Sums of powers of binomial coefficients via Legendre polynomials, Part 2

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In this paper we modify a formula of Carlitz and find the novel formula (equivalent to an identity of Strehl)

$$\sum_{k=0}^{n}\binom{n}{k}^3 = \sum_{0\le k\le n/2}\binom{n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k} \tag{1}$$

which stands in contrast to MacMahon's well-known formula

$$\sum_{k=0}^{n}\binom{n}{k}^3 = \sum_{0\le k\le n/2}\binom{n}{2k}\binom{2k}{k}\binom{n+k}{k}2^{n-2k}. \tag{2}$$

We also show how to transform the one into the other, using the identity

$$\sum_{k\le j\le n/2}\binom{n}{2j}\binom{2j}{j}\binom{j}{k}2^{n-2j} = \binom{n}{k}\binom{2n-2k}{n}. \tag{3}$$

Next, we examine sums of fourth powers and recall some results of MacMahon for general powers. Finally, we present some problems for further study involving generalized hypergeometric sums.

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1. Introduction. In part 1 of this paper [9] we have discussed some sums of binomial coefficient powers of the form

$$S_n(p,x) = \sum_{k=0}^{n} \binom{n}{k}^p x^k, \text{ where } n \geq 0, \tag{1.1}$$

We gave new proofs of the two formulas of Carlitz [1]

$$\sum_{k=0}^{n} \binom{n}{k}^3 = ((x^n))\,(1 - x^2)^n\, P_n\left(\frac{1+x}{1-x}\right) \tag{1.2}$$

and

$$\sum_{k=0}^{n} \binom{n}{k}^4 = ((x^n))\,(1 - x)^{2n} \left\{P_n\left(\frac{1+x}{1-x}\right)\right\}^2 , \tag{1.3}$$

and we found the new formulas

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 = ((x^n))\,(1 - x)^{2n}\, P_n\left(\frac{1+x}{1-x}\right), \tag{1.4}$$

$$\sum_{k=0}^{n} \binom{n}{k}^5 = ((x^n))\,(1 - x)^n\, P_n\left(\frac{1+x}{1-x}\right) \sum_{k=0}^{n} \binom{n}{k}^3 x^k . \tag{1.5}$$

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[1] Dedicated to the memory of Professor Truman A. Botts (1917-2005)

Additionally, we simplified (1.4) to yield Dixon's [4] formula

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \binom{2n}{n}\binom{3n}{n} = (-1)^n \frac{(3n)!}{n!^3}. \tag{1.6}$$

MacMahon [11] proved this by noting that the desired sum is the coefficient of $(xyz)^n$ in the expansion of $(y-z)^n(z-x)^n(x-y)^n$.

We continue to use the notation $((x^n))\,f(x)$ for the coefficient of x^n in the series expansion of the function $f(x)$.

Our proofs used several of the standard forms for the Legendre polynomials as exhibited in my first publication [5] and elementary formulas in my book [7].

In the present paper we show how to obtain a novel formula for the sum of the cubes of the binomial coefficients by modifying formula (1.2) of Carlitz. This formula for the sum of cubes is different from one found by MacMahon [11] whose discoveries all followed from techniques he had previously developed [10] for finding coefficients in products of powers of multinomials. We also show by a binomial identity that our new formula is equivalent to MacMahon's formula. The formula is equivalent to one found by Strehl [13, eq. (29)].

We then modify Carlitz's (1.3) and examine representations for sums of fourth powers.

Finally we also discuss further avenues for research.

2. Modification of Carlitz's formula for sums of cubes.

We recall the formula (3.8) in [9].

$$P_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k} 2^{-2k} x^{n-2k} (x^2 - 1)^k . \qquad (2.1)$$

By simple algebra this yields

$$P_n\left(\frac{1+x}{1-x}\right) = (1-x)^{-n} \sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k} (1+x)^{n-2k} x^k . \qquad (2.2)$$

Then Carlitz's formula (1.2) yields

$$\sum_{k=0}^{n} \binom{n}{k}^3 = ((x^n)) (1 - x^2)^n (1-x)^{-n} \sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k} (1+x)^{n-2k} x^k$$

$$= ((x^n)) \sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k} (1+x)^{2n-2k} x^k . \qquad (2.3)$$

Then we find

$$\sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k} (1+x)^{2n-2k} x^k$$

$$= \sum_{0\le k\le n/2} \binom{n}{2k}\binom{2k}{k} \sum_{0\le j\le 2n-2k} \binom{2n-2k}{j} x^{j+k}$$

$$= \sum_{0\le 2k\le n} \binom{n}{2k}\binom{2k}{k} \sum_{2k\le j\le 2n} \binom{2n-2k}{j-2k} x^{j-k}$$

From this we can find the coefficient of x^n by choosing $j = n+k$, and therefore we find the novel formula

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{0\le k\le n/2} \binom{n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k} , \qquad (2.4)$$

$$= \sum_{0\le k\le n/2} \binom{n}{k}^2 \binom{2n-2k}{n-2k} , \qquad (2.5)$$

$$= \sum_{0\le k\le n/2} \binom{n}{k}\binom{n-k}{k}\binom{2n-2k}{n-k} . \qquad (2.6)$$

$$= \sum_{0\le k\le n/2} \frac{n!(2n-2k)!}{(n-2k)!k!^2(n-k)!^2} . \qquad (2.7)$$

Thus new formula stands in contrast to the well-known formula

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{0\le k\le n/2} \binom{n}{2k}\binom{2k}{k}\binom{n+k}{k} 2^{n-2k}, \qquad (2.8)$$

$$= \sum_{0\le k\le n/2} \binom{n}{k}\binom{n-k}{k}\binom{n+k}{k} 2^{n-2k}, \qquad (2.9)$$

$$= \sum_{0\le k\le n/2} \frac{(n+k)!}{(n-2k)!k!^3} 2^{n-2k}, \qquad (2.10)$$

found by MacMahon [11, p.281-282]. This is the special case of MacMahon's formula (6.7) in my book [7] where we take $x = y = 1$.

Formula set (2.4)-(2.7) does not appear in MacMahon's paper, and does not appear in my book [7]. However, the formula may clearly be rewritten in the form

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{n/2\le k\le n} \binom{n}{k}^2 \binom{k}{n} , \qquad (2.10a)$$

a formula published by Volker Strehl [13. (29)] which he proves easily with two applications of the Chu-Vandermonde convolution. He shows a tie-in with the Apery numbers.

MacMahon's formula (2.8)-(2.10) is printed incorrectly at the bottom of page 281 of his paper, with missing powers of 2, as

$$1 + \binom{p}{1}^3 + \binom{p}{2}^3 + \cdots$$

$$= 2^p + \binom{p+1}{3}\frac{3!}{(1!)^3} + \binom{p+2}{6}\frac{6!}{(2!)^3} + \cdots \quad + \text{a final term.}$$

However, he then gives his more general formula

$$\sum_{k=0}^{n}\binom{n}{k}^3 x^k y^{n-k} = \sum_{0\le k\le n/2}\binom{n}{2k}\binom{2k}{k}\binom{n+k}{k}x^k y^k (x+y)^{n-2k}, \qquad (2.11)$$

correctly, which is number (6.7) in my book [7].

MacMahon arrives at this by examining the coefficient of $(xyz)^n$ in the product $(ay + bz)^n(az + bx)^n(ax + by)^n$.

This well-known formula of MacMahon, of course, when $x = 1$, $y = 1$, gives us (2.8).

3. Formula (2.4) implies MacMahon's (2.8).

Formula (6.33) in my book [7] says that

$$\sum_{k\le j\le n/2}\binom{n}{2j}\binom{2j}{j}\binom{j}{k}2^{n-2j} = \binom{n}{k}\binom{2n-2k}{n}, \qquad (3.1)$$

so that by using this in formula (2.5) we get

$$\sum_{k=0}^{n}\binom{n}{k}^3 = \sum_{0\le k\le n/2}\binom{n}{k}^2\binom{n}{k}^{-1}\sum_{k\le j\le n/2}\binom{n}{2j}\binom{2j}{j}\binom{j}{k}2^{n-2j}$$

$$= \sum_{0\le k\le n/2}\binom{n}{k}\sum_{k\le j\le n/2}\binom{n}{2j}\binom{2j}{j}\binom{j}{k}2^{n-2j}$$

$$= \sum_{0\le j\le n/2} \binom{n}{2j}\binom{2j}{j} 2^{n-2j} \sum_{0\le k\le j} \binom{n}{k}\binom{j}{k}$$

$$= \sum_{0\le j\le n/2} \binom{n}{2j}\binom{2j}{j} 2^{n-2j} \sum_{0\le k\le j} \binom{n}{k}\binom{j}{j-k}$$

$$= \sum_{0\le j\le n/2} \binom{n}{2j}\binom{2j}{j} 2^{n-2j} \binom{n+j}{j},$$

by the Vandermonde convolution, and thus we have precisely (2.8). Since the steps are reversible, then the new formula (2.4) and MacMahon's (2.8) are equivalent.

4. Modification of Carlitz's formula for sums of biquadrates.

Using (2.2) to modify (1.3) we find

$$\sum_{k=0}^{n}\binom{n}{k}^4 = ((x^n))\left(\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k}{k}(1+x)^{n-2k}x^k\right)^2. \tag{4.1}$$

Write this as $((x^n))\,S^2$. Then, to evaluate this we proceed as follows:

$$S = \sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k}{k}(1+x)^{n-2k}x^k$$

$$= \sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k}{k}x^k\sum_{j=0}^{n-2k}\binom{n-2k}{j}x^j$$

$$= \sum_{\substack{0\le 2k\le n\\ k\le j\le n-k}} \frac{n!}{k!^2 j!(n-2k-j)!}x^{k+j}$$

$$= \sum_{j=0}^{n} x^j \sum_{k=0}^{j}\frac{n!}{k!^2(j-k)!(n-j-k)!} = \sum_{j=0}^{n}\binom{n}{j}x^j\sum_{k=0}^{j}\binom{j}{k}\binom{n-j}{k}$$

$$= \sum_{j=0}^{n} \binom{n}{j} x^j \sum_{k=0}^{j} \binom{j}{j-k} \binom{n-j}{k}$$

$$= \sum_{j=0}^{n} \binom{n}{j} x^j \binom{n}{j} = \sum_{j=0}^{n} \binom{n}{j}^2 x^j .$$

by the Chu-Vandermonde formula.

Then we find at once that

$$((x^n))\, S^2 = \sum_{j=0}^{n} \binom{n}{j}^4 ,$$

which however is the sum we started with, so that approaching (4.1) in this manner we learn nothing new.

It is worth noting that for fourth powers, MacMahon [11, p.285] found the double summation

$$\sum_{k=0}^{n} \binom{n}{k}^4 = \sum_{i} \sum_{j} \frac{(2n-i-j)!}{(n-2i-j)!(n-i-2j)!i!^2 j!^2} , \qquad (4.2)$$

which is evidently not widely known.

He obtained this by noting that the desired sum is the coefficient of $(xyzu)^n$ in the expansion of $(y+z)^n (z+u)^n (u+x)^n (x+y)^n$. He was also able to obtain a formula analogous to (2.11) which may be written as

$$\sum_{k=0}^{n} \binom{n}{k}^4 x^k y^{n-k}$$

$$= \sum_i \sum_j \sum_k \frac{(2n-i-j-k)!}{(i!)^3 (j!)^3 (k!)^3 (n-2i-j-k)!(n-i-2j-k)!} x^{i+j} y^{n-i-j-k} (x-y)^k . \quad (4.3)$$

He observes that when $x = y$ then $k = 0$, and the formula reduces to (4.2).

For the sum of general powers of the binomial coefficient, MacMahon notes that

$$\sum_{k=0}^{n} \binom{n}{k}^p = ((x_p^{\,n})) \frac{1}{1 - x_1 + x_2 - x_3 + \cdots (-1)^p x_{p-1}}, \quad (4.4)$$

where $x_1, x_2, x_3, \ldots$ are the elementary symmetric functions of the quantities $x, y, z, y, \ldots$.

5. Alternative formulations of identities and other problems.

Carlitz [2, (14)] has noted that (2.8) above, of MacMahon, may be written for arbitrary a in the form

$$\sum_{k=0}^{\infty} \binom{a}{k}^3 = 2^a \sum_{k=0}^{\infty} (-1)^k \binom{-1-a}{k} \binom{\frac{1}{2}a}{k} \binom{-\frac{1}{2}+\frac{1}{2}a}{k}, \quad (5.1)$$

Then when we set $a = n$, with n a non-negative integer, it is easy to see that this becomes (2.8). As noted by Carlitz [2] formula (5.1) is a consequence of Saalschütz's theorem.

Moreover, as noted by Carlitz and Gould [3], a generalized version of (2.11) of MacMahon is

$$\sum_{k=0}^{\infty} \binom{a}{k}^3 u^k = (1+u)^a \sum_{r=0}^{\infty} \frac{(-\frac{a}{2})_r (-\frac{a}{2}+\frac{1}{2})_r (a+1)_r}{r!\, r!\, r!} \frac{4^r u^r}{(1+u)^{2r}}$$

$$= \sum_{r=0}^{\infty} \binom{a}{2r}\binom{2r}{r}\binom{a+r}{r} u^r (1+u)^{a-2r}, \qquad (5.2)$$

which follows from formula (12) in Carlitz [2].

Next we wish to observe that our new formula (2.4) may be rephrased in the form

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \binom{2n}{n} \sum_{0 \le k \le n/2} \binom{n}{k} \binom{\frac{n}{2}}{k} \binom{\frac{n}{2} - \frac{1}{2}}{k} \binom{n - \frac{1}{2}}{k}^{-1}. \qquad (5.3)$$

This raises the question as to what may be said of the expression

$$_3F_2 \left[\begin{matrix} -a, -a/2, & -a/2 + 1/2 ; 1 \\ 1, & -a + 1/2 \end{matrix} \right], \qquad (5.4)$$

in generalized hypergeometric notation in analogy to (5.1). This will be left for a later paper.

Now, we used formula (3.1) to transform the new formula (2.4) into formula (2.8) of MacMahon, and conversely. It would be of interest, if possible, to find an extension of (3.1) that would be analogous to (2.11) of MacMahon.

Using the Chu-Vandermonde formula on the term $\binom{n+k}{k}$ in (2.11) we find

$$\sum_{k=0}^{n} \binom{n}{k}^3 x^k y^{n-k} = \sum_{0 \le k \le n/2} \binom{n}{2k}\binom{2k}{k}\binom{n+k}{k} x^k y^k (x+y)^{n-2k}$$

$$= \sum_{j=0}^{n} \binom{n}{j} \sum_{0 \le k \le n/2} \binom{n}{2k}\binom{2k}{k}\binom{k}{j} x^k y^k (x+y)^{n-2k} \qquad (5.5)$$

but we have not seen any way to evaluate the inner sum in closed form except when $x = y = 1$ as in the case of (3.1). Indeed it would be highly desirable to explore the series

$$\sum_{0\le k\le n/2} \binom{n}{2k}\binom{2k}{k}\binom{k}{j} x^{n-2k} \qquad (5.6)$$

for values of x other than $x = 2$. This series may be expanded using the higher derivatives of $P_n(x)$. Cf. Gould [5]. In fact, from (2.1), we have

$$x^{-n} P_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k} 2^{-2k} z^k,$$

where now $z = 1 - \frac{1}{x^2}$, so that we find

$$2^n D_z^j \left(x^{-n} P_n(x) \right) = \sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k} 2^{n-2k} D_z^j (z^k),$$

and recalling that $D_z^j (z^k) = j! \binom{k}{j} z^{k-j}$,

we find

$$\sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k}\binom{k}{j} 2^{n-2k} z^{k-j} = \frac{1}{j!} 2^n D_z^j \left(x^{-n} P_n(x) \right),$$

and consequently

$$\sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k}\binom{k}{j} 2^{n-2k} z^k = \frac{1}{j!} 2^n z^j D_z^j \left(x^{-n} P_n(x) \right). \qquad (5.7)$$

We could apply the standard Leibniz formula to the product $x^{-n} P_n(x)$ and then use an old formula of Reinhold Hoppe, which says that

$$D_z^j f(x) = \sum_{k=0}^{j} D_x^k f(x) \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-r} \binom{k}{r} x^{k-r} D_z^j x^r, \qquad (5.8)$$

where x is a function of z to get the higher derivatives needed. See Gould

[8] for references and proof of (5.8).

Thus in principle we have a way to write (5.6) as a linear combination of the higher derivatives $P_n(x)$. The details are complicated.

Another problem is to determine if any new formula for the sum of biquadrates exists, in analogy to the one we have presented here for the cubes.

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