On the vertex distinguishing equitable edge-coloring of graphs [1]

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Abstract

Abstract: A vertex-distinguishing edge-coloring (VDEC) of a simple graph G which contains no more than one isolated vertex and no isolated edge is *equitable* (VDEEC) if the absolute value of difference between the number of edges colored by color i and the number of edges colored by color j is at most one. The minimal number of colors needed such that G has a VDEEC is called the vertex distinguishing equitable chromatic index of G. In this paper we propose two conjectures after investigating VDEECs on some special families of graphs, such as the stars, fans, wheels, complete graphs, complete bipartite graphs etc.

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1 Simple introduction and concepts

As we have known, many problems can be regarded as graph coloring problems, such as time tabling and scheduling, frequency assignment, register allocation, labeling a point set, computer security and electronic banking, integers assignment and so on. Of course, many problems of coloring involve the question of how to color objects efficiently. These questions lead

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to consideration of computational complexity of algorithms. After investigating various graph colorings of decades, most graph colorings, as far as we have known, have been verified to be very difficult and challenging because many of them have been proved being **NP**-complete problems. In particular, for some special graphs, some problems can be performed in polynomial times.

It is interesting that the frequency assignment problem is somewhat similar to that investigated by the distinguishing methods. In 1973, Meyer [8] introduced the notion of equitable (vertex) coloring of graphs and proposed the conjecture that the equitable chromatic number of a connected graph G, which is neither a complete graph nor an odd cycle, is at most $\Delta(G)$. Motivated, we study the vertex-distinguishing equitable edge coloring on graphs.

In the following argument, all graphs are simple and undirected. Let integers $m > 0$ and $n > 0$. For simplicity, we use the symbol $[n]^0$ to indicate an integer set $\{0, 1, 2, \dots, n\}$ for integer $n > 0$, and $[n] = [n]^0 \setminus \{0\}$, and $[m, n] = \{m, m+1, \dots, n\}$ where integers n and m satisfies that $n > m > 0$.

For a simple graph G, an edge k-coloring f from $E(G)$ to $[k]$ is *proper* if $f(uv) \neq f(vx)$ for all pairs of incident edges $uv, vx \in E(G)$. The notation $C(u)$ denotes the set of all labels $f(uv)$ of edges uv which are incident to the vertex $u \in V(G)$, and we often write the color set $C = [k]$ as well as $\overline{C(u)} = C \setminus C(u)$. Another symbol $n_i(G)$ is used to indicate the number of all vertices of degree i in G, throughout this paper.

Definition 1. [6] Let f be a proper edge k-coloring of a simple graph G and let $C(u) = \{f(uv) \mid uv \in E(G)\}$ for every $u \in V(G)$. The coloring f is called a *vertex distinguishing edge k-coloring* (k-VDEC) of G if $C(u) \neq C(v)$ for distinct vertices $u, v \in V(G)$. The number $\min\{k \mid k\text{-VDEC of } G\}$ is called *vertex distinguishing chromatic index* of G, denoted by $\chi'_{vd}(G)$.

In particular, a graph G is called a *vdec*-graph if it has a k-VDEC for some positive integer k. In the paper [13], the authors defined that the *combinatorial degree* of a *vdec*-graph G, $\mu(G)$, is defined by the number $\min\{\mu \mid \binom{\mu}{i} \geq n_i, \delta \leq i \leq \Delta\}$. They conjectured that for a *vdec*-graph G, there exists that $\chi'_{vd}(G) \leq \mu(G) + 1$ (cf. [13]).

Definition 2. [8] Suppose that G is a *vdec*-graph, so it has a k-VDEC f for some positive integer k. Let $E_i = \{e \in E(G) \mid f(e) = i\}$ for each $i \in [k]$. We sat f a *vertex-distinguishing equitable edge k-coloring* (k-VDEEC) if $\big||E_i| - |E_j|\big| \leq 1$ for distinct integers i and j with respect to $1 \leq i, j \leq k$. The *vertex distinguishing equitable chromatic index* of G, denoted as $\chi'_{vde}(G)$, is defined by the number $\min\{k \mid k\text{-VDEEC of } G\}$.

The graphs, mentioned here, are simple and finite; and the standard notations of graph theory used here have been introduced in [5]. We need an operation on graphs in the paper in the following. The join graph of two disjoint graphs G and H, denoted by $G \vee H$, is a graph with its vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set

$$E(G \vee H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}.$$

2 Main results

The following lemma is an immediate consequence of Definition 2.

Lemma 1. *For a simple graph G which has no isolated edge and at most one isolated vertex, there exists that $\chi'_{vd}(G) \geq \mu(G)$.*

Theorem 2. *For a star S_n on $n+1$ vertices ($n \geq 2$), we have $\chi'_{vde}(S_n) = n$.*

Proof. Let S_n be a star with vertex set $V(S_n) = \{v_i \mid i \in [n]^0\}$ and edge set $E(S_n) = \{v_0v_i \mid i \in [n]\}$. Obviously, we have $\mu(S_n) = n$. To verify the correctness of this theorem, by Lemma 1, it is sufficient to provide a n-VDEEC of S_n. Define an edge coloring f being as $f(v_0v_i) = i$ for every $i \in [n]$. It follows from Definition 2 that f is truely a n-VDEEC of S_n. □

Theorem 3. *For a complete graph K_n on $n \geq 3$ vertices, we have $\chi'_{vde}(K_n) = n$ if $n \equiv 1 \pmod 2$, and $\chi'_{vde}(K_n) = n+1$ if $n \equiv 0 \pmod 2$.*

Proof. Let K_n be a complete graph on n vertices with vertex set $V(K_n) = \{v_i \mid i \in [n]\}$.

Case 1. Let $n \equiv 1 \pmod 2$ and $n \geq 3$, and let $C = [n-1]^0$. It is obvious that $\chi'_{vde}(K_n) \geq n$, so that it is sufficient to find a n-VDEEC of K_n. Define an edge coloring f for K_n in this way: $f(v_iv_j) = i + j - 2 \pmod n$ for $i \in [n-1]$ and $j \in [i+1, n]$. Therefore, we have $\overline{C(v_i)} = \{2(i-1)\}$ for $i \in [(n+1)/2]$, and $\overline{C(v_i)} = \{2i-n-2\}$ for $i \in [1+(n+1)/2, n]$. Thus, f is a $n-VDEC$ of K_n. Since $|E_i| = n-1$ for $i \in [n]$, it means that f is a n-VDEEC of K_n.

Case 2. $n \equiv 0 \pmod 2$ and $n \geq 4$.

Assume that $\chi'_{vde}(K_n) = n$. Since $|C(v_i)| = n-1$ for any vertex $v_i \in V(K_n)$, without loss of generalization, we have $\overline{C(v_i)} = \{i\}$ for $i \in [n]$. So the color i is appears at every vertex $v_j \in V(K_n) \setminus \{v_i\}$. Furthermore, the graph $K_n - v_i$ has a perfect matching with the color i, that means $|V(K_n - v_i)| = n-1$ is even, a contradiction. Therefore, $\chi'_{vde}(K_n) \geq n+1$. We, now, give all $(n+1)$-VDEECs of K_n by its even order n (≥ 4) in the following. Let $C = [n]^0$.

For $n = 4$, we define an edge coloring f for K_n by $f(v_1v_i) = i+1 \pmod 5$ for $i = 2, 3, 4$; $f(v_2v_3) = 1$, $f(v_2v_4) = 4$ and $f(v_3v_4) = 2$.

For $n = 6$, let f be as: $f(v_1v_i) = i + 1 \pmod 7$ for $i \in [6] \setminus \{1\}$; $f(v_2v_3) = 0$, $f(v_2v_4) = 4$, $f(v_2v_5) = 1$ and $f(v_2v_6) = 5$; $f(v_3v_4) = 1$, $f(v_3v_5) = 5$, $f(v_3v_6) = 2$, $f(v_4v_5) = 2$, $f(v_4v_6) = 6$ and $f(v_5v_6) = 3$.

For $n \geq 8$, we construct an edge coloring f for K_n in the following form: $f(v_1v_i) = i + 1(\text{mod } (n+1))$ for $i \in [n] \setminus \{1\}$, and there are two cases as follows:

(1) If $i = 2, 4, \ldots, n-2$, we have $f(v_iv_{i+1}) = 2+i+n/2$, $f(v_iv_{i+2}) = i+2$; $f(v_iv_{i+3}) = 3 + i + n/2$, $f(v_iv_{i+4}) = i + 3, \ldots, f(v_iv_n) = 1 + (n+i)/2$.

(2) If $i = 3, 5, \ldots, n-1$, we have $f(v_iv_{i+1}) = 2+i+n/2$, $f(v_iv_{i+2}) = i+2$, $f(v_iv_{i+3}) = 3 + i + n/2$, $f(v_iv_{i+4}) = i + 3$, $f(v_iv_n) = (i+1)/2$.

Obviously, f is $(n+1)$-VDEEC of K_n for all integers $n \geq 8$. □

Remark: If $n = 8 + 6k$ for $k = 0, 1, \ldots$, we are not able to obtain a $(n+1)$-VDEEC f' of K_n by deleting a vertex from K_{n+1} which has a $(n+1)$-VDEEC f, that is, $f' = f$. Otherwise, there exists $\overline{C(u)} = \overline{C(v)}$ for some vertices u and v.

Theorem 4. *Let F_n be a fan on $n+1$ vertices. Then $\chi'_{vde}(F_2) = 3$, $\chi'_{vde}(F_3) = 4$ and $\chi'_{vde}(F_n) = n$ for $n \geq 4$.*

Proof. It is not hard to see $\mu(F_2) = 3$, $\mu(F_3) = 4$, and $\mu(F_n) = n$ for $n \geq 4$. In order to prove the theorem, it is sufficient to give a $\mu(F_n)$-VDEEC of the fan F_n. Let $V(F_n) = \{v_i \mid i \in [n]^0\}$ be the vertex set of the fan F_n, so its edge set is $E(F_n) = \{v_0v_i \mid i \in [n]\} \cup \{v_iv_{i+1} \mid i \in [n]\}$.

Case 1. Since $F_2 = K_3$, the result is evident by Theorem 3.

Case 2. We define an edge coloring f of F_3 by: $f(v_0v_i) = i$ for $i = 1, 2, 3$; $f(v_1v_2) = 3$ and $f(v_2v_3) = 4$. Obviously, the coloring f is a 4-VDEEC of F_3.

Case 3. For case $n \geq 4$, let the color set $C = [n]$. There is an edge coloring f defined as that $f(v_0v_i) = i$ for $i \in [n]$; $f(v_1v_2) = n$; and $f(v_iv_{i+1}) = i - 1$ for $i \in [2, n-1]$.

Furthermore, f is a n-VDEC of F_n since $C(v_0) = [n]$, $C(v_1) = \{1, n\}$, $C(v_n) = \{n-2, n\}$ and $C(v_i) = \{i-2, i-1, i\}$ for $i \in [3, n-1]$. Notice that $|E_{n-2}| = 1$ and $|E_i| = 2$ for $i \in [n] \setminus \{n-2\}$. Therefore, the edge coloring f is a n-VDEEC of F_n.

The proof of the theorem is completed according to the above cases verified. □

Theorem 5. *For a wheel W_n of order $n+1$, then $\chi'_{vde}(W_3) = 5$, and $\chi'_{vde}(W_n) = n$ for $n \geq 4$.*

Proof. Let W_n be a wheel on $n+1$ vertices, with vertex set $V(W_n) = \{v_i \mid i \in [n]^0\}$, and edge set $E(W_n) = \{v_0v_i \mid i \in [n]\} \cup \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. For $W_3 = K_4$, the result is true according to Theorem 3. Obviously, we have $\mu(W_n) = n$ for $n \geq 4$. We have an edge coloring f of the wheel W_n in the following. Defining $f(v_0v_i) = i$ for $i \in [n]$; $f(v_1v_2) = n$; $f(v_iv_{i+1}) = i-1$ for $i \in [2, n-1]$; $f(v_{n-1}v_n) = n-1$. Since the rest part of the proof is as the same as the proof of Theorem 4, so the coloring f is a n-VDEEC of W_n. □

Theorem 6. *For a complete bipartite graph $K_{m,n}$ with $m \geq n \geq 2$, then $\chi'_{vde}(K_{m,n}) = m+1$ for $m > n \geq 2$, and $\chi'_{vde}(K_{m,n}) = m+2$ for $m = n \geq 2$.*

Proof. For case $m > n \geq 2$, let $C = [m]^0$. Observe that

$$\mu(K_{m,n}) = \min\left\{\mu \middle| \binom{\mu}{m} \geq n,\ \binom{\mu}{n} \geq m\right\} = m+1,$$

it is sufficient to find a $(m+1)$-VDEEC of $K_{m,n}$. Let $V(K_{m,n}) = \{u_i \mid i \in [n]\} \cup \{v_i \mid i \in [n]\}$ and $E(K_{m,n}) = \{u_iv_j \mid i \in [m],\ j \in [n]\}$.

There is an edge coloring f of $K_{m,n}$ described in the form: $f(u_iv_j) = i+j-1 (\bmod\ (m+1))$ for $i \in [n]$ and $j \in [m]$. It is easy to see that $C(u) \neq C(v)$ for all $u, v \in V(K_{m,n})$ and $u \neq v$. Furthermore, we have $|E_i| = n-1$ if $i \in [n-1]^0$ and otherwise $|E_j| = n$. Hence, f is a $(m+1)$-VDEEC of $K_{m,n}$.

Consider case $m = n \geq 2$. Let $C = [m+1]^0$. Since $\mu(K_{m,m}) = m+2$, so it enables us to define an edge coloring f of $K_{m,m}$ as this: $f(u_1v_j) = j$ for $j \in [m]$; $f(u_iv_j) = i+j \ (\bmod\ (m+2))$ for $i \in [2, m]$ and $j \in [m]$. Thereby, there is the coloring adjacent matrix of $K_{m,m}$ depicted as follows.

$$M_c = \begin{pmatrix} 1 & 2 & \dots & m \\ 3 & 4 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ m+1 & 0 & \dots & n-2 \end{pmatrix} = (c_{ij})_{n\times n},$$

where $f(u_iv_j) = c_{ij}$ for $i, j \in [n]$. According to the matrix M_c, we know $C(u) \neq C(v)$ for distinct vertices $u, v \in V(K_{m,m})$. Notice that $|E_i| = m-1$ for $i = 0, 1, m, m+1$, and $|E_i| = m-2$ otherwise, it turns out that f is a $(m+2)$-VDEEC of $K_{m,m}$. The proof of this theorem is completed by the cases above. □

Theorem 7. *$\chi'_{vde}(P_n \vee S_n) = 2n+1$ for $n \in [3]$, and $\chi'_{vde}(P_n \vee S_n) = 2n$ for $n \geq 4$.*

Proof. It is easy to see that $\mu(P_n \vee S_n) = 2n+1$ for $n \in [3]$, and $\mu(P_n \vee S_n) = 2n$ for $n \geq 4$. To prove the results, thus, it is sufficient to provide exactly a $\mu(P_n \vee S_n)$-VDEEC of $P_n \vee S_n$.

For each $n \in [4]$, it is easy to give a $\mu(P_n \vee S_n)$-VDEEC f of $P_n \vee S_n$, so we omit to provide f since it is very simple.

For case $n \geq 5$, let the path $P_n = u_1u_2\dots u_n$ and let $C = [2n-1]^0$. let $V(S_n) = \{v_i \mid i \in [n]^0\}$ and $E(S_n) = \{v_0v_i \mid i \in [n]\}$. We define an edge coloring f in the following. First, we let $f(v_0v_i) = i$, $f(v_0u_i) = n+i (\text{mod } 2n)$ for each $i \in [n]$. For other edges, we have the following cases.

Case 1. $n \equiv 0 \pmod 2$.

If $n = 6$, define $f(u_iv_j) = n + i + j (\text{mod } 2n)$ for $i = 1, 2, 3$ and $j \in [6]$; $f(u_iv_j) = i + j - 3$ for $i = 4, 5, 6$ and $j \in [6]$; $f(u_iu_{i+1}) = i + 4$ for $i = 1, 2$; and define $f(u_iu_{i+1}) = i + 5$ for $i = 3, 4, 5$.

If $n > 6$, define $f(u_iv_j) = n + i + j \pmod{2n}$ for $i \in [n/2]$ and $j \in [n]$; $f(u_iv_j) = i + j - n/2, i \in [1 + n/2, n]$ and $j \in [n]$; and define $f(u_iu_{i+1}) = n + i - 2$ for $i \in [n-1]$.

Case 2. $n \equiv 1 \pmod 2$.

We have an edge coloring f defined as that $f(u_iv_j) = n+i+j \pmod{2n}$ for $i \in [(n + 1)/2]$ and $j \in [n]$; $f(u_iv_j) = i + j - (n + 1)/2$ for $i \in [1+(n+1)/2, n]$ and $j \in [n]$; $f(u_iu_{i+1}) = i+(n+1)/2$ for $i \in [(n-1)/2-1]$; and $f(u_iu_{i+1}) = i + 1 + (n + 1)/2$ for $i \in [(n - 1)/2, n - 1]$.

Obviously, the above edge labels show that f is a $2n$-VDEC of $P_n \vee S_n$. Next, we show that f is truly a $(2n)$-VDEEC of $P_n \vee S_n$ by checking $||E_i| - |E_j|| \leq 1$ for $i \neq j$.

If $n = 6$, it easy to see that f is equitable. If n is an even integer not less than 6, we have $|E_i| = (n+2)/2$ for $i \in [n-1]^0 \cup \{2n-2, 2n-1\}$, and $|E_i| = (n + 4)/2$ otherwise.

If n is of odd, there is $|E_n| = (n + 1)/2$ and $|E_i| = (n + 3)/2$ for other $i \neq n$. □

Theorem 8. *$\chi'_{vde}(C_3 \vee S_3) = 7$ and $\chi'_{vde}(C_n \vee S_n) = 2n$ for $n \geq 4$.*

Proof. There are obvious facts that $\mu(C_3 \vee S_3) = 7$ and $\mu(C_n \vee S_n) = 2n$ for $n \geq 4$. We will define some edge colorings for $C_n \vee S_n$.

For $n = 3$ or 4, we can omit to provide the 7-VDEC or 8-VDEC of $C_n \vee S_n$ since it is easy to do. Let the cycle $C_n = u_1u_2 \ldots u_nu_1$ in the consideration, and $V(S_n) = \{v_i \mid i \in [n]^0\}$, $E(S_n) = \{v_0v_i \mid i \in [n]\}$. Let $C = [2n - 1]^0$.

For $n \geq 5$ and n is of even, we construct a mapping f from $E(C_n \vee S_n)$ to C which is as the same one in the proof of Theorem 7, and followed to color u_nu_1 by color 2. Furthermore, there are $|E_i| = (n + 2)/2$ for $i \in ([n-2]\setminus\{2\}) \cup \{2n-2, 2n-1\}$ and $|E_j| = (n+4)/2$ for $j \in [n-1, 2n-3]$.

For the case of odd n and $n \geq 5$, we define f by the way: $f(v_0v_i) = n+i$, $f(v_0u_i) = i$ for $i \in [n]$. Let $f(u_iv_j) = n + i + j \pmod{(2n)}$ for $i \in [n]$ and $j \in [(n-1)/2]$; $f(u_iv_j) = i+j-(n-1)/2$ for $i \in [n]$ and $j \in [1+(n-1)/2, n]$; $f(u_iu_{i+1}) = i - 1$ for $i \in [(n + 3)/2]$; and $f(u_iu_{i+1}) = n + i - 1$ for $i \in [1 + (n + 3)/2, n]$, where $u_{n+1} = u_1$.

It is easy to estimate $|E_i| = (n + 3)/2$ for $i \in [2n - 1]^0$. Also, f is a $(2n)$-VDEEC of $C_n \vee S_n$, as desired. We complete the proof of this theorem. □

3 Conjectures

In [14], the authors obtained the following results.

• Let P_n denote a path n vertices, $n \geq 2$, we have $\chi'_{vde}(P_n \vee P_n) = n+3$ for $2 \leq n \leq 6$; and $\chi'_{vde}(P_n \vee P_n) = n+4$ for $n \geq 7$.

• For $n \geq 3$, we have $\chi'_{vde}(P_n \vee C_n) = 6$ for $n = 3$; and $\chi'_{vde}(P_n \vee C_n) = n+4$ for $n \geq 4$.

• For $n \geq 3$, $\chi'_{vde}(C_n \vee C_n) = n+4$.

Based on the above results, we pose the following conjectures about $\chi'_{vde}(G)$.

Conjecture 1. *For a* vdec-*graph, then* $\chi'_{vde}(G) \leq \mu(G) + 1$.

Conjecture 2. *For a* vdec-*graph, then* $\chi'_{vde}(G) = \chi'_{vd}(G)$*(if* $\chi'_{vd}(G) = \mu(G) + 1$*).*

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