A COMBINATORIAL SHIFTING METHOD ON MULTICOMPLEXES WITH APPLICATIONS TO SIMPLICIAL COMPLEXES AND SIMPLE GRAPHS

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ABSTRACT. We introduce a combinatorial shifting operation on multicomplexes that carries similar properties required for the ordinary shifting operation on simplicial complexes. A linearly colored simplicial complex is called shifted if its associated multicomplex is stable under defined operation. We show that the underlying simplicial subcomplex of a linearly shifted simplicial complex is shifted in the ordinary sense, while the ordinary and linear shiftings are not interrelated in general. Separately, we also prove that any linearly shifted complex must be shellable with respect to the order of its facets induced by the linear coloring. As an application, we provide a characterization of simple graphs whose independence complexes are linearly shifted. The class of graphs obtained constitutes a superclass of threshold graphs.

1. INTRODUCTION

In this paper, we extend the combinatorial shifting technique to linearly colored simplicial complexes by introducing a shifting operation on multicomplexes. Recall that a simplicial complex Δ on $[n]$ is called *shifted* if $F\backslash\{j\} \cup \{i\} \in \Delta$ for every face F of Δ, whenever there exist $i < j$ with $i \notin F$ and $j \in F$, while a *shifting operation* is a map $\Delta \mapsto \text{Shift}(\Delta)$ that associates with a simplicial complex Δ on $[n]$ a simplicial complex $\text{Shift}(\Delta)$ on $[n]$ satisfying the following conditions:

(S1) $\text{Shift}(\Delta)$ is shifted.
(S2) $\text{Shift}(\Delta) = \Delta$ if Δ is shifted.
(S3) $f(\Delta) = f(\text{Shift}(\Delta))$.
(S4) $\text{Shift}(\Delta') \subset \text{Shift}(\Delta)$ if $\Delta' \subset \Delta$,

where $[n] = \{1, 2, \ldots, n\}$ and $f(\Delta)$ denotes the f-vector of Δ. If Δ is a simplicial complex on the vertex set V, the shifting operation may be applied to Δ after possibly choosing a bijection $V \to [n]$, where $|V| = n$. Of course, such a choice will substantially affect the resulting shifted complex.

Key words and phrases. Linear coloring, multicomplex, graph complex, combinatorial shifting.

In general, any surjective map $\kappa: V \to [k]$ is called a vertex coloring of Δ, while the images of faces of Δ under κ may not produce a simplicial complex on $[k]$. However, assuming κ to be a linear coloring, we can always construct a unique multicomplex over $[k]$ which carries enough information to recover Δ back. We recall that a vertex coloring $\kappa: V \to [k]$ of Δ is called *linear* provided that for every two vertices u, v of Δ having the same color, we have either $\mathcal{F}(u) \subset \mathcal{F}(v)$ or $\mathcal{F}(v) \subset \mathcal{F}(u)$, where $\mathcal{F}(u)$ denotes the set of facets of Δ containing the vertex u (see Definition 2.1 for an equivalent formulation.) We proved in [2] that there is a one-to-one correspondence between linearly colored simplicial complexes using k colors and all multicomplexes over $[k]$.

We introduce a combinatorial shifting operation on multicomplexes as a map $\Gamma \mapsto \mathcal{C}(\Gamma)$ that associates a multicomplex Γ on $[k]$ with a multicomplex $\mathcal{C}(\Gamma)$ on $[k]$ which satisfies similar properties required for an ordinary shifting operation and may be stated as follows (see Theorem 4.8):

(M1) $\mathcal{C}(\Gamma)$ is shifted.
(M2) $\mathcal{C}(\Gamma) = \Gamma$ if Γ is shifted.
(M3) $f(\Gamma) = f(\mathcal{C}(\Gamma))$.

As in the ordinary case, $\mathcal{C}(\Gamma)$ not only depends on Γ but also on the order of the operations that are applied.

Once we associate a multicomplex Γ to a simplicial complex Δ, we shift Γ and define $\mathcal{S}(\Delta)$ to be the associated simplicial complex of $\mathcal{C}(\Gamma)$. We then call the map $\Delta \to \mathcal{S}(\Delta)$, a *linear combinatorial shifting* of Δ with respect to chosen linear coloring. The linear shifting somewhat behaves differently. For instance, a linear shifting of a pure simplicial complex may result in a nonpure complex. However, many combinatorial and topological properties of linearly shifted complexes are easy to detect. For example, any such complex is shellable (see Proposition 3.9). Furthermore, we provide examples that are not shifted in the ordinary sense but turn out to be linearly shifted.

As an application, we consider the independence complexes of simple graphs, and provide a structural characterization of graphs whose independence complexes are linearly shifted. The class of graphs obtained forms a superclass of threshold graphs.

We organize the paper as follows: In Section 2 we review basic definitions of multisets and multicomplexes, and recall useful facts about linear colorings of simplicial complexes. We then explain how to associate a multicomplex to a simplicial complex and vice versa. Then, in Section 3, we introduce linearly shifted simplicial complexes and verify some of their topological and combinatorial properties. The Section 4 is devoted to the construction and study of the linear combinatorial shifting operation. Finally, we investigate the structures of simple graphs whose independence complexes are linearly shifted in Section 5.

2. Linear Colorings

Let $\mathbb{N}$ denote the set of non-negative integers. We denote by $\mathbb{N}^k$ the lattice of k-tuples of elements of $\mathbb{N}$ ordered componentwise, i.e., $\mathfrak{n} \leq \mathfrak{m}$ in $\mathbb{N}^k$ if $n(i) \leq m(i)$ for all $1 \leq i \leq k$, where $\mathfrak{m} = (m(1), \ldots, m(k))$ and $\mathfrak{n} = (n(1), \ldots, n(k))$.

A *multiset* on $[k]$ is a function $\mathfrak{m}\colon [k] \to \mathbb{N}$, where $\mathfrak{m}(i)$ is regarded as the number of repetitions of $i \in [k]$. Any multiset $\mathfrak{m}$ on $[k]$ can be identified with a unique vector $(m(1), \ldots, m(k))$ of $\mathbb{N}^k$ by assigning $\mathfrak{m}(i) = m(i)$ for all $i \in [k]$, and via this assignment, we will always mean by a multiset over $[k]$, a vector in the lattice $\mathbb{N}^k$. As a consequence, a submultiset $\mathfrak{n}$ of $\mathfrak{m}$ is a multiset satisfying $\mathfrak{n} \leq \mathfrak{m}$. The *size* of a multiset $\mathfrak{m} \in \mathbb{N}^k$ is defined to be $\|\mathfrak{m}\| := \sum_{i=1}^{k} m(i)$.

For a given multiset $\mathfrak{m}$ over $[k]$, we define $\langle \mathfrak{m} \rangle := \{\mathfrak{n} \in \mathbb{N}^k \colon \mathfrak{n} \leq \mathfrak{m}\}$, and a subset $\Gamma \subset \mathbb{N}^k$ is said to be a *multicomplex* over $[k]$ provided that $\langle \mathfrak{m} \rangle \subset \Gamma$ for all $\mathfrak{m} \in \Gamma$. Unless otherwise stated, we will always consider finite multicomplexes over $[k]$. If Γ is such a multicomplex, then it is clear that there exists an antichain $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_s\}$ in Γ such that $\Gamma = \langle \mathfrak{m}_1 \rangle \cup \ldots \cup \langle \mathfrak{m}_s \rangle$, in which case the elements $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ are called the *facets* of Γ and we sometimes say that Γ is the multicomplex generated by $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$. A multicomplex is said to be *pure* if its facets are of same size. If Γ is a multicomplex, its *f-vector* $f(\Gamma) = (f_0, f_1, \ldots)$ is defined by $f_i := |\{\mathfrak{m} \in \Gamma \colon \|\mathfrak{m}\| = i + 1\}|$.

It is also evident from the definitions that ordinary subsets A of $[k]$ may be identified with multisets of $\mathbb{N}^k$ via their characteristic functions $\chi_A := (\chi_A(1), \ldots, \chi_A(k))$ given by $\chi_A(i)$ equals 0 if $i \notin A$ or 1 if $i \in A$. In other words, they corresponds to those elements $\mathfrak{m} \in \mathbb{N}^k$ with $m(i) \leq 1$ for each $1 \leq i \leq k$. It follows that if Γ is a multicomplex over $[k]$, then there exists a simplicial complex $\mathcal{U}(\Gamma)$ with vertex set $[k]$ defined by $G \in \mathcal{U}(\Gamma)$ if $\chi_G \in \Gamma$. This simplicial complex $\mathcal{U}(\Gamma)$ is called the *underlying simplicial complex* of Γ.

We next recall some definitions and facts about linear colorings of simplicial complexes, more details can be found in [2].

Definition 2.1. Let Δ be a simplicial complex with vertex set V. A surjective map $\kappa\colon V \to [k]$ is called a *k-linear coloring* of Δ if the collection $\{\mathcal{F}(v) \colon \kappa(v) = i\}$ is linearly ordered by inclusion for all $i \in [k]$, where $\mathcal{F}(v)$ denotes the set of facets of Δ containing v. We say that a simplicial complex is k-linear colorable if it admits a k-linear coloring map. The least integer k where Δ is k-linear colorable is called the *linear chromatic number* of Δ and denoted by $\mathrm{lchr}(\Delta)$.

When Δ is k-linear colorable, we may associate a unique multicomplex in $\mathbb{N}^k$ as follows. When F is face of Δ, we define the multiset $\mathfrak{m}_F = (m_F(1), \dots, m_F(k))$ in $\mathbb{N}^k$ by $m_F(i) := |\{v \in F\colon \kappa(v) = i\}|$ for each $1 \leq i \leq k$. In other words, $\mathfrak{m}_F$ gives the color combinations of the vertices of F under κ. It is then easy to verify that the collection $\{\mathfrak{m}_F\colon F \in \Delta\}$ is a multicomplex over $[k]$. We call this multicomplex the *associated multicomplex* of the couple (Δ, κ) and denoted it by $\Gamma(\Delta, \kappa)$.

This gives us an assignment $(\Delta, \kappa) \to \Gamma(\Delta, \kappa)$ from the set of linearly colored simplicial complexes to multicomplexes. Indeed, this assignment is surjective. Let Γ be a multicomplex over $[k]$, and define $r_i := \max\{m(i)\colon \mathfrak{m} \in \Gamma\}$ and set $V_i := \{a_t^i\colon 1 \leq t \leq r_i\}$. We next define a k-linear colorable simplicial complex $\Delta(\Gamma)$ on $V := \cup_{i=1}^k V_i$ as follows. We first associate a subset $F_\mathfrak{m}$ of V to every multiset $\mathfrak{m} = (m(1), \dots, m(k)) \in \Gamma$ by

$$F_\mathfrak{m} := \bigcup_{i=1}^{k} \{a_1^i, a_2^i, \dots, a_{m(i)}^i\}.$$

Now, $\Delta(\Gamma)$ is the k-linear colorable simplicial complex generated by the subsets $F_\mathfrak{m}$ for which $\mathfrak{m}$ is a facet of Γ, and the linear coloring map $\kappa\colon V \to [k]$ is given by $\kappa(a_t^i) := i$.

As an example we illustrate in Figure 1 the associated simplicial complexes of the multicomplexes

$$\Gamma = \langle(2,0,1)\rangle \cup \langle(1,2,0)\rangle \cup \langle(1,1,1)\rangle \quad \text{and}$$
$$\Gamma' = \langle(1,1,0,0)\rangle \cup \langle(1,0,1,0)\rangle \cup \langle(1,0,0,1)\rangle \cup \langle(0,1,1,0)\rangle \cup \langle(0,1,0,1)\rangle$$

where each number attached to a vertex denotes its color.

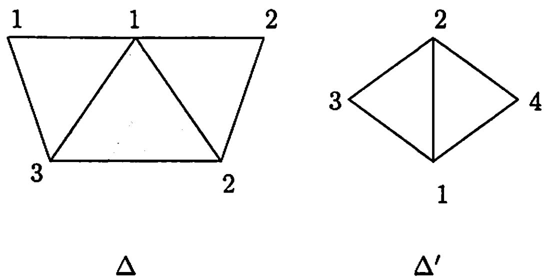

FIGURE 1. The associated simplicial complexes of multicomplexes.

The construction described above gives a unique simplicial complex $\Delta(\Gamma)$ associated to any multicomplex Γ, and it is also evident that the assignment $\Gamma \to \Delta(\Gamma)$ is inverse to the assignment $(\Delta, \kappa) \to \Gamma(\Delta, \kappa)$.

One of the important consequence of requiring a simplicial complex Δ to be k-linear colorable is that it provides a strong deformation of the complex to a subcomplex having as many vertices as the number of colors used. This subcomplex (called the representative subcomplex of the couple (Δ, κ) in [2]) is an isomorphic copy of the underlying simplicial complex of the associated multicomplex embedded into Δ.

Theorem 2.2. [see Section 3 of [2]] *Let Δ be a k-linear colorable simplicial complex and let $\Gamma = \Gamma(\Delta, \kappa)$ be the associated multicomplex. Then Δ and the underlying simplicial complex $\mathcal{U}(\Gamma)$ are homotopy equivalent.*

3. Linearly Shifted Complexes

The notion of combinatorial shifting goes back to Erdös, Ko and Rado, and was introduced to combinatorial and algebraic theory of simplicial complexes by Kalai. For the combinatorial aspects of the theory, we refer readers to [3] and [6], while the surveys [4] and [5] provide recent developments on the algebraic side. We here extend the combinatorial shifting technique in the frame work of linear colorings. It is interesting to note that some complexes that are not shifted turn out to be linearly shifted, and the structure of such complexes is easy to detect as in the ordinary case.

Given a multiset $\mathfrak{m} = (m(1), \dots, m(k)) \in \mathbb{N}^k$, we let $\rho_j(\mathfrak{m}) := \sum_{i=j}^{k} im(i)$ for all $1 \leq j \leq k$, and define a partial order on $\mathbb{N}^k$ by

$$(3.1) \qquad \mathfrak{n} \leq_P \mathfrak{m} :\iff \|\mathfrak{n}\| = \|\mathfrak{m}\| \text{ and } \rho_j(\mathfrak{n}) \leq \rho_j(\mathfrak{m}) \text{ for each } 1 \leq j \leq k.$$

On the other hand, $\leq_{RL}$ will denote the reverse lexicographical order on $\mathbb{N}^k$ defined by $\mathfrak{n} \leq_{RL} \mathfrak{m}$, if in the vector difference $\mathfrak{m} - \mathfrak{n} = (m(1) - n(1), \dots, m(k) - n(k))$, the right-most nonzero element is positive. We note that the reverse lexicographical order is a linear extension of the partial order $\leq_P$.

Definition 3.2. Let $\mathcal{M} \subseteq \mathbb{N}^k$ be given. Then $\mathcal{M}$ is called *shifted* (resp. *compressed*) if $\mathfrak{m} \in \mathcal{M}$ and $\mathfrak{n} \leq_P \mathfrak{m}$ (resp. $\mathfrak{n} \leq_{RL} \mathfrak{m}$) implies that $\mathfrak{n} \in \mathcal{M}$.

Definition 3.3. A simplicial complex Δ is called *k-linearly shifted* (resp. *k-linearly compressed*) if it admits a k-linear coloring κ such that the associated multicomplex $\Gamma = \Gamma(\Delta, \kappa)$ is shifted (resp. compressed).

Remark 3.4. We remark that the definition of k-linearly shifted complexes does not cover the ordinary shifted complexes. In other words, if Δ is a simplicial complex with vertex set $[n]$, it is never n-linearly shifted, while it could be shifted in the ordinary sense. On the contrary, a non-shifted simplicial complex may turn out to be linearly shifted. In fact, the simplicial complex Δ depicted in Figure 1 is 2-linearly shifted, while it is not shifted in the ordinary sense. On the other hand, the 1-dimensional simplicial complex Δ' illustrated in Figure 1 is shifted but not linearly shifted since it only admits a trivial linear coloring.

Since the reverse lexicographical order is a linear extension of the partial order, the following is immediate.

Corollary 3.5. *Any k-linearly compressed simplicial complex is k-linearly shifted.*

The converse of Corollary 3.5 does not generally hold as illustrated in the following example.

Example 3.6. Let Δ_1 and Δ_2 be the simplicial complexes associated to the following multicomplexes:

$$\Gamma(\Delta_1) = \langle(3,0,0)\rangle \cup \langle(2,1,0)\rangle \quad \text{and}$$
$$\Gamma(\Delta_2) = \langle(3,0,0)\rangle \cup \langle(2,1,0)\rangle \cup \langle(2,0,1)\rangle \cup \langle(1,2,0)\rangle \cup \langle(1,1,1)\rangle,$$

respectively. It is clear that Δ_1 is 2-linearly compressed and Δ_2 is 3-linearly shifted. However, Δ_2 is not 3-linearly compressed, since $(0,3,0) \notin \Gamma(\Delta_2)$ while $(0,3,0) <_{RL} (1,1,1) \in \Gamma(\Delta_2)$. Indeed, Δ_2 admits no 3-linear coloring for which it is 3-linearly compressed.

Theorem 3.7. *If Δ is a k-linearly shifted simplicial complex with respect to κ, then $\mathcal{U}(\Gamma) = \mathcal{U}(\Gamma(\Delta, \kappa))$ is a shifted simplicial complex over $[k]$.*

Proof. We recall that the multicomplex associated to $\mathcal{U}(\Gamma)$ is given by $\Gamma' = \{\chi_G : G \in \mathcal{U}(\Gamma)\}$ which is embedded into Γ. Since Γ is shifted, so is Γ'. In other words, the underlying complex $\mathcal{U}(\Gamma)$ is shifted as claimed. □

Let Δ be a k-linear colorable simplicial complex with the k-linear coloring map κ and let $\Gamma = \Gamma(\Delta, \kappa)$ be the associated multicomplex. We recall that the reverse lexicographical order $\leq_{RL}$ provides a linear order on the set of facets $\mathcal{F}(\Gamma)$, and since there is a bijection between $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Gamma)$, the set of facets of Δ and Γ respectively, it also induces a linear order on $\mathcal{F}(\Delta)$. Therefore, we may lexicographically order the facets of Δ by

$$F <_{RL} F' :\Leftrightarrow \mathfrak{m}_F <_{RL} \mathfrak{m}_{F'}. \tag{3.8}$$

We next verify that the prescribed order on $\mathcal{F}(\Delta)$ is in fact a shelling order whenever Δ is linearly shifted. However, we first recall the definition of shellability. A simplicial complex Δ is said to be *shellable* if there exists an order $F_1, \ldots, F_n$ of the facets of Δ such that for every i and l with $1 \leq i < l \leq n$, there is a j with $1 \leq j < l$ and an $x \in F_l$ such that $F_i \cap F_l \subset F_j \cap F_l = F_l \backslash \{x\}$.

Proposition 3.9. *Every k-linearly shifted simplicial complex is shellable.*

Proof. Let Δ be a k-linearly shifted with respect to a k-linear coloring map κ and let $\Gamma = \Gamma(\Delta, \kappa)$ be the associated multicomplex. Assume that $F_1, F_2, \ldots, F_t$ is the resulting order of facets according to the induced order, i.e., $F_i <_{RL} F_j$ if and only if $i < j$, and let F_i and F_j be two facets of Δ such that $i < j$. We write $\mathfrak{m}_{F_j} = (m(1), \ldots, m(k))$ and $\mathfrak{m}_{F_i} = (n(1), \ldots, n(k))$, and since $\mathfrak{m}_{F_i} <_{RL} \mathfrak{m}_{F_j}$, there exists $t \in [k]$ such that $m(t) > n(t)$ and $m(r) = n(r)$ for all $r > t$. So, let $v \in F_j \backslash F_i$ be a vertex with $\kappa(v) = t$ and define $\mathfrak{m}' := (m(1), \ldots, m(t-2), m(t-1)+1, m(t)-1, m(t+1), \ldots, m(k))$. It follows that $\mathfrak{m}' \leq_P \mathfrak{m}_{F_j}$ so that $\mathfrak{m}' \in \Gamma(\Delta)$, since Δ is k-linearly shifted. We then let G be any face of Δ for which $\mathfrak{m}' = \mathfrak{m}_G$ and let F_l be the earliest facet of Δ with respect to $<_{RL}$ containing G. In other words, F_l is the lexicographically earliest facet such that $\mathfrak{m}' \leq \mathfrak{m}_{F_l}$, i.e., $\mathfrak{m}' \in \langle \mathfrak{m}_{F_l} \rangle$. If we write $\mathfrak{m}_{F_l} = (s(1), \ldots, s(k))$, then we must have $s(t) = m(t) - 1$ and $s(r) = m(r)$ for all $r > t$, since otherwise $\mathfrak{m}_{F_j} \in \langle \mathfrak{m}_{F_l} \rangle$ so that F_j can not be a facet by the definition of a k-linear coloring. We also note that $s(q) \geq m(q)$ for any $q < t$ which implies $\mathfrak{m}_{F_l} \cap \mathfrak{m}_{F_j} = (m(1), \ldots, m(t-1), m(t)-1, m(t+1), \ldots, m(k))$, that is, $\mathfrak{m}_{F_l} \cap \mathfrak{m}_{F_j} = \mathfrak{m}_{F_j \backslash \{v\}}$.

Therefore, $\mathfrak{m}_{F_l} <_{RL} \mathfrak{m}_{F_j}$, that is $l < j$. On the other hand, since $v \notin G$ and $s(t) = m(t) - 1$, it follows that $v \notin F_l$. Furthermore, we have $\mathfrak{m}_{F_i} \cap \mathfrak{m}_{F_j} \leq \mathfrak{m}_{F_l} \cap \mathfrak{m}_{F_j}$; hence, $F_i \cap F_j \subset F_l \cap F_j = F_j \backslash \{v\}$, which means that $<_{RL}$ induces a shelling for Δ. □

We next describe an alternative way to decide whether a k-linear coloring of a pure simplicial complex induces a shelling.

Definition 3.10. Let Δ be a k-linear colorable pure simplicial complex with the k-linear coloring map κ. For any face $G \in \Delta$, we say that G has a *perfect descent* at $v \in G$ if there exists a vertex $w \notin G$ such that $\kappa(w) < \kappa(v)$ and $G \backslash \{v\} \cup \{w\} \in \Delta$. We denote the set of vertices of G at which it has a perfect descent by $D_p(G)$.

Theorem 3.11. *A k-linear coloring κ of a pure simplicial complex Δ induces a shelling if and only if $F <_{RL} F'$ implies that $D_p(F') \not\subseteq F$ for any two facets $F, F' \in \mathcal{F}(\Delta)$.*

Proof. Let $F_1, F_2, \ldots, F_t$ be the resulting order of facets according to the induced order, i.e., $F_i <_{RL} F_j$ if and only if $i < j$.

Assume first that $<_{RL}$ provides a shelling order, and F_i, F_j are any two facets such that $i < j$. It then follows that there exists an l with $1 \leq l < j$ and $v \in F_j$ such that $F_i \cap F_j \subseteq F_l \cap F_j = F_j \backslash \{v\}$. Let w be the vertex of F_l not contained in F_j. The fact $\mathfrak{m}_{F_l} <_{RL} \mathfrak{m}_{F_j}$ guaranties that $\kappa(w) < \kappa(v)$, and since $F_l = F_j \backslash \{v\} \cup \{w\} \in \Delta$, it follows that $v \in D_p(F_j)$; hence, $D_p(F_j) \not\subseteq F_i$ as claimed.

For the sufficiency, suppose that $<_{RL}$ satisfies the required condition, and let F_i, F_j be any two facets such that $i < j$. Since $D_p(F_j) \nsubseteq F_i$, there exists $x \in D_p(F_j)$ for which $x \notin F_i \cap F_j$. On the other hand, by the definition, there is a $y \notin F_j$ such that $\kappa(y) < \kappa(x)$ and $F_h := F_j \backslash \{x\} \cup \{y\} \in \Delta$. It follows that $\mathrm{m}_{F_h} <_{RL} \mathrm{m}_{F_j}$, i.e., $h < j$, and $F_i \cap F_j \subseteq F_h \cap F_j = F_j \backslash \{x\}$. Therefore, $<_{RL}$ is a shelling. □

Corollary 3.12. *If Δ is a k-linearly shifted simplicial complex, then it is homotopy equivalent to wedge of spheres and the Betti numbers of Δ are given by*

$$\beta_i(\Delta) = |\{F \in \mathcal{U}(\Gamma(\Delta)) \colon |F| = i+1 \text{ and } F \cup \{1\} \notin \mathcal{U}(\Gamma(\Delta))\}|, \tag{3.13}$$

for $i \leq k$ and $\beta_i(\Delta) = 0$ if $i > k$.

Proof. The first claim follows from Proposition 3.9 together with the fact that any shellable complex is homotopy equivalent to wedge of spheres. For the later, the simplicial complexes Δ and $\mathcal{U}(\Gamma(\Delta))$ are homotopy equivalent by Theorem 2.2, and $\mathcal{U}(\Gamma(\Delta))$ is an ordinary shifted simplicial complex over $[k]$. Therefore, the claim follows from Theorem 3 of [1]. □

4. Linear Combinatorial Shifting

In this section, we introduce a combinatorial shifting operation on multicomplexes that carries similar properties required for the ordinary shifting operation on simplicial complexes.

Let $\mathcal{M}$ be a set of multisets over $[k]$. If $\mathrm{m} = (m(1), \ldots, m(k)) \in \mathcal{M}$ and $1 \leq i < j \leq k$ are given with $m(j) \geq 1$, we define a new multiset $\mathrm{m}_{ij} = (m_{ij}(1), \ldots, m_{ij}(k))$ by $m_{ij}(i) := m(i) + 1$, $m_{ij}(j) := m(j) - 1$ and $m_{ij}(r) := m(r)$ if $r \in [k] \backslash \{i, j\}$. This provides us an operation $\mathrm{m} \mapsto C_{ij}(\mathrm{m})$ for all $i, j \in [k]$ with $i < j$ on the set $\mathcal{M}$ defined by

$$C_{ij}(\mathrm{m}) := \begin{cases} \mathrm{m}_{ij}, & \text{if } m(j) \geq 1 \text{ and } \mathrm{m}_{ij} \notin \mathcal{M}, \\ \mathrm{m}, & \text{otherwise}, \end{cases} \tag{4.1}$$

and we write C_{ij}^r for the composition of C_{ij} with itself r-times for $r \geq 1$.

Definition 4.2. A subset $\mathcal{M}$ of $\mathbb{N}^k$ is called *stable* if $C_{ij}(\mathrm{m}) = \mathrm{m}$ for all $\mathrm{m} \in \mathcal{M}$ and $1 \leq i < j \leq k$.

Proposition 4.3. *A subset $\mathcal{M} \subset \mathbb{N}^k$ is shifted if and only if it is stable.*

Proof. Assume that $\mathcal{M}$ is shifted. Let $\mathrm{m} \in \mathcal{M}$ and $1 \leq i < j \leq k$ be given with $m(j) \geq 1$. Since $\mathrm{m}_{ij} \leq_P \mathrm{m}$ and $\mathcal{M}$ is shifted, we have $\mathrm{m}_{ij} \in \mathcal{M}$; hence $C_{ij}(\mathrm{m}) = \mathrm{m}$. Thus, $\mathcal{M}$ is stable.

Conversely, suppose $\mathcal{M}$ is stable and let $\mathrm{n} \leq_P \mathrm{m}$ be given such that $\mathrm{m} \in \mathcal{M}$. Let us write $\mathrm{n} = (n(1), \ldots, n(k))$ and $\mathrm{m} = (m(1), \ldots, m(k))$. If $t \in [k]$ is the integer such that $n(t) < m(t)$ and $n(s) = m(s)$ for all $s > t$, we define

$\mathfrak{m}_t := (m_t(1), \ldots, m_t(k))$ by $m_t(t) = n(t), m_t(t-1) = m(t-1)+m(t)-n(t)$ and $m_t(r) = m(r)$ for $r \neq t, t-1$. We note that $\mathfrak{m}_t \in \mathcal{M}$, since $\mathcal{M}$ is stable. On the other hand, we have $\mathfrak{n} \leq_P \mathfrak{m}_t$ together with $n(s) = m_t(s)$ for all $s \geq t$. We replace $\mathfrak{m}$ with $\mathfrak{m}_t$ and apply the same process to $\mathfrak{n} \leq_P \mathfrak{m}_t$. It is now clear that there exists a $q \leq t$ such that $\mathfrak{n} = \mathfrak{m}_q \in \mathcal{M}$ which completes the proof. □

An important consequence of Proposition 4.3 is that if $\mathfrak{n}, \mathfrak{m}$ are two multisets with $\mathfrak{n} \leq_P \mathfrak{m}$, then $\mathfrak{n}$ can be obtained from $\mathfrak{m}$ by a finite number of applications of the operations C_{ij} for some $i < j$. This implies the following useful fact.

Corollary 4.4. *For any finite set $\mathcal{M}$, there exists a sequence of pairs of integers $(i_1, j_1), \ldots, (i_q, j_q)$ with $1 \leq i_s < j_s \leq k$ such that*

$$C_{i_q j_q}(C_{i_{q-1} j_{q-1}}(\ldots(C_{i_1 j_1}(\mathcal{M})\ldots)))$$

is shifted, where $C_{ij}(\mathcal{M}) := \{C_{ij}(\mathfrak{m}) : \mathfrak{m} \in \mathcal{M}\}$ for each $1 \leq i < j \leq k$.

Even though, any finite set can be brought to a shifted set by a finite number of applications of the operations C_{ij} by Corollary 4.4, the resulting shifted set is not uniquely determined as in the ordinary case, that is, it is dependent on the choices of ij and the order of operations. Any set that is obtained from $\mathcal{M}$ by a sequence of operations as described in Corollary 4.4 will be denoted by $\mathcal{C}(\mathcal{A})$. As an example, we compute $\mathcal{C}(\mathcal{A})$ for $\mathcal{A} = \{(1,1,0,1), (1,2,0,0), (0,0,2,0)\}$ as follows.

$$\begin{aligned}
\mathcal{C}(\mathcal{A}) &= C_{12}(C_{13}(C_{13}(C_{14}(\mathcal{A})))) \\
&= C_{12}(C_{13}(C_{13}(\{(2,1,0,0), (1,2,0,0), (0,0,2,0)\}))), \\
&= C_{12}(C_{13}(\{(2,1,0,0), (1,2,0,0), (1,0,1,0)\})) \\
&= C_{12}(\{(2,1,0,0), (1,2,0,0), (2,0,0,0)\}), \\
&= \{(3,0,0,0), (2,1,0,0), (2,0,0,0)\}).
\end{aligned}$$

It is important to note that when Γ is a multicomplex on $[k]$, it is not generally true that $C_{ij}(\Gamma)$ is a multicomplex for all $1 \leq i < j \leq k$. To illustrate an example, we let $\Gamma = \langle(1,0,2)\rangle \cup \langle(0,1,0)\rangle$ so that it is a multicomplex on $[3]$. However, $C_{23}(\Gamma)$ is not a multicomplex, since $(1,0,1) \notin C_{23}(\Gamma)$, while $(1,0,1) \leq (1,1,1) \in C_{23}(\Gamma)$. We also remark that $C_{23}(\Gamma)$ is not shifted as well. We clarify this fact as follows.

Proposition 4.5. *Let Γ be a multicomplex on $[k]$ and let $i, j \in [k]$ be given with $i < j$. If $C_{ij}(\Gamma)$ is not a multicomplex, then it is not shifted.*

Proof. Assume that $C_{ij}(\Gamma)$ is not a multicomplex. Therefore, there exist $\mathfrak{m} \in C_{ij}(\Gamma)$ and $\mathfrak{n} \leq \mathfrak{m}$ such that $\mathfrak{n} \notin C_{ij}(\Gamma)$. Without loss of generality, we may further suppose that $\|\mathfrak{m}\| = \|\mathfrak{n}\| + 1$. There are two possible cases and we look at each separately.

Case 1: Assume that $\mathfrak{m} \in \Gamma$. Thus, $\mathfrak{n} \in \Gamma$, since Γ is a multicomplex. On the other hand, we must have $C_{ij}(\mathfrak{m}) = \mathfrak{m}$ and $C_{ij}(\mathfrak{n}) = \mathfrak{n}_{ij}$ by the choices of $\mathfrak{m}$ and $\mathfrak{n}$. If we write $\mathfrak{m} = (m(1), \ldots, m(k))$ and $\mathfrak{n} = (n(1), \ldots, n(k))$, it follows that either $m(j) = 0$ so that $n(j) = 0$ and $C_{ij}(\mathfrak{n}) = \mathfrak{n}$, a contradiction, or $m(j) \geq 1$ and $\mathfrak{m}_{ij} \in \Gamma$. For the latter, we have $\mathfrak{n}_{ij} = (n(1), \ldots, n(i)+1, \ldots, n(j)-1, \ldots, n(k)) \leq \mathfrak{m}_{ij} \in \Gamma$; hence, $\mathfrak{n}_{ij} \in \Gamma$ which implies again $C_{ij}(\mathfrak{n}) = \mathfrak{n}$, a contradiction.

Case 2: Suppose that there exists $\mathfrak{m}' \in \Gamma$ such that $C_{ij}(\mathfrak{m}') = \mathfrak{m}'_{ij} = \mathfrak{m} \notin \Gamma$. Since $\mathfrak{n} \leq \mathfrak{m}$ and $\|\mathfrak{m}\| = \|\mathfrak{n}\| + 1$, there is an $s \in [k]$ such that $n(s) = m(s) - 1$ and $n(r) = m(r)$ for all $r \in [k]\backslash\{s\}$. If $s \neq i$, we then define $\mathfrak{n}' = (m'(1), \ldots, m'(s)-1, \ldots, m'(k))$. Since $\mathfrak{n}' \leq \mathfrak{m}' \in \Gamma$, we deduce that $\mathfrak{n}' \in \Gamma$. Furthermore, $C_{ij}(\mathfrak{n}') = \mathfrak{n}'_{ij} = \mathfrak{n}$ so that $\mathfrak{n} \in C_{ij}(\Gamma)$ that contradicts to the choice of $\mathfrak{n}$. On the other hand, if $s = i$, we conclude that $\mathfrak{n} = (m'(1), \ldots, m'(i), \ldots, m'(j) - 1, \ldots, m'(k))$. Thus, $\mathfrak{n} \leq \mathfrak{m}'$ and $\mathfrak{n} \in \Gamma$. Since $\mathfrak{n} \notin C_{ij}(\Gamma)$, we also have $C_{ij}(\mathfrak{n}) = \mathfrak{n}_{ij} \notin \Gamma$, where $\mathfrak{n}_{ij} = (m'(1), \ldots, m'(i)+1, \ldots, m'(j)-2, \ldots, m'(k))$. Now, if $C_{ij}(\Gamma)$ is shifted, it follows that $C_{ij}(C_{ij}(\Gamma)) = C_{ij}(\Gamma)$. In particular, $C_{ij}(\mathfrak{m}) = C_{ij}(\mathfrak{m}'_{ij}) = \mathfrak{m}'_{ij}$ that implies $(\mathfrak{m}'_{ij})_{ij} \in C_{ij}(\Gamma)$. We note that $m'(j) \geq 2$, since $\mathfrak{n}_{ij}$ is a multiset. Finally, we must have an $\mathfrak{m}'' \in \Gamma$ such that $C_{ij}(\mathfrak{m}'') = (\mathfrak{m}'_{ij})_{ij}$. However, such a multiset $\mathfrak{m}'' \in \Gamma$ satisfies $\mathfrak{n}_{ij} \leq \mathfrak{m}''$; thus, $\mathfrak{n}_{ij} \in \Gamma$, a contradiction. This completes the proof. □

Another way to look at Proposition 4.5 is that if $C_{ij}(\Gamma)$ is shifted for some $i < j$, then it has to be a multicomplex on $[k]$. In fact, we have the following.

Proposition 4.6. *Let Γ be a multicomplex on $[k]$ and let let $i, j \in [k]$ be given with $i < j$. Then there exists a positive integer $r_j \geq 1$ such that $C_{ij}^{r_j}(\Gamma)$ is a multicomplex.*

Proof. For any $i < j$, we define $r_j := \max\{m(j) : \mathfrak{m} \in \Gamma\}$ and claim that $C_{ij}^{r_j}(\Gamma)$ is a multicomplex. We first note that the set $C_{ij}^{r_j}(\Gamma)$ is stable under the operation C_{ij}, that is, if $\mathfrak{a} \in C_{ij}^{r_j}(\Gamma)$, then $C_{ij}(\mathfrak{a}) = \mathfrak{a}$, since $a(j) \leq r_j$, where $\mathfrak{a} = (a(1), \ldots, a(k))$. Therefore, if $\mathfrak{c} = (c(1), \ldots, c(k)) \in C_{ij}^{r_j}(\Gamma)$ and $\mathfrak{b} = (b(1), \ldots, b(k))$ is a multiset such that $\mathfrak{b} \leq_P \mathfrak{c}$ and $c(s) = b(s)$ for all $s \in [k]\backslash\{i, j\}$, then $\mathfrak{b} \in C_{ij}^{r_j}(\Gamma)$.

To verify the claim, let $\mathfrak{n}, \mathfrak{m}$ be given such that $\mathfrak{m} \in C_{ij}^{r_j}(\Gamma)$ and $\mathfrak{n} \leq \mathfrak{m}$. Again, without loss of generality, we may suppose that $\|\mathfrak{m}\| = \|\mathfrak{n}\| + 1$. Since $\mathfrak{m} \in C_{ij}^{r_j}(\Gamma)$, there exists an $\mathfrak{m}' \in \Gamma$ such that $C_{ij}^{r_j}(\mathfrak{m}') = \mathfrak{m}$. If we write $\mathfrak{m}' = (m'(1), \ldots, m'(k))$, then $1 \leq m'(j) \leq r_j$ and $\mathfrak{m} = \mathfrak{m}'_t$ for some $t \leq m'(j)$.

If $t = 0$, that is, $\mathfrak{m} = \mathfrak{m}' \in \Gamma$, we obtain that $\mathfrak{n} \in \Gamma$, since $\mathfrak{n} \leq \mathfrak{m}$ and Γ is a multicomplex. Moreover, the fact $C_{ij}^{r_j}(\mathfrak{m}) = \mathfrak{m}$ implies that $\mathfrak{m}_l \in \Gamma$ for all $0 \le l \le m(j)$ which in particular forces $\mathfrak{n}_s \in \Gamma$ for any $s \le n(j) \le m(j)$, since $\mathfrak{n}_s \le \mathfrak{m}_s \in \Gamma$. Therefore, we conclude that $C_{ij}^{r_j}(\mathfrak{n}) = \mathfrak{n}$ as expected.

Assume that $t \ge 1$. Since $\|\mathfrak{m}\| = \|\mathfrak{n}\| + 1$, there exists a $v \in [k]$ such that $n(v) = m(v) - 1$ and $n(r) = m(r)$ for all $r \in [k]\backslash\{v\}$. If $v \ne i$, we define $\mathfrak{n}' := (m'(1), \dots, m'(v) - 1, \dots, m'(k))$ so that $\mathfrak{n}' \le \mathfrak{m}'$; hence, $\mathfrak{n}' \in \Gamma$. It follows that $\mathfrak{n} \le_P C_{ij}^{r_j}(\mathfrak{n}')$; thus, $\mathfrak{n} \in C_{ij}^{r_j}(\Gamma)$, since $C_{ij}^{r_j}(\Gamma)$ is stable under the operation C_{ij}. Furthermore, if $v = i$, we set $\mathfrak{n}' := (m'(1), \dots, m'(j) - 1, \dots, m'(k))$, and note that if $m'(j) = 1$, we must have $\mathfrak{n} = \mathfrak{n}'$. It follows similarly that $\mathfrak{n} \le_P C_{ij}^{r_j}(\mathfrak{n}')$ so that $\mathfrak{n} \in C_{ij}^{r_j}(\Gamma)$ as claimed. □

By combining Propositions 4.6 with Corollary 4.4, we obtain the following important result.

Corollary 4.7. *If Γ is a multicomplex on $[k]$, then there exists a sequence of pairs of integers $(i_1, j_1), \dots, (i_q, j_q)$ with $1 \le i_s < j_s \le k$ such that*

$$\mathcal{C}(\Gamma) = C_{i_q j_q}(C_{i_{q-1} j_{q-1}}(\dots(C_{i_1 j_1}(\Gamma)\dots)))$$

is a shifted multicomplex on $[k]$.

Theorem 4.8. *For any multicomplex Γ on $[k]$, the multicomplex $\mathcal{C}(\Gamma)$ satisfies the followings.*

(M1) $\mathcal{C}(\Gamma)$ *is shifted.*
(M2) $\mathcal{C}(\Gamma) = \Gamma$ *if Γ is shifted.*
(M3) $f(\Gamma) = f(\mathcal{C}(\Gamma))$.

Proof. The condition $(M2)$ is a consequence of Proposition 4.3, and for $(M3)$, we note that $|C_{ij}(\mathcal{M})| = |\mathcal{M}|$ for any set $\mathcal{M} \subset \mathbb{N}^k$ and $1 \le i < j \le k$. □

Definition 4.9. Let Δ be a k-linear colorable simplicial complex with the k-linear coloring map κ and let $\Gamma = \Gamma(\Delta, \kappa)$ be the associated multicomplex. We define $\mathcal{C}_\kappa(\Delta)$ to be the simplicial complex associated to the multicomplex $\mathcal{C}(\Gamma)$. This simplicial complex $\mathcal{C}_\kappa(\Delta)$ is called the *k-linear shifting* of Δ with respect to κ. In particular, if κ is an lchr(Δ)-linear coloring of Δ, we write $\mathcal{S}_\kappa(\Delta) := \mathcal{C}_\kappa(\Delta)$.

Example 4.10. We illustrate in Figure 2, a 2-linear colorable simplicial complex and its 2-linear shifting with respect to given coloring. It is interesting to note that contrary to ordinary case, a k-linear combinatorial shifting of a pure simplicial complex may results a nonpure simplicial complex. Furthermore, we remark that $f(\Delta) = f(\mathcal{S}_\kappa(\Delta))$ in this case.

Corollary 4.11. *Let Δ be a k-linear colorable simplicial complex with the k-linear coloring map κ. Then the simplcial complex $\mathcal{C}_\kappa(\Delta)$ is k-linearly shifted. Moreover, if Δ is k-linearly shifted, then $\mathcal{C}_\kappa(\Delta)$ and Δ are isomorphic.*

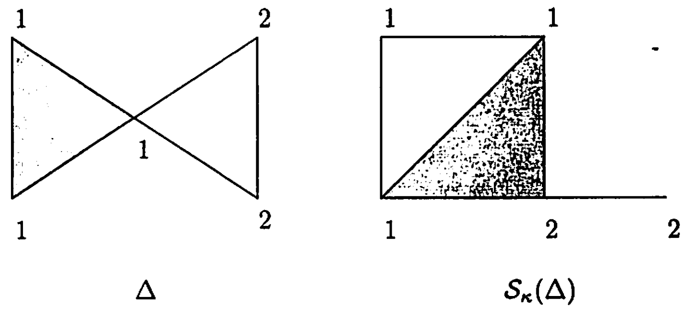

FIGURE 2. A 2-linear shifting of a 2-linear colorable complex.

Proof. The former follows from the fact that the simplicial complex $\mathcal{C}_\kappa(\Delta)$ is associated to a shifted multicomplex over $[k]$, and the latter is a consequence of the fact that a shifted multicomplex is invariant under shifting operations. □

It is obvious that the assignment $\Delta \mapsto \mathcal{S}_\kappa(\Delta)$ is not generally a shifting operation, i.e., it may not satisfy the conditions $(S3)-(S4)$ given in Section 1. However, since there are cases where it turns out to be a shifting operation, one may still argue as follows.

Problem 4.12. When is the association $\Delta \mapsto \mathcal{S}_\kappa(\Delta)$ a shifting operation?

For instance, if a simplicial complex is disconnected, the prescribed assignment is not a shifting operation. To provide an example, let Δ be the simplicial complex consisting of the disjoint union of two 1-dimensional simplexes. The linear chromatic number of Δ is 2, while the f-vectors of Δ and $\mathcal{S}_\kappa(\Delta)$ are clearly different.

5. Linearly shifted independence complexes

In this final section, we provide a characterization of simple graphs whose independence complexes are linearly shifted. The work presented here may be considered as parallel to that of Klivans [7] and [8], where she provided such a characterization for the ordinary shifting operation.

By a simple graph $G = (V, E)$, we mean an undirected graph without loops and multiple edges. The (open) *neighborhood* of a vertex $v \in V$ is defined to be $\mathcal{N}(v) := \{u \in V : (u, v) \in E\}$, while $\mathcal{N}[v] := \mathcal{N}(v) \cup \{v\}$ denotes the closed neighborhood of v. A subset $U \subseteq V$ is said to be an *independent set* of G, if there is no edge in G among the vertices contained by U, and the *independence complex* $\mathcal{I}(G)$ of G is defined to be the simplicial complex on V whose faces are formed by its independent sets.

We recall that a (vertex) coloring of G is a surjective mapping $\nu\colon V \to [n]$ such that $\nu(u) \neq \nu(v)$ whenever $(u,v) \in E$. The least integer n for which G admits a coloring is called the (vertex) *chromatic number* of G and denoted by $\chi(G)$. It follows that if $\kappa\colon V \to [k]$ is a linear coloring of $\mathcal{I}(G)$, then it is also a (vertex) coloring of G, since $\{u,v\} \notin \mathcal{I}(G)$ whenever $(u,v) \in E$. Therefore, we have the inequality $\chi(G) \leq \mathrm{lchr}(\mathcal{I}(G))$. We now provide the required condition for a coloring of a graph to be a linear coloring of its independence complex.

Proposition 5.1. *A coloring $\kappa\colon V \to [k]$ of $G = (V,E)$ is a linear coloring of $\mathcal{I}(G)$ if and only if either $\mathcal{N}(v) \subseteq \mathcal{N}(u)$ or $\mathcal{N}(u) \subseteq \mathcal{N}(v)$ holds for every $u,v \in V$ with $\kappa(u) = \kappa(v)$.*

Proof. Assume that whenever $\kappa(u) = \kappa(v)$ for any two vertices $u,v \in V(G)$, then one of the inclusions $\mathcal{N}(v) \subseteq \mathcal{N}(u)$ or $\mathcal{N}(u) \subseteq \mathcal{N}(v)$ holds. Let $u,v \in V(G)$ be two such vertices and let $\mathcal{N}(u) \subseteq \mathcal{N}(v)$. To verify that $\mathcal{F}_{\mathcal{I}(G)}(v) \subseteq \mathcal{F}_{\mathcal{I}(G)}(u)$, let F be a facet of $\mathcal{I}(G)$ containing v. Assume that $z \in F$ be given. If $(z,u) \in E$, then $z \in \mathcal{N}(u) \subset \mathcal{N}(v)$; hence, $(z,v) \in E$, a contradiction. Therefore, $(z,u) \notin E$, i.e., $F \cup \{u\} \in \mathcal{I}(G)$. Since F is facet, we must have $u \in F$.

Suppose that $\kappa\colon V \to [k]$ be a linear coloring of G, and let $\kappa(x) = \kappa(y)$ for $x,y \in V$. By the definition of linearity, we may assume without losing generality that $\mathcal{F}_{\mathcal{I}(G)}(x) \subseteq \mathcal{F}_{\mathcal{I}(G)}(y)$. We claim that $\mathcal{N}(y) \subseteq \mathcal{N}(x)$ must hold. To see that let $z \in \mathcal{N}(y)$ be given such that $z \notin \mathcal{N}(x)$. In other words, $(z,x) \notin E$. It follows that the set $\{x,z\}$ forms a face of $\mathcal{I}(G)$. If F is a facet of $\mathcal{I}(G)$ containing this face, we then have $y \in F$ by our assumption. Therefore, $(z,y) \notin E$; hence, $z \notin \mathcal{N}(y)$, a contradiction. □

Klivans proved in [8] that for a simple graph $G = (V,E)$, the independence complex $\mathcal{I}(G)$ is shifted in the ordinary sense if and only if G is a threshold graph. We recall that a simple graph is called *threshold* if one of the inclusions $\mathcal{N}(x) \subset \mathcal{N}[y]$ or $\mathcal{N}(y) \subset \mathcal{N}[x]$ is satisfied for any two vertices. Alternatively, a graph is threshold if and only if it can be constructed from the one-vertex graph by repeatedly adding a disjoint vertex or a starred vertex. Following her notation, we will continue to denote these operations by D and S respectively, where the operation D adds a new disjoint vertex to a graph, while S adds a new vertex adjacent to all former vertices of the graph.

We next prove that the independence complexes of threshold graphs are linearly shifted, whereas the converse holds no longer as shown in Example 5.4.

Proposition 5.2. *If G is a threshold graph, then $\mathcal{I}(G)$ is linearly shifted.*

Proof. Let $G = (V,E)$ be a threshold graph. Then we necessarily have $\chi(G) = \mathrm{lchr}(\mathcal{I}(G))$, since any vertex coloring of G is a linear coloring of $\mathcal{I}(G)$ in this case. Indeed, let $\mu\colon V \to [k]$ be a vertex coloring of G, and let $\mu(u) = \mu(v)$ for some $u, v \in V$. Since G is threshold, we may assume without loss of generality that $\mathcal{N}(u) \subseteq \mathcal{N}[v]$. However, the equality $\mu(u) = \mu(v)$ implies that $(u, v) \notin E$. In other words, $\mathcal{N}(u) \subseteq \mathcal{N}(v)$; hence, μ is a linear coloring of $\mathcal{I}(G)$ by Proposition 5.1.

Therefore, it is enough to show that $\mathcal{I}(G)$ is $\chi(G)$-linearly shifted. We consider the string representation of G formed by D's and S's which is written from left to right, and assume that the number of staring operations in this representation is equal to k; hence, $\chi(G) = k + 1$. We define a linear coloring $\kappa\colon V \to [k+1]$ by labeling the vertices represented by D operations by 1, and label the vertices corresponding to S operations by 2 through $k+1$ from left to right. We note that κ is linear, since it is clearly a vertex coloring of G. Furthermore, the associated multicomplex $\Gamma(\mathcal{I}(G), \kappa)$ is shifted, since $\mathcal{I}(G)$ contains no edges between vertices represented by S operations. Thus, $\mathcal{I}(G)$ is $\chi(G)$-linearly shifted with respect to the linear coloring κ. □

Example 5.3. We illustrate the threshold graph G and its independence complex in Figure 3, whose representation is given by $G = DDSSDS$.

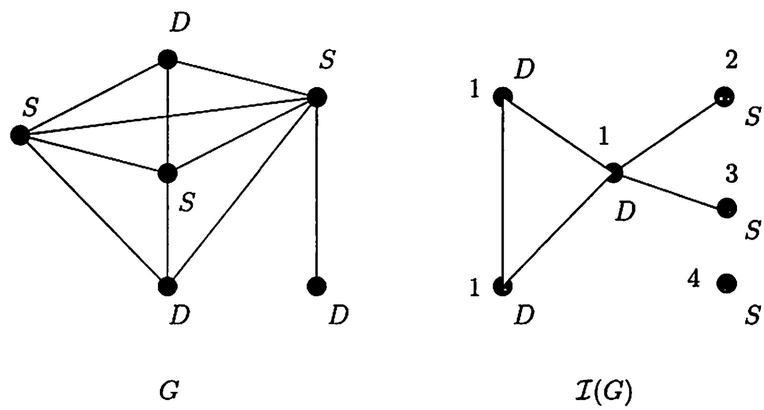

FIGURE 3

Example 5.4. For the simple graph G depicted in Figure 4, its independence complex $\mathcal{I}(G)$ is 2-linearly shifted, while G is not threshold. Furthermore, the linear shiftedness of the independence complex is not generally inherited by the induced subgraphs of a linearly shifted graph. For example, the independence complex of the subgraph of G induced by $\{x, y, u, z\}$ is not linearly shifted.

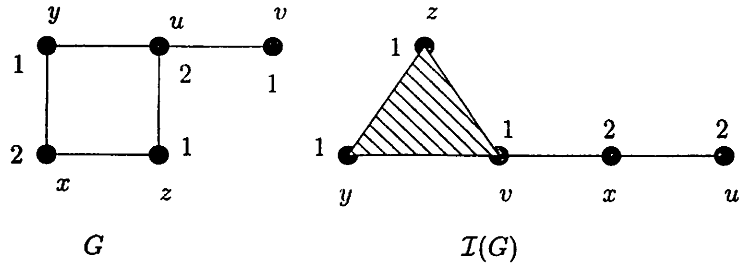

FIGURE 4

We remark that the equality $\chi(G) = \mathrm{lchr}(\mathcal{I}(G))$ for a threshold graph is not a coincidence, as we shall prove next that it holds for all simple graphs with linearly shifted independence complexes.

Proposition 5.5. *If $\mathcal{I}(G)$ is linearly shifted, then* $\chi(G) = \mathrm{lchr}(\mathcal{I}(G))$.

Proof. Let $G = (V, E)$ be a simple graph with $|V| = m$, and assume that $\mathcal{I}(G)$ is n-linearly shifted with respect to an n-linear coloring $\nu: V \to [n]$ of $\mathcal{I}(G)$, and suppose that $\chi(G) = k$ for some $k < n$. Then there exists a pair of vertices $u, v \in V$ with $(u, v) \notin E$ such that $\nu(u) \neq \nu(v)$, while the mapping $\overline{\nu}: V \to [n]\backslash\{\nu(v)\}$ defined by $\overline{\nu}(x) := \nu(x)$ if $x \in V\backslash\{v\}$ and $\overline{\nu}(v) := \nu(u)$ is a vertex coloring of G. We may assume without loss of generality that $\nu(u) = i < j = \nu(v)$. Since $\overline{\nu}$ is a vertex coloring, we must have $\nu(z) \neq i$ for any $z \in \mathcal{N}(v)$. It follows that the multiset

$$\mathrm{m}_{ij} := (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^n,$$

where 1's are i^{th} and j^{th}-position, is contained by the associated multicomplex $\Gamma(\mathcal{I}(G), \nu)$. Since it is shifted, we have $\mathrm{m}_{ii} = (0, \ldots, 0, 2, 0, \ldots, 0) \in \Gamma(\mathcal{I}(G), \nu)$. Therefore, there exists a vertex $w_1 \in V\backslash\{v, u\}$ such that $\nu(w_1) = i$. Note that $w_1 \notin \mathcal{N}(v)$ for such a vertex. Then the set $\{v, u, w_1\}$ forms a face in $\mathcal{I}(G)$. In other words, the multiset

$$\mathrm{m}_{iij} := (0, \ldots, 0, 2, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^n$$

is also contained by $\Gamma(\mathcal{I}(G), \nu)$. Once again, since $\Gamma(\mathcal{I}(G), \nu)$ is shifted, it must also contain m_{iii}, that is, there must exists a vertex $w_2 \in V\backslash\{v, u, w_1\}$ with $w_2 \notin \mathcal{N}(v)$ such that $\nu(w_2) = i$; hence, the set $\{v, u, w_1, w_2\}$ forms a face in $\mathcal{I}(G)$. Continuing in this way, we reach a contradiction, since G is a finite graph. □

We note that the equality $\chi(G) = \mathrm{lchr}(\mathcal{I}(G))$ for a simple graph G is not sufficient for guaranteeing its independence complex to be linearly shifted, as one may easily construct a counterexample.

We next investigate the local structures of simple graphs with linearly shifted independence complexes in order to provide a characterization. Assume that $G = (V, E)$ is a simple graph with $|V| = m$ such that $\mathcal{I}(G)$ is k-linearly shifted with respect to a k-linear coloring $\kappa \colon V \to [k]$, where $k = \chi(G)$. We may disregard the case where $k = 1$ for which $E = \emptyset$; thus, $\mathcal{I}(G)$ is just a simplex on m vertices. We therefore let $k > 1$. Since κ is eventually a vertex coloring of G, it partitions V into independent sets as $V = V_1 \cup \ldots \cup V_k$, where $V_i = \{v \in V : \kappa(v) = i\}$, and we write $|V_i| = r_i$ for $1 \leq i \leq k$. Since κ is a linear coloring, the vertices of each set $V_i = \{x^i_1, \ldots, x^i_{r_i}\}$ can be ordered in such a way that $\mathcal{N}(x^i_j) \subseteq \mathcal{N}(x^i_l)$ whenever $j < l$. We further note that only the set V_1 can have a disjoint vertex, i.e., a vertex which is not incident to any edge of G by the linear shiftedness of $\mathcal{I}(G)$. Moreover, for any given $i, j \in [k]$ with $i < j$, there exists an $1 < l \leq r_i$ such that $(x^i_l, x^j_1) \in E$, since otherwise the set $\{x^i_1, \ldots, x^i_{r_i}, x^j_1\}$ forms a face in $\mathcal{I}(G)$ so that the multiset

$$\mathfrak{m}_{r_ij} := (0, \ldots, 0, r_i, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^k$$

belongs to $\Gamma(\mathcal{I}(G), \kappa)$, however $\mathfrak{m} := (0, \ldots, 0, r_i + 1, 0, \ldots, 0) \notin \Gamma(\mathcal{I}(G), \kappa)$, while $\mathfrak{m} \leq_P \mathfrak{m}_{r_ij}$ that contradicts to the shiftedness of $\Gamma(\mathcal{I}(G), \kappa)$. By a similar reasoning, we conclude that whenever the set

$$\{x^i_1, x^i_2, \ldots, x^i_h, x^j_1, x^j_2, \ldots, x^j_l\}$$

is a face of $\mathcal{I}(G)$ for some $0 \leq h \leq r_i$, $l \leq r_j$ and $i, j \in [k]$ with $i < j$, then $h + l \leq r_i$, and the set

$$\{x^i_1, x^i_2, \ldots, x^i_{h+t}, x^j_1, x^j_2, \ldots, x^j_{l-t}\}$$

also forms a face in $\mathcal{I}(G)$ for all $1 \leq t \leq l$. This last fact also forces an inequality $r_i \geq r_j + e(i, j)$ on the sizes of blocks of the partition, where $i, j \in [k]$ with $i < j$, and $e(i, j) := \max\{h \in [0, r_i] : (x^i_s, x^j_t) \notin E$ for any $s \leq h$ and $t \leq r_j\}$.

We claim that these conditions suffice to characterize the class of simple graphs with linearly shifted independence complexes.

Theorem 5.6. *For a simple graph $G = (V, E)$, the independence complex $\mathcal{I}(G)$ is linearly shifted if and only if either $E = \emptyset$ or there exists a disjoint partition $V = V_1 \cup \ldots \cup V_k$ into $\chi(G) = k$ independent sets such that*

(a) *each set $V_i = \{x^i_1, \ldots, x^i_{r_i}\}$ can be ordered in such a way that $\mathcal{N}(x^i_s) \subseteq \mathcal{N}(x^i_t)$ whenever $s < t$,*

(b) *if $\{x^i_1, x^i_2, \ldots, x^i_h, x^j_1, x^j_2, \ldots, x^j_l\} \in \mathcal{I}(G)$ for some $h \in [0, r_i]$ and $l \leq r_j$, then $\{x^i_1, x^i_2, \ldots, x^i_{h+t}, x^j_1, x^j_2, \ldots, x^j_{l-t}\} \in \mathcal{I}(G)$ for any $1 \leq t \leq l$ and $i, j \in [k]$ with $i < j$.*

Proof. We only need to verify the sufficiency of the given conditions. So, let $G = (V, E)$ admit such a partition. It readily follows that the mapping

$\kappa: V \to [k]$ defined by $\kappa(x_s^i) := i$ is a linear coloring of $\mathcal{I}(G)$ by the condition (a). We claim that $\Gamma = \Gamma(\mathcal{I}(G), \kappa)$ is a shifted multicomplex. This is in turn equivalent to showing that Γ is stable under the operation C_{ij} for all $i, j \in [k]$ with $i < j$ by Proposition 4.3. Let $i, j \in [k]$ be given with $i < j$ and let $\mathfrak{n} = (n(1), \ldots, n(k)) \in \Gamma$ such that $n(j) \geq 1$. Thus, we have to verify that $C_{ij}(\mathfrak{n}) = \mathfrak{n}$, that is, $\mathfrak{n}_{ij} \in \Gamma$. Since $\mathfrak{n} \in \Gamma$, the set

$$\bigcup_{\substack{h=1 \\ n(h)>0}}^{k} \{x_1^h, \ldots, x_{n(h)}^h\}$$

forms a face in $\mathcal{I}(G)$. In particular, the set $\{x_1^i, x_2^i, \ldots, x_{n(i)}^i, x_1^j, x_2^j, \ldots, x_{n(j)}^j\} \in \mathcal{I}(G)$; hence, we conclude that $\{x_1^i, x_2^i, \ldots, x_{n(i)+1}^i, x_1^j, x_2^j, \ldots, x_{n(j)-1}^j\} \in \mathcal{I}(G)$ by the condition (b), that is,

$$\mathfrak{n}_{ij} = (n(1), \ldots, n(i-1), n(i)+1, n(i+1), \ldots, n(j-1), n(j)-1, n(j+1), \ldots, n(k)) \in \Gamma$$

as claimed. □

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