Cyclically Decomposing the Complete Graph into Cycles with Pendent Edges

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Abstract

Let C_m be a cycle on m (≥ 3) vertices and let $\Theta_{n-m}C_m$ denote the class of graphs obtained from C_m by adding $n-m$ (≥ 1) distinct pendent edges to the vertices of C_m. In this paper it is proved that for every T in $\Theta_{n-m}C_m$, the complete graph K_{2n+1} can be cyclically decomposed into the isomorphic copies of T. Moreover, if m is even, then for every positive integer p, the complete graph K_{2pn+1} can also be cyclically decomposed into the isomorphic copies of T.

1 Introduction

Let G be a simple graph. A *G-decomposition* of $K_n = (V, E)$ is a collection C of edge-disjoint subgraphs of K_n such that each member of C is isomorphic to G, and each edge of E appears exactly once in a member of C. Here K_n is a complete graph on n vertices. A G-decomposition of K_n is also called a (K_n, G)-*design* [4]. In particular, if G is a cycle with m (≥ 3) vertices (namely, an m-cycle), then a (K_n, G)-design is known as an m-cycle system of order n. There have been many results considering the existence of cyclic m-cycle system of order n. See, for example, [2, 3, 6, 7, 10-14].

Throughout this paper, we consider the complete graph with odd number of vertices. Let Π be an automorphism group of a (K_n, G)-design, i.e., a group of permutations on $n = |V|$ vertices leaving the collection C invariant. If there is an automorphism $\pi \in \Pi$ of order n, then the (K_n, G)-design is called *cyclic*. For a cyclic (K_n, G)-design, the vertex set V can be identified with Z_n, i.e., the residue group of integers modulo n. Thus the automorphism can be represented by

$$\pi: i \to i+1 \pmod{n} \text{ or } \pi: (0, 1, \ldots, n-1)$$

on the vertex set $V = Z_n$.

Let Γ be a member of the collection C. By $\Gamma + k$ we mean the graph in which each vertex is renamed as the addition of the corresponding vertex in Γ and an integer $k \in Z_n$ modulo n. An *orbit* of Γ is a set of distinct Γ in the collection $\{\Gamma + i \mid i \in Z_n\}$. The *length* of an orbit is its cardinality, i.e., the minimum positive integer k such that $C + k = C$. An orbit of Γ with length n is said to be *full* and otherwise *short*. A *base graph* (or *starter*) of an orbit O is a member in O that is chosen arbitrarily. Any cyclic (K_n, G)-design should be generated from full or short base graphs.

Let O_j $(1 \leq j \leq r)$ be orbits of a cyclic (K_n, G)-design and $\Gamma_j \in O_j$ be a base graph of O_j. The list of differences in Γ_j is the multiset $\partial\Gamma_j = \{\pm(x-y) \mid \{x, y\} \text{ is any edge in } \Gamma_j\}$. Given a set $B = \{\Gamma_1, \ldots, \Gamma_r\}$, the list of differences from B is defined as the union of differences of $\partial\Gamma_j$ $(1 \leq j \leq r)$, i.e., $\partial B = \bigcup_{j=1}^{r} \partial\Gamma_j$.

It should be mentioned that any cyclic or 1-rotational graph decomposition using the method of partial differences which generalize the above differences has been introduced in [1].

Theorem 1.1. *Let G be a graph and let B be a set of full base graphs, each isomorphic to G, such that $\partial B = Z_{2n+1} - \{0\}$. Then there exists a cyclic (K_{2n+1}, G)-design.*

Let $\Theta_{n-m}C_m$ denote the class of graphs obtained from C_m by adding $n - m$ (≥ 1) distinct pendent edges to the vertices of C_m.

In this paper the main result is the following:

Theorem 1.2. *Let m and n be any positive integers with $n > m \geq 3$. Then there exists a cyclic (K_{2n+1}, T)-design for any $T \in \Theta_{n-m}C_m$. Moreover, if m is even, then for any positive integer p, there exists a cyclic (K_{2pn+1}, T)-design for any $T \in \Theta_{n-m}C_m$.*

2 Definitions and Preliminaries

Let $G = (V, E)$ be a simple graph with n edges. A *proper labeling* of G is an injection $f: V \to \{0, 1, \ldots, 2n\}$ such that the induced labeling $f^*: E \to \{1, 2, \ldots, n\}$ on the edges, defined through $f^*(\{u, v\}) = |f(u) - f(v)|$, sends

E bijectively onto $\{1, 2, \ldots, n\}$; and the proper labeling f is said to be λ-*proper* if there exists an integer λ such that, for each edge $\{u, v\}$ in G, either $f(v) \le \lambda < f(u)$ or $f(u) \le \lambda < f(v)$.

As an example, consider the graphs, shown in Figure 1, that have a proper labeling and a 3-proper labeling, respectively. It should be noted that a proper labeling is a stronger version of a ρ-labeling [9].

(1) A graph with a proper labeling (2) A graph with a 3-proper labeling

Figure 1.

The concept of a graceful valuation and an α-valuation was first introduced by Rosa [9] (known as graceful labeling and α-labeling). A *graceful labeling* of G is an injection f of $V(G)$ into the set $\{0, 1, \ldots, n\}$ with the property: if for each edge $\{u, v\}$ in G, the value $f^*(\{u, v\})$ of the edge $\{u, v\}$ is defined by $f^*(\{u, v\}) = |f(u) - f(v)|$ then f^* is a bijection of $E(G)$ onto the set $\{1, 2, \ldots, n\}$. A graceful labeling is an *α-labeling* if there is an integer λ $(0 \le \lambda \le n - 1)$ such that for each edge $\{u, v\}$, $\min\{f(u), f(v)\} \le \lambda < \max\{f(u), f(v)\}$.

Clearly, a graph admitting an α-labeling or a λ-proper labeling must be bipartite, and a graceful labeling (α-labeling) is a proper labeling (λ-proper labeling).

Proposition 2.1. *If a graph G with n edges has a proper labeling, then there exists a cyclic (K_{2n+1}, G)-design.*

Proof. Since the graph G has a proper labeling, the graph G itself is a full base graph with $\partial G = \{\pm 1, \pm 2, \ldots, \pm n\}$. Then the assertion follows from Theorem 1.1. □

Remark. The consequence in Proposition 2.1 is a special case of Theorem 7 in [9].

Let G be a bipartite subgraph of K_v with edge set $\{\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}\}$, where $a_i < b_i$ for $1 \le i \le n$. By $G \oplus k$ we mean the graph with edge set $\{\{a_1, b_1 + k\}, \{a_2, b_2 + k\}, \ldots, \{a_n, b_n + k\}\}$, where the value $b_i + k$ is understood modulo v.

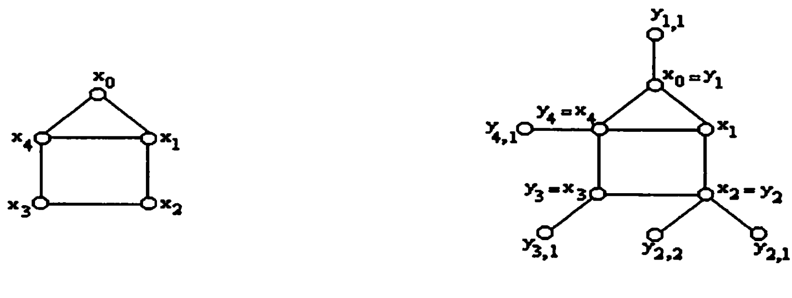
Proposition 2.2. *If a graph G with n edges has a λ-proper labeling, then there exists a cyclic (K_{2pn+1}, G)-design for every positive integer $p \geq 1$.*

Proof. It is clear that G is bipartite since G has a λ-proper labeling. Let $\{\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}\}$ be the edge set of G satisfying $b_i - a_i = i$ for $1 \leq i \leq n$. Consider the graphs $G \oplus in$, $1 \leq i \leq p-1$. Obviously, $G \cong G \oplus in$ for $1 \leq i \leq p-1$ and $\partial(G \oplus in) = \{\pm(in+1), \pm(in+2), \ldots, \pm(i+1)n\}$. Choose the graphs $G, G \oplus n, \ldots, G \oplus (p-1)n$ as the base graphs and the proof then follows from Theorem 1.1. □

Remark. Proposition 2.2 is a generalization of theorem 8 in [9].

Notice that not every graph has proper labelings. It can be proved that if G is an Eulerian graph with $|E(G)| \equiv 1$ or $2 \pmod 4$, then G has no proper labeling.

By $\Theta_k G$ we mean the class of graphs obtained from G by adding any k (≥ 1) distinct pendent edges to the vertices of G. Let $x_0, x_1, \ldots, x_s$ $(s \leq n)$ be the vertices of G. The vertices in $\Theta_k G$ that have additional end vertices are denoted by $y_1, y_2, \ldots, y_p$ $(1 \leq p \leq s+1)$, and let $y_{j,1}, y_{j,2}, \ldots, y_{j,q_j}$ $(q_j \geq 1)$ be the additional end vertices adjacent to the vertex y_j, $1 \leq j \leq p$. Clearly, $\sum_{i=1}^{p} q_i = k$. The graph, depicted in Figure 2, is an easy example.



(1) A graph G (2) A member in $\Theta_5 G$

Figure 2.

Proposition 2.3. *If a graph G with n edges has a graceful labeling, then there exists a cyclic $(K_{2(n+k)+1}, T)$-design for every positive integer k and every $T \in \Theta_k G$.*

Proof. Let h be a graceful labeling of G. Without loss of generality, assume that $h(x_i) < h(x_{i+1})$ and $h(y_j) < h(y_{j+1})$ for $0 \le i \le s-1$ and $1 \le j \le p-1$. Let f be a labeling of $\Theta_k G$ defined as

$$f(u) = \begin{cases} h(x_i), & \text{if } u = x_i,\ 0 \le i \le s; \text{ and} \\ h(y_j) + n + t + \sum_{i=1}^{j-1} q_i, & \text{if } u = y_{j,t},\ 1 \le j \le p \text{ and } 1 \le t \le q_j, \end{cases}$$

where each vertex u is in T.

An easy computation shows that the labeling f is a proper labeling of T, and the proof follows from Proposition 2.1. □

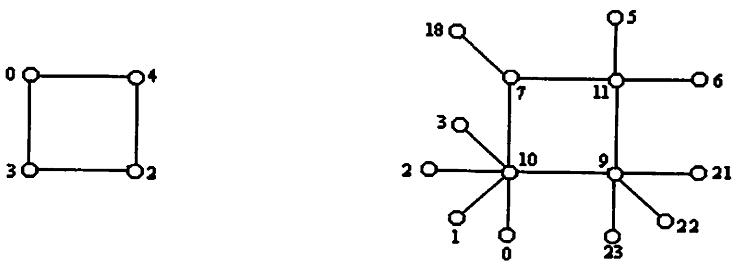
Proposition 2.4. *Let p and k be any positive integers. If a graph G with n edges has an α-labeling, then for every $T \in \Theta_k G$, there exists a cyclic $(K_{2p(n+k)+1}, T)$-design.*

Proof. Let h be an α-labeling of G. So G is bipartite on $X = \{x_i \mid h(x_i) \le \lambda, 0 \le i \le r-1\}$ and $Y = \{x_i \mid h(x_i) > \lambda, r \le i \le s\}$, where r is a positive integer less than or equal to s. Also, assume that $h(x_i) < h(x_{i+1})$ and $h(y_j) < h(y_{j+1})$, $0 \le i \le s-1$ and $1 \le j \le p-1$. Clearly, $h(x_0) = 0$, $h(x_{r-1}) = \lambda$, and $h(x_r) = \lambda + 1$, $h(x_s) = n$. Let $c = \min\{i \mid h(y_i) > \lambda\}$ and let $\ell = \sum_{i=c}^{p} q_i$, where q_i is the number of end vertices of the vertex y_i. Let us introduce a labeling f of T given by

$$f(u) = \begin{cases} h(x_i) + n + \ell - \lambda - 1, & \text{if } u = x_1, 0 \le i \le s; \\ h(y_j) + 2n + 2\ell + t - \lambda - 1 + \sum_{i=1}^{j-1} q_i, & \text{if } u = y_{j,t} \text{ and } h(y_j) \le \lambda, \\ & \quad 1 \le t \le q_j; \text{ and} \\ h(y_j) + t - \lambda - 2 + \sum_{i=c}^{j-1} q_i, & \text{if } u = y_{j,t} \text{ and } h(y_j) > \lambda, \\ & \quad 1 \le t \le q_j, \end{cases}$$

where each vertex u is in T.

By routine computation, it can be verified that f is a $(n + \ell - 1)$-proper labeling of T. By virtue of Proposition 2.2, the desired result follows. □



(1) (2)

Figure 3.

As an example, consider the cycle C_4 with an α-labeling and a member T in $\Theta_{10}C_4$ with a 9-proper labeling, shown in Figure 3-(1) and 3-(2), respectively.

By a path we mean a tree with exactly two end vertices. A caterpillar is a tree with the property that the removal of its end vertices leaves a path; similarly, a lobster is a tree with the property that the removal of its end vertices leaves a caterpillar.

Since any path and caterpillar have α-labelings [9], Proposition 2.4 implies the following.

Corollary 2.5. *Let T_ℓ be a caterpillar or lobster with n edges. Then for any positive integer p, there exists a cyclic (K_{2pn+1}, T_ℓ)-design.*

Note that the same result as Corollary 2.5 can also be found in [5]. Moreover, in 1963, G. Ringel [8] posed the conjecture that for every tree T with n edges, there exists a (K_{2n+1}, T)-design. From the fact that K_{2pn+1} can be cyclically decomposed into isomorphic copies of a caterpillar or lobster with n edges, the Ringel's conjecture seems to be further generalized to the following:

Conjecture. *Suppose that T is a tree with n edges. Then for any positive integer p, there exists a cyclic (K_{2pn+1}, T)-design.*

Before proving the main result, we also need a preliminary lemma.

Lemma 2.6. [9]
(1) *The m-cycle C_m has an α-labeling if and only if $m \equiv 0 \pmod 4$.*
(2) *The m-cycle C_m has a graceful labeling if and only if $m \equiv 0$ or $3 \pmod 4$.*

3 Main result

To obtain the main result, it is sufficient to show that each m-cycle has a graceful labeling, and in particular, if m is even, that the m-cycle has an α-labeling.

By virtue of Lemma 2.6, the $4m$- and $(4m+3)$-cycles have an α-labeling and a graceful labeling, respectively, but the $(4m+2)$- and $(4m+1)$-cycles do not. In order to prove that for $m \equiv 2$ (or 1) (mod 4), each member in $\Theta_{n-m}C_m$ has an α-labeling (or graceful labeling), we shall use the class $\Theta_1 C_{4m+2}$ (or $\Theta_1 C_{4m+1}$) instead of C_{4m+2} (or C_{4m+1}).

Assume $\Theta_1 C_{4m+2} \ni G = (u_0, v_0, u_1, v_1, \ldots, u_{2m}, v_{2m}; w)$ with vertex set $\{u_i, v_i, w \mid i \in Z_{2m+1}\}$ and edge set $\{\{u_i, v_i\}, \{v_i, u_{i+1}\}, \{u_0, w\} \mid i \in Z_{2m+1}\}$; similarly, $\Theta_1 C_{4m+1} \ni G^* = (u_0, v_0, u_1, v_1, \ldots, u_{2m-1}, v_{2m-1}, u_{2m}; w)$ with vertex set $\{u_i, v_i, u_{2m}, w \mid i \in Z_{2m}\}$ and edge set $\{\{u_i, v_i\}, \{v_i, u_{i+1}\}, \{u_{2m}, u_0\}, \{u_0, w\} \mid i \in Z_{2m}\}$.

Let f and g be respectively the labelings of G and G^* defined as

$$f(x) = \begin{cases} i, & \text{if } x = u_i, 0 \le i \le 2m; \\ 4m+1 = i, & \text{if } x = v_i, 0 \le i \le m-2; \\ 4m - i, & \text{if } x = v_i, m-1 \le i \le 2m-1; \\ 4m+3, & \text{if } x = v_{2m}; \text{ and} \\ 4m+2, & \text{if } x = w; \end{cases}$$

and

$$g(x) = \begin{cases} i, & \text{if } x = u_i, 0 \le i \le 2m-1; \\ 4m+2, & \text{if } x = u_{2m}; \\ 4m - i, & \text{if } x = v_i, 0 \le i \le m-2; \\ 4m-1-i, & \text{if } x = v_i, m-1 \le i \le 2m-1; \text{ and} \\ 4m+1, & \text{if } x = w; \end{cases}$$

A routine verification shows that f and g are certainly an α-labeling and a graceful labeling of G and G^*, respectively.

Combining these results with Propositions 2.3 and 2.4, we have the desired result.

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