(K_{2s}) \qquad r \leq s;$$

$$\chi_\ell(K_r \times K_{2s}) > \chi_\ell(K_{2s}) \qquad r > s.$$

Proof. Since $K_r \times K_{2s}$ contains K_{2s} as a subgraph, we have $\chi_\ell(K_{2s}) \leq \chi_\ell(K_r \times K_{2s})$. To prove the statement it is enough to show that for $r \leq s$

there is a local coloring of $K_r \times K_{2s}$ of value $\chi_\ell(K_{2s})$, while for $r > s$ there is not such a coloring of $K_r \times K_{2s}$. To see this, first we show a fact for each local coloring c of K_{2s} of value $\chi_\ell(K_{2s}) = 3s - 1$.

Claim. In each local coloring c of K_{2s} of value $3s - 1$, the set of colors to be used is $A = \{1, 2, 4, 5, \ldots, 3s - 2, 3s - 1\}$.

Proof of claim. We prove the claim in following two parts:

(1) In a local coloring c of K_{2s} of value $\chi_\ell(K_{2s}) = 3s-1$, if $A = \{a_1, a_2, \ldots, a_{2s}\}$ is an increasing ordered set of colors to be used of color set $\{1, 2, \ldots, 3s-1\}$, then for each i, $1 \leq i \leq 2s - 1$, we have $a_{i+1} - a_i \leq 2$.

Assume for some j, $a_{j+1} - a_j \geq 3$. Now we define a new local coloring c' of K_{2s} as follows. For some j, define

$$c'(v) = \begin{cases} c(v) & \text{If } c(v) \leq a_j, \\ c(v) - 1 & \text{If } c(v) \geq a_{j+1}. \end{cases}$$

It is obvious that c' is a local coloring of K_{2s} of value less than the value of c, which is a contradiction. Hence the fact (1) is true.

(2) If color $a \in A$, then one of the colors $a - 1$ or $a + 1$ is also in A, while both are not in A.

Assume $a - 1$, $a + 1 \notin A$. Therefore all colors in A are less than $a - 2$ or greater than $a + 2$. If u be a vertex of color a in coloring c, then the assignment c on $V(K_{2s}) - \{u\}$ is also a local coloring of value at most $3s - 1$ for $K_{2s} - \{u\}$ which is a complete graph K_{2s-1}. Now for K_{2s-1} we define a new local coloring c' as follows. For each vertex $v \in V(K_{2s-1})$, define

$$c'(v) = \begin{cases} c(v) & \text{If } c(v) \leq a - 2, \\ c(v) - 2 & \text{If } c(v) \geq a + 2. \end{cases}$$

Note that if $a = 3s-1$ then $c' = c$ on $V(K_{2s}) - \{u\}$. It is easy to see that c' is a local coloring of K_{2s-1} of value $3s - 3$, whence $\chi_\ell(K_{2s-1}) = 3s - 2$, so it is a contradiction. Moreover if both of colors $a - 1$ and $a + 1$ are in A, then the vertices of colors $a - 1$, a and $a + 1$ induced subgraph K_3, which contradicts that c is a local coloring. Hence the fact (2) is true.

Now since the value of c is $3s - 1$, $1 \in A$. So by fact (2), we have $2 \in A$. Since $2 \in A$ and $1 \in A$, $3 \notin A$. Now by fact (1) we have $4 \in A$. Continuing this process by similar reason we conclude that $A = \{1, 2, 4, 5, 7, 8, \ldots, 3s - 2, 3s - 1\}$. Hence the claim is proved.

We consider graph $K_r \times K_{2s}$ as a $r \times 2s$ array such that each entry represent a vertex of the graph, each row is a representative of a copy of K_{2s} and each column is a representative of a copy of K_r. For simply we denote the vertex v_{ij} as a vertex represented by entry ij in the array. By the above claim in each local coloring of $K_r \times K_{2s}$ of value $\chi_\ell(K_{2s}) = 3s-1$, in each row i, $1 \leq i \leq r$, we have the set A of colors to be used for local coloring K_{2s}; which $A = \{1, 2, 4, 5, 7, 8, \ldots, 3s-2, 3s-1\}$. We denote the set of colors that can be used to color vertex v_{ij} by S_{ij}.

Now we prove that if $r > s$ then $\chi_\ell(K_r \times K_{2s}) > \chi_\ell(K_{2s})$. By contrary assume that $r > s$ and $\chi_\ell(K_r \times K_{2s}) = \chi_\ell(K_{2s})$. By the above notation the set of colors can be used to color vertex v_{i1} in column 1 is S_{i1}, and $S_{11} = A$, so $|S_{11}| = 2s$. By the fact (2), there is a vertex v_{1j}, $1 \leq j \leq 2s$, such that $c(v_{1j}) = c(v_{11}) + 1$ or $c(v_{1j}) = c(v_{11}) - 1$. Since the vertices v_{11}, v_{1j} and v_{21} induced a path P_3, the vertex v_{21} can not be colored with the same color used for the vertices v_{11} and v_{1j}. Therefore $|S_{21}| = 2s - 2$. By the same argument we have $|S_{i1}| = 2s - 2(i-1)$. To have a coloring of value $3s - 1$, we must have $|S_{r1}| = 2s - 2(r-1) \geq 1$, which gives the condition $r \leq s$. This contradicts the assumption $r > s$. Therefore for $r > s$, $\chi_\ell(K_r \times K_{2s}) > \chi_\ell(K_{2s})$.

Now for $r \leq s$, we provide a local coloring of $K_r \times K_{2s}$ of value $\chi_\ell(K_{2s}) = 3s - 1$. For the first row of the array we have $2s$ sets $S_{11} = S_{12} = \cdots = S_{1,2s}$ which $|S_{1j}| = 2s$, $1 \leq j \leq 2s$. Therefore an SDR of the family $(S_{1j}, \ldots S_{1,2s})$ is a local coloring for the first row of the array. For the second row, we have sets $S_{21}, S_{22}, \ldots, S_{2,2s}$ such that $|S_{2j}| = 2s - 2$, $1 \leq j \leq 2s$. The set S_{2j}, $1 \leq j \leq 2s$, is the set of colors that can be used to color the vertices v_{2j} in the second row of the array. Each color of A is contained in exactly $2s - 2$ of the sets S_{2j}, $1 \leq j \leq 2s$. Therefore an SDR of the family $(S_{2j}, \ldots, S_{2,2s})$ exists and is local coloring for the vertices in the second row of the array. By continuing this process we conclude that an SDR for the family $(S_{ij}, \ldots, S_{i,2s})$ exists, because $|S_{ij}| = 2s - 2(i-1)$ and each elements is contained in exactly $2s - 2(i-1)$ of the sets S_{ij}. This SDR gives us a local coloring for the ith row of the array. Therefore for $r \leq s$, we have a local coloring of $K_r \times K_{2s}$ of value $\chi_\ell(K_{2s})$. □

Theorem 8. *For any two positive integers with $r \leq s+1$, we have*

$$\chi_\ell(K_r \times K_{2s+1}) = \chi_\ell(K_{2s+1}).$$

Proof. Since K_{2s+1} is a subgraph of $K_r \times K_{2s+1}$, we have $\chi_\ell(K_r \times K_{2s+1}) \geq \chi_\ell(K_{2s+1})$. On the other way, $\chi_\ell(K_r \times K_{2s+1}) \leq \chi_\ell(K_{s+1} \times K_{2s+1})$. In Figure 1 we arise a local coloring of $K_{s+1} \times K_{2s+1}$ of value

$\chi_\ell(K_{2s+1})$. Hence $\chi_\ell(K_r \times K_{2s+1}) = \chi_\ell(K_{2s+1}) = 3s+1$. Each entry represents a vertex of the graph and the symbols represent the color of corresponding vertex of the entry in the given local coloring.

The symbols a_i are the same as explained in the proof of Theorem 7, where $a_{2s} = \chi_\ell(K_{2s}) = 3s-1$. □

a_1	a_2	...	a_4	a_5	...	a_{2s-1}	a_{2s}	$a_{2s}+2$
$a_{2s-1}+2$	$a_{2s}+2$	...	a_2	a_3	...	a_{2s-3}	a_{2s-2}	$a_{2s-2}+2$
$a_{2s-3}+2$	$a_{2s-2}+2$	...	$a_{2s}+2$	a_1	...	a_{2s-5}	a_{2s-4}	$a_{2s-4}+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$
a_3+2	a_4+2	...	a_6+2	...	...	a_1	a_2	a_2+2
a_1+2	a_2+2	...	a_4+2	...	...	$a_{2s-1}+2$	$a_{2s}+2$	a_1

Figure 1: A local coloring of $K_{s+1} \times K_{2s+1}$ of value $\chi_\ell(K_{2s+1})$.

It is known that the upper bound for $\chi_\ell(G)$ in Theorem A is attainable for infinitely many values of $\chi(G)$ and that the lower bound is attainable for $\chi(G) \leq 4$. The more general question in [3] is:

Problem. For which pairs a, b of integers with $a \leq b \leq 2a-1$, does there exist a graph G with $\chi(G) = a$ and $\chi_\ell(G) = b$?

In the following theorem we provide a partial answer to this question.

Theorem 9. *For any two positive integers with $\lfloor \frac{3n-1}{2} \rfloor \leq m \leq 2n-1$, there exists a graph G with $\chi(G) = n$ and $\chi_\ell(G) = m$.*

Proof. Let $n = r+s$ and $G = K_{\underbrace{2,\ldots,2}_{r},\underbrace{1,\ldots,1}_{s}}$. Graph G is a complete n partite graph with r parts of size 2 and s parts of size 1. By Theorem D, $\chi_\ell(G) = 2r + \lfloor \frac{3s-1}{2} \rfloor$. Now for each $\lfloor \frac{3n-1}{2} \rfloor \leq m \leq 2n-1$, let $s = 4n-2m-1$ and $r = 2m-3n+1$. Therefore we have $\chi_\ell(G) = m$. □

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