Eccentricity sequences and eccentricity sets in digraphs

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Abstract

The eccentricity $e(v)$ of a vertex v in a strongly connected digraph G is the maximum distance from v. The eccentricity sequence of a digraph is the list of eccentricities of its vertices given in nondecreasing order. A sequence of positive integers is a *digraphical eccentric sequence* if it is the eccentricity sequence of some digraph. A set of positive integers S is a *digraphical eccentric set* if there is a digraph G such that $S = \{e(v),\ v \in V(G)\}$. In this paper, we present some necessary and sufficient conditions for a sequence S to be a digraphical eccentric sequence. In some particular cases, where either the minimum or the maximum value of S is fixed, a characterization is derived. We also characterize digraphical eccentric sets.

Keywords: eccentricity, eccentric sequence, eccentric set.

1 Introduction

Given a graph [resp., digraph] G it is 'easy' to compute the list of its eccentricities. Nevertheless, the converse problem, which consists to determine if given a sequence S there is a graph [resp., digraph] G such that S is the eccentricity sequence of G, seems to be difficult. Lesniak [4] investigated eccentricity sequences of graphs and obtained some necessary and sufficient

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conditions for a sequence to be eccentric. Moreover, Lesniak [4] characterized eccentricity sequences of trees (see also Behzad and Simpson [1] for a simpler proof). In the directed case, Harminc and Ivančo [3] characterized eccentricity sequences of tournaments. However, the general problem of finding a 'constructive characterization' of eccentric sequences (either in the undirected or in directed case) remains open. Besides, Behzad and Simpson [1] studied eccentricity sets in graphs, which are constituted by all the different values of an eccentric sequence, and derived a characterization of such sets (see also Mao and Liu [5]).

The main purpose of this paper is to investigate eccentricity sequences and sets in digraphs. Several necessary and sufficient conditions for a sequence to be the eccentricity sequence of a digraph are presented in Section 2. A characterization of eccentric sets in digraphs is also given in Section 3.

Definitions and terminology

A *directed graph* or *digraph* G consists of a finite nonempty set $V(G)$ of objects called *vertices* and a set $E(G)$ of ordered pairs of vertices called *arcs*. The *order* of G is the cardinality of $V(G)$. If (u, v) is an arc, it is said that u is *adjacent to* v and also that v is *adjacent from* u. The set of vertices which are adjacent from [to] a given vertex v is denoted by $N^+(v)$ [$N^-(v)$]. A *path* of length h from a vertex u to a vertex v ($u \to v$ path) in G is a sequence of $h+1$ distinct vertices $u = u_0, u_1, \ldots, u_{h-1}, u_h = v$ such that each pair (u_{i-1}, u_i) is an arc of G. A digraph G is *(strongly) connected* if there is a $u \to v$ path for any pair of vertices u and v of G. The length of a shortest $u \to v$ path is the *distance from* u *to* v, denoted by $\mathrm{dist}(u, v)$. If there is no path from u to v, we write $\mathrm{dist}(u, v) = \infty$. The set of vertices at distance l from [to] u is denoted by $\Gamma_l^+(u)$ [$\Gamma_l^-(u)$]. The *eccentricity* of a vertex u, denoted by $e(u)$, is the maximum distance from u to any vertex in G. The *radius* of G, $\mathrm{rad}(G)$, is the minimum eccentricity of the vertices in G; the *diameter*, $\mathrm{diam}(G)$, is the maximum eccentricity of the vertices in G.

From here on, by a *sequence* we will always mean a finite nondecreasing sequence of positive integers. A sequence $S : s_1, s_2, \ldots, s_p$, where $p \geq 2$, is a *digraphical eccentric sequence* [resp., *(graphical) eccentric sequence*] if there exists a digraph [resp., graph] G of order p whose vertices can be labelled $v_1, v_2, \ldots, v_p$ so that $e(v_i) = s_i$.

A finite nonempty set S of positive integers is a *digraphical eccentric set* [resp., *(graphical) eccentric set*] if there exists a digraph [resp., graph] G such that $S = \{e(v),\ v \in V(G)\}$; that is, the eccentricity of each vertex of G belongs to the set S and each element in S is the eccentricity of some vertex of G.

Reader is referred to Chartrand and Lesniak [2] for additional graph concepts.

2 Digraphical eccentric sequences

In the undirected case, some necessary conditions for a sequence to be an eccentric sequence were established by Lesniak [4].

Theorem 2.1 ([4]). *If $S : s_1, s_2, \dots, s_p$ is a graphical eccentric sequence then the following properties hold:*

(i) $s_1 \le p/2$;

(ii) *If k is an integer such that $s_1 < k \le s_p$, then $s_i = s_{i+1} = k$ for some i $(2 \le i \le p-1)$;*

(iii) $s_p \le \min\{p-1, 2s_1\}$.

Taking into account that in the directed case the 'distance function' is not longer a metric, since the symmetry property does not hold, we determine which of the previous conditions are necessary for a sequence to be a digraphical eccentric sequence.

Proposition 2.1. *If $S : s_1, s_2, \dots, s_p$ is a digraphical eccentric sequence then the following properties hold:*

(i) $s_{i+1} \le s_i + 1$ *for all* i $(1 \le i \le p-1)$;

(ii) $s_p \le p-1$.

Proof. Let G be a strongly connected digraph of order p whose vertices can be labelled $v_1, v_2, \dots, v_p$ so that $e(v_i) = s_i$, $i = 1, 2, \dots, p$. Since the diameter of G is at most $p-1$, we have $s_p = \operatorname{diam}(G) \le p-1$.

To prove *(i)*, let us assume that there are two consecutive terms of the (nondecreasing) sequence S, s_i and s_{i+1}, such that $s_{i+1} \ge s_i + 2$. Let us consider the vertex v_i of G with $e(v_i) = s_i$ and let us partition the set of vertices of G according to their distance to v_i; that is $V(G) = \cup_{j=0}^{k} \Gamma_j^-(v_i)$, where $k = \max\{\operatorname{dist}(v, v_i),\ v \in V(G)\}$. Applying the triangular law of distance, for every vertex $v \in \Gamma_1^-(v_i)$ we have $e(v) \le e(v_i) + 1$. Since by assumption there is no vertex with eccentricity equal to $s_i + 1$, $e(v) \le s_i$ for all vertices $v \in \Gamma_1^-(v_i)$. Similarly, if all vertices in $\Gamma_j^-(v_i)$, $1 \le j \le k-1$, have eccentricity $\le s_i$ then all vertices in $\Gamma_{j+1}^-(v_i)$ have also eccentricity $\le s_i$, since any vertex in $\Gamma_{j+1}^-(v_i)$ is adjacent to at least one vertex in $\Gamma_j^-(v_i)$. As a consequence of this inductive argument, all vertices in $V(G)$ have eccentricity $\le s_i$, which contradicts the fact that $e(v_{i+1}) \ge s_i + 2$. □

We point out that the proof of condition *(ii)* is similar to the one provided by Mao and Liu [5] for the undirected case. We also remark that Proposition 2.1 can be extended to the case where S is the eccentricity sequence of a non strongly connected digraph, $S : s_1, \ldots, s_{p'}, s_{p'+1}, \ldots, s_p$, with $s_1 \leq \cdots \leq s_{p'} < \infty$ and $s_{p'+1} = \cdots = s_p = \infty$. In such a case, $s_{p'} \leq p - 1$ and $s_{i+1} \leq s_i + 1$ $(1 \leq i \leq p' - 1)$.

The conditions given in Proposition 2.1 are necessary for a sequence to be digraphical eccentric but, as we will see, they are not sufficient (e.g. the sequence $4, 4, 5, 6, 6, 6, 6$ is not a digraphical eccentric sequence).

In the undirected case, Lesniak [4] proved that a sequence S is eccentric if and only if some subsequence S' (with the same distinct values) is eccentric. However, since S' may be the full sequence S, such a result does not provide an algorithm to determine whether a given sequence S is eccentric.

Next we prove that Lesniak's result also holds in the directed case.

Theorem 2.2. *A sequence $S : s_1, s_2, \ldots, s_p$, with m distinct values, is a digraphical eccentric sequence if and only if some subsequence of S, with m distinct values, is a digraphical eccentric sequence.*

Proof. If S is a digraphical eccentric sequence then S can be considered as the desired subsequence.

To prove the converse, let us consider a digraph G' whose eccentricity sequence is S', where S' is a subsequence of S with the same m distinct values, $t_1, t_2, \ldots, t_m$. For each of these values t_i, let us take a vertex v_i of G' with $e(v_i) = t_i$ and let us define n_i as the number of occurrences (multiplicity) of t_i in S less the number of occurrences of t_i in S'. In order to preserve the eccentricity sequence of G', while adding vertices in G', we 'replace', in turn, each vertex v_i by a complete digraph K_{n_i+1} of order $n_i + 1$, as Lesniak did for the undirected case (see [4, Theorem 1]).

Thus, in G' we replace v_1 by a copy of K_{n_1+1} in such a way that each vertex of K_{n_1+1} becomes adjacent to every vertex in $N^+_{G'}(v_1)$ and adjacent from every vertex in $N^-_{G'}(v_1)$. The resulting digraph G_1 has as many vertices with eccentricity t_1 as indicates the multiplicity of t_1 in S. Following the same procedure, we can construct a chain of digraphs $G_1, G_2, \ldots, G_m$, where G_{i+1} is obtained from G_i by 'replacing' the vertex v_i by a copy of K_{n_i+1} $(i = 1, 2, \ldots, m-1)$. The final digraph G_m has S as its eccentricity sequence. □

To illustrate, with an example, how to apply Theorem 2.2, let us consider the sequence $S : 1, 1, 2, 2, 3, 4, 4, 4$ and the subsequence $S' : 1, 1, 2, 3, 4$. Figure 1 shows two digraphs, G' and G, with eccentricity sequence S' and S, respectively, where G has been constructed from G' by following the proof of this theorem.

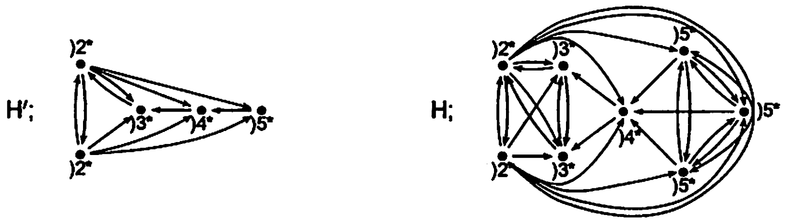

Figure 1: An example of application of Theorem 2.2. (Numbers in brackets indicate vertex eccentricities).

We point out that Theorem 2.2 also holds in the case where $\max(S) = \infty$. Besides, we notice that if a sequence $S : s_1, \dots, s_p$, with $\max(S) < \infty$, is a digraphical eccentric sequence, then $\tilde{S} : s_1, \dots, s_p, \infty, \overset{m}{\dots}, \infty$ is a digraphical eccentric sequence too. Thus, given a digraph G with eccentricity sequence S, we can consider the digraph $G \cup N_m$ with additional arcs from each vertex of G to each vertex of the null graph N_m.

Corollary 2.1. *A constant sequence S of length $p \geq 2$, $S : s, s, \overset{p}{\dots}, s$, with $s \in \mathbb{Z}^+$, is a digraphical eccentric sequence if and only if $s \leq p - 1$.*

Proof. From Proposition 2.1, if S is a digraphical eccentric sequence then $s_p = s \leq p - 1$.

To proof the converse, we take any subsequence S' of length $s+1$ $(\leq p)$, which represents the eccentricity sequence of a directed cycle of order $s+1$, and we apply Theorem 2.2. □

Next, we derive the digraphical eccentric character of some particular types of sequences.

Proposition 2.2. *Let s_1 and m be positive integers. All the following sequences are digraphical eccentric sequences:*

(1) $s_1, s_1+1, \dots, s_1+i-1, s_1+i, \overset{s_1+1}{\dots}, s_1+i, s_i+i+1, \dots, s_1+m$,
where $0 \leq i \leq m$;

(2) $s_1, s_1+1, \dots, s_1+i-1, s_1+i, \overset{s_1}{\dots}, s_1+i,$
$s_i+i+1, \dots, s_1+j-1, s_1+j, s_1+j, s_1+j+1, \dots, s_1+m$,
where $0 \leq i < j \leq m$;

(3) $s_1, s_1+1, \dots, s_1+j-1, s_1+j, s_1+j,$
$s_1+j+1, \dots, s_1+i-1, s_1+i, \overset{s_1}{\dots}, s_1+i, s_1+i+1, \dots, s_1+m$,
where $0 \leq j \leq \max\{i+1-s_1, m-i-1\}$.

Proof. In each case we will construct a digraph with the given eccentricity sequence. Let us consider the auxiliary digraph $\tilde{G}_i$ with vertex set $V(\tilde{G}_i) =$

$\{1, 2, \ldots, s_1 + m + 1\}$ and arc set

$$E(\tilde{G}_i) = \{(k+1, k),\ 1 \le k \le s_1 + m\} \bigcup \{(1, k),\ 2 \le k \le i+1 \text{ or } s_1 + i + 1 \le k \le s_1 + m + 1\}$$

(see Figure 2). It can be checked that

$$e(k) = \begin{cases} \mathrm{dist}(k, i+2) = s_1 + k - 1, & \text{if } 1 \le k \le i+1, \\ \mathrm{dist}(k, k+1) = s_1 + i, & \text{if } i+2 \le k \le s_1 + i - 1, \\ \mathrm{dist}(k, k+1) = k, & \text{if } s_1 + i \le k \le s_1 + m, \\ \mathrm{dist}(k, 1) = s_1 + m, & \text{if } k = s_1 + m + 1. \end{cases}$$

Therefore, the eccentricity sequence of $\tilde{G}_i$ is

$$s_1, s_1+1, \ldots, s_1+i-1, s_1+i, \overset{s_1}{\ldots}, s_1+i, s_i+i+1, \ldots, s_1+m-1, s_1+m, s_1+m,$$

which coincides with sequence (2) when $j = m$ and $0 \le i < m$.

It can be seen that the addition to $\tilde{G}_i$ of any subset of arcs incident from vertex $s_1 + m + 1$ to vertices of the set $\{s_1 + i, s_1 + i + 1, \ldots, s_1 + m - 1\}$ only changes the eccentricity of vertex $s_1 + m + 1$. Thus, if we add to $\tilde{G}_i$ the set of arcs

$$E'_{i,j} = \{(s_1 + m + 1, k),\ s_1 + j \le k \le s_1 + m - 1\}, \text{ where } 0 \le i \le j < m,$$

we get a digraph G whose eccentricity sequence coincides with (2), if $i < j$, and is equal to (1), if $i = j$, since $e_G(s_1 + m + 1) = \mathrm{dist}_G(s_1 + m + 1, 1) = s_1 + j$. Analogously, in the case $0 \le j \le \max\{i + 1 - s_1, m - i - 1\}$, if we add to $\tilde{G}_i$ the set of arcs $E'_{i,j}$, where

$$E'_{i,j} = \{(s_1 + m + 1, k),\ s_1 + j \le k \le i + 1 \text{ or } s_1 + i + 1 \le k \le s_1 + m - 1\},$$

if $j \le i + 1 - s_1$, and

$$E'_{i,j} = \{(s_1 + m + 1, k),\ s_1 + j + i + 1 \le k \le s_1 + m - 1 \text{ or } k = i + 1)\},$$

otherwise, we get a digraph G whose eccentricity sequence coincides with (3). □

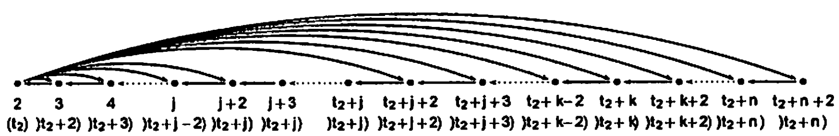

Figure 2: The digraph $\tilde{G}_i$ and its eccentricities.

Using the previous results, we characterize eccentric sequences of digraphs with radius $r \le 3$.

Corollary 2.2. *Every sequence S of length $p \geq 2$, $S: s_1, s_2, \ldots, s_p$, satisfying that $s_1 \leq 3$, $s_{i+1} \leq s_i + 1$ $(i = 1, \ldots, p-1)$ and $s_p \leq p-1$ is a digraphical eccentric sequence.*

Proof. Let S be a sequence, with m distinct values, satisfying the given conditions. We will distinguish different cases, according to the value of s_1, and for each of them we will consider the 'minimal' subsequences of S, with the same distinct values, and show that all of them are digraphical eccentric sequences.

In the case $s_1 = 1$, S must contain a subsequence like

$$S'_{1,i} : 1, 2, \ldots, i-1, i, i, i+1, \ldots, m, \quad \text{where } 1 \leq i \leq m,$$

since length$(S) = p$ and $\max(S) = m \leq p-1$. From Proposition 2.2, $S'_{1,i}$ is a digraphical eccentric sequence, since it is a sequence of type (1) with $s_1 = 1$.

In the case $s_1 = 2$, S must contain a subsequence like

$$\begin{aligned} S'_{2,i} &: 2, 3, \ldots, i-1, i, i, i, i+1, \ldots, m+1 \quad \text{or} \\ S'_{2,i,j} &: 2, 3, \ldots, i-1, i, i, i+1, \ldots, j-1, j, j, j+1, \ldots, m+1. \end{aligned}$$

From Proposition 2.2, both sequences $S'_{2,i}$ and $S'_{2,i,j}$ are digraphical eccentric sequences, since they are sequences of type (1) and (2), respectively (with $s_1 = 2$).

In the case $s_1 = 3$, S must contain a subsequence like

$$\begin{aligned} S'_{3,i} &: 3, 4, \ldots, i-1, i, i, i, i, i+1, \ldots, m+2 \quad \text{or} \\ S'_{3,i,j} &: 3, 4, \ldots, i-1, i, i, i, i+1, \ldots, j-1, j, j, j+1, \ldots, m+2 \quad \text{or} \\ S''_{3,i,j} &: 3, 4, \ldots, j-1, j, j, j+1, \ldots, i-1, i, i, i, i+1, \ldots, m+2 \quad \text{or} \\ S'_{3,i,j,k} &: 3, 4, \ldots, i-1, i, i, i+1, \ldots, j-1, j, j, j+1, \ldots, k-1, k, k \\ &\quad\; k+1, \ldots, m+2. \end{aligned}$$

Proposition 2.2 guarantees that sequences $S'_{3,i}$ and $S'_{3,i,j}$ are digraphical eccentric sequences. When $j \leq i-2$ or $j \leq m-i+4$, Proposition 2.2 can also be applied to sequence $S''_{3,i,j}$. So, it remains to consider the case $j = i-1$ and $2i > m+5$; that is,

$$S''_{3,i,i-1} : 3, 4, \ldots, i-2, i-1, i-1, i, i, i, i+1, \ldots, m+2,$$

where $(m+5)/2 < i \leq m+2$. If $\max\{4, (m+5)/2\} < i < m+2$ then $S''_{3,i,i-1}$ turns out to be the eccentricity sequence of the digraph G with vertex set $V(G) = \{1, 2, \ldots, m+3\}$ and arc set

$$\begin{aligned} E(G) &= \{(k+1, k),\ 1 \leq k \leq m+2 \text{ and } k \neq i-3\} \bigcup \{(i-2, i-4)\} \\ &\quad \bigcup \{(1, k),\ 2 \leq k \leq m+3 \text{ and } k \neq i, i+1\}. \end{aligned}$$

In the particular case $i = 4$, $S''_{3,4,3} : 3,3,4,4,4$ is the eccentricity sequence of the digraph G with vertex set $V(G) = \{1,2,3,4,5\}$ and arc set

$$E(G) = \{(1,2),(1,5),(2,1),(3,2),(3,4),(4,3),(5,4)\}.$$

In the other extreme case, $i = m+2$, the sequence

$$S''_{3,m+2,m+1} : 3,4,\ldots,m,m+1,m+1,m+2,m+2,m+2$$

turns out to be the eccentricity sequence of the digraph G with vertex set $V(G) = \{1,2,\ldots,m+3\}$ and arc set

$$\begin{aligned} E(G) &= \{(k+1,k),\ 1 \le k \le m+2\} \bigcup \{(m-1,m),(m+1,m+2)\} \\ &\quad \bigcup \{(1,k),\ 2 \le k \le m-1 \text{ or } k = m+3\}. \end{aligned}$$

Finally, we show that the sequence $S'_{3,i,j,k}$ is a digraphical eccentric sequence too. We construct the digraph G with vertex set $V(G) = \{1,2,\ldots,m+3\}$ and arc set

$$\begin{aligned} E(G) &= \{(l+1,l),\ 1 \le l \le m+2\} \bigcup \{(m+3,l),\ k \le l \le m+1\} \\ &\quad \bigcup \{(1,l),\ 2 \le l \le m+3 \text{ and } l \ne i-1,i,j\}, \end{aligned}$$

if $j \ne i+1$, and

$$\begin{aligned} E(G) &= \{(l+1,l),\ 1 \le l \le m+2\} \\ &\quad \bigcup \{(m+3,l),\ i+1 \le l \le m+1 \text{ and } l \ne k\} \\ &\quad \bigcup \{(1,l),\ 2 \le l \le m+3 \text{ and } l \ne i-1,i,k\}, \end{aligned}$$

if $j = i+1$. In any case, it can be checked that the eccentricity sequence of G is $S'_{3,i,j,k}$.

Hence, since S contains a subsequence, with the same distinct values, that it is a digraphical eccentric sequence, applying Theorem 2.2 we conclude that S is a digraphical eccentric sequence. □

In order to prove that certain sequences S, with $\max(S) = \text{length}(S)-1$, are not digraphical eccentric sequences, the following result can be of some help.

Lemma 2.1. *Let G be a digraph of order $p \ge 2$, radius r and diameter $p-1$. We assume that the vertices of G, $V(G) = \{1,2,\ldots,p\}$, are labelled in such a way that $P : 1,2,\ldots,p$ is a shortest path in G.*

(i) If $(i,1)$ is an arc of G, then $i \ge r+1$ or $i \le p-r$. In addition, if $i < p$ then G has at most $p-i+1$ vertices with eccentricity equal to $p-1$.

(ii) *Suppose that* $(p, 1)$ *is an arc of* G. *If* $(i+g, i)$ *is an arc of* G, *other than* $(p, 1)$, *then* $g \geq r$ *or* $g \leq p-1-r$. *In addition,* G *has at most* $p-g$ *vertices with eccentricity equal to* $p-1$.

Proof. If there is an arc incident from vertex i to vertex 1, then $e(i) \leq \max\{i-1, p-i\}$. Since $e(i) \geq r$, it follows that $i \geq r+1$ or $i \leq p-r$. Moreover, if $i < p$ then $e(j) < p-1$, for each $j = 2, \ldots, i$, and consequently G has at most $p-i+1$ vertices with eccentricity $p-1$.

Now, let us assume that G contains these two cycles:

$$C : 1, 2, \ldots, p-1, p, 1 \quad \text{and} \quad C' : i, i+1, \ldots, i+g, i.$$

Then, $r \leq e(i+g) \leq \max\{g, p-g-1\}$ and, consequently, $g \geq r$ or $g \leq p-1-r$. Moreover, if $(i+g, i) \neq (p, 1)$ then $e(j) < p-1$, for each $j = i+1, \ldots, i+g$, and consequently G has at most $p-g$ vertices with eccentricity $p-1$. □

Theorem 2.3. *Let* $S : s_1, s_2, \ldots, s_p$ *be the following sequence:*

$$s_1 = \cdots = s_k = p-2 \quad \text{and} \quad s_{k+1} = \cdots = s_p = p-1,$$

where $1 \leq k \leq p-1$ *and* $p \geq 7$. *Then,* S *is a digraphical eccentric sequence if and only if* $k \leq \lfloor p/2 \rfloor$ *or* $k = p-2, p-1$.

Proof. First, we show that the sequence $S : p-2, \overset{k}{\ldots}, p-2, p-1, \overset{p-k}{\ldots}, p-1$, when $1 \leq k \leq \lfloor p/2 \rfloor$ or $k = p-1, p-2$, with $p \geq 3$, is the eccentricity sequence of a digraph that can be constructed from the directed graph cycle Z_p by adding certain arcs. Let $V(Z_p) = \{1, 2, \ldots, p\}$ and $E(Z_p) = \{(1,2), \ldots, (p-1, p), (p, 1)\}$ be the vertex set and the arc set of Z_p, respectively. If we add the arc $(1, 3)$ to Z_p we obtain a digraph G_{p-2} whose vertex eccentricities are

$$e(1) = p-2,\ e(2) = e(3) = p-1,\ e(4) = \cdots = e(p) = p-2,$$

which corresponds to the case $k = p-2$. Moreover, if we add the arc $(2, 1)$ to G_{p-2} we get a digraph G_{p-1} with vertex eccentricities

$$e(1) = e(2) = p-2,\ e(3) = p-1,\ e(4) = \cdots = e(p) = p-2,$$

which corresponds to the case $k = p-1$. Furthermore, taking into account that adding to Z_p an arc of the form $(i+1, i)$ only decreases the eccentricity of vertex $i+1$ to $p-2$, we can add $k \leq \lfloor p/2 \rfloor$ independent arcs to Z_p,

$(i_1+1, i_1), (i_2+1, i_2), \ldots, (i_k+1, i_k)$, where $i_{j+1} > i_j+1$ and $j = 1, \ldots, k-1$, and we get a digraph with k vertices with eccentricity $p-2$ and $p-k$ vertices with eccentricity $p-1$.

Next, we prove that the sequence $S : p-2, \overset{k}{\ldots}, p-2, p-1, \overset{p-k}{\ldots}, p-1$, when $k > \lfloor p/2 \rfloor$ and $p-k \geq 3$ ($p \geq 7$), is not realizable as an eccentricity sequence. Let us suppose that there is a digraph G whose eccentricity sequence is S. Since G has order p and diameter $p-1$, let us list its vertices, $V(G) = \{1, 2, \ldots, p\}$, in such a way that $P : 1, 2, \ldots, p$ is a shortest path in G; that is, there are no arcs of the form $(i, i+h)$ with $h > 1$. Clearly, there must be at least one incident arc to 1, $(i, 1) \in E(G)$, which according to Lemma 2.1 it must be incident from either $i = 2$ or $i = p-1$ or $i = p$, since G has radius $p-2$.

We start by assuming that $(p, 1)$ is an arc of G. Since G has at least one vertex with eccentricity $< p-1$, there must be an arc incident from a vertex $i+g$ to a vertex i in G. From Lemma 2.1 we derive that $g = 1$ or $g = p-2$. If $g = p-2$ then G contains either the cycle $C : 1, 2, \ldots, p-1, 1$ or $C : 2, 3, \ldots, p, 2$. In any case, G has at most two vertices with eccentricity $p-1$, which is impossible (the multiplicity of $p-1$ in S is $p-k \geq 3$). Therefore, all remaining arcs in G must be of the form $(i+1, i)$. But these arcs must be independent, since a path of the form $i+2, i+1, i$ it would imply that $e(i+2) < p-2$. Hence, there are at most $\lfloor p/2 \rfloor$ digons (2-cycles), each of them constituted by a unique vertex with eccentricity $p-2$. This contradicts the fact that the number of such vertices in G is $k > \lfloor p/2 \rfloor$.

It remains to consider the case when G is not hamiltonian; that is $(p, 1)$ is not an arc of G. So, the only arcs incident to 1 can be $(2, 1)$ or $(p-1, 1)$. If $(p-1, 1) \in E(G)$ then the number of vertices with eccentricity $p-1$ is at most two (vertices 1 and p), which is impossible. Hence, G must contain the digon $1, 2, 1$. Let us consider the digraph $G' = G - 1$, derived from G by deleting vertex 1, which has diameter $p-2$ and radius $r \geq p-3$. By applying Lemma 2.1 to G', we know that all arcs incident to 2 must come from vertices $i \in \{3, 4, p-2, p-1, p\}$ (we keep the vertex labelling from G). It can be checked that the cases $i \in \{3, 4, p-2, p-1\}$ would imply the existence in G of a vertex with eccentricity $\leq p-3$, which contradicts the assumption that $\text{rad}(G) = p-2$. In addition, the case $i = p$ would imply that G has at most two vertices with eccentricity $p-1$, which is also impossible. □

As a consequence, the sequence $S : 5, 5, 5, 5, 6, 6, 6$ is the shortest sequence satisfying that $\min(S) = \max(S) - 1 = \text{length}(S) - 2$, which is not a digraphical eccentric sequence.

Theorem 2.4. *Let $S : s_1, s_2, \ldots, s_p$ be the following sequence:*

$$s_1 = \cdots = s_k = p-3, \quad s_{k+1} = p-2 \quad \text{and} \quad s_{k+2} = \cdots = s_p = p-1,$$

where $1 \leq k \leq p-2$ *and* $p \geq 7$. *Then,* S *is a digraphical eccentric sequence if and only if* $k = 1$ *or* $p-4 \leq k \leq p-2$.

Proof. First, for each sequence S_k,

$$S_k : p-3, \overset{k}{\ldots}, p-3, p-2, p-1, \overset{p-k-1}{\ldots}, p-1 \quad (p \geq 5),$$

where $k = 1$ or $p-4 \leq k \leq p-2$, we construct a digraph G_k whose eccentricity sequence is S_k. Let us take the directed graph cycle Z_p with vertex set $V(Z_p) = \{1, 2, \ldots, p\}$ and arc set $E(Z_p) = \{(1,2), \ldots, (p-1,p), (p,1)\}$. Adding the arc $(3,1)$ to Z_p, we get the digraph G_1 whose vertex eccentricities are

$$e(1) = p-1, \; e(2) = p-2, \; e(3) = p-3, \; e(4) = \cdots = e(p) = p-1,$$

as it corresponds to the case $k = 1$. Moreover, adding the arcs $(p,3)$ and $(p,2)$ to G_1, we obtain the digraph G_{p-2} whose vertex eccentricities are

$$e(1) = p-1, \; e(2) = p-2, \; e(3) = e(4) = \cdots = e(p) = p-3,$$

which solves the case $k = p-2$. Besides, adding the arc $(p,3)$ to Z_p, we get the digraph G_{p-4} with vertex eccentricities

$$e(1) = e(2) = e(3) = p-1, \; e(4) = p-2, \; e(5) = e(6) = \cdots = e(p) = p-3,$$

which corresponds to the case $k = p-4$. Adding the arc $(4,2)$ to G_{p-4}, we obtain the digraph G_{p-3} with vertex eccentricities

$$e(1) = e(2) = p-1, \; e(3) = p-2, \; e(4) = e(5) = \cdots = e(p) = p-3,$$

which solves the case $k = p-3$.

Next, we prove that the sequence S,

$$S : p-3, \overset{k}{\ldots}, p-3, p-2, p-1, \overset{p-k-1}{\ldots}, p-1, \quad \text{with } p \geq 7 \quad \text{and} \quad 1 < k < p-4,$$

is not realizable as an eccentricity sequence. Let us suppose that there is a digraph G whose eccentricity sequence is S. Let us list the vertices of G, $V(G) = \{1, 2, \ldots, p\}$, in such a way that $P : 1, 2, \ldots, p$ is a shortest path in G. Since $\mathrm{rad}(G) = p-3$, there must be an arc $(i,1) \in E(G)$, with $i = 2, 3$ or $p-2 \leq i \leq p$.

First, let us assume that $(p,1) \in E(G)$. Since $G \neq Z_p$, there must be an arc $(i+g, i) \in E(G)$, where $(i+g, i) \neq (p,1)$. From Lemma 2.1, $g = 1, 2$ or $g = p-3, p-2$. If $g \geq p-3$ then, applying Lemma 2.1, G would have at most three vertices with eccentricity $p-1$, which is impossible ($k < p-4$). So, $g = 1, 2$:

- If $g = 2$ then G contains the 3-cycle $C : i, i+1, i+2, i$. Taking into account that $\mathrm{rad}(G) = p-3$, it follows that $e(i+2) = p-3$. This implies that there is no arc $(i+h, i-h') \in E(G)$ with $h \geq 2$ and $h' \geq 1$, apart from $(p, 1)$. We can see that $e(i) = p-1$, since otherwise $\mathrm{dist}(i, i-1) \leq 2$ and, consequently, $e(i+2) \leq \max\{3, p-4\}$, which is impossible ($p \geq 7$). Moreover, from $e(i+1) \leq p-2$ and $e(i) \leq e(i+1)+1$, we conclude that $e(i+1) = p-2$. So, since G has a unique vertex with eccentricity $p-2$, there is no other 3-cycle in G.

- If $g = 1$ then G contains the 2-cycle $C : i, i+1, i$ and, consequently, $e(i+1) \leq p-2$. Taking into account that $p \geq 7$ we can derive that either $e(i+1) = p-2$ or $e(i) = p-2$. Therefore, G has at most two 2-cycles, in which case they must share a vertex.

Hence, assuming that G is hamiltonian, the uniqueness of a vertex with eccentricity $p-2$ implies that G has a unique vertex with eccentricity $p-3$. But this situation is excluded ($k > 1$).

It remains to consider the case $(p, 1) \notin E(G)$. Using Lemma 2.1 and taking into account that G has $k < p-4$ vertices with eccentricity $p-3$, we can derive that either $(2,1) \in E(G)$ or $(3,1) \in E(G)$. In both cases it follows that $e(1) = p-1$, $e(2) = p-2$, $e(3) = e(4) = \cdots = e(k+2) = p-3$ and

$$e(k+3) = \cdots = e(p-2) = e(p-1) = e(p) = p-1,$$

since $e(i) \leq e(i+1)+1$ and G has $p-k-1 \geq 4$ vertices with eccentricity $p-1$.

Let us analyze the possible situations:

- If the subdigraph of G induced by the vertex set $\{1, 2, 3\}$ is connected, then there must be an arc incident from a vertex $j \in \{4, 5, \ldots, p\}$ to a vertex in $\{1, 2, 3\}$. Notice that $j \geq p-2$, since otherwise $e(j) < p-3$ ($p \geq 7$). Taking into account that $e(p-2) = e(p-1) = e(p) = p-1$, we conclude that $(p, 1) \in E(G)$, which is impossible.

- If $(2,1) \in E(G)$ and $(3,1), (3,2) \notin E(G)$, then there must be an arc incident from $\{4, 5, \ldots, p\}$ to $\{1, 2\}$. Analogously to the previous case, a contradiction is derived.

Hence, no digraph exists with eccentricity sequence equal to S. □

As a consequence, $S : 4, 4, 5, 6, 6, 6, 6$ is not a digraphical eccentric sequence. In fact, this is the first shortest sequence, given in lexicographic order, which satisfies the necessary conditions given in Proposition 2.1 and it is not a digraphical eccentric sequence.

3 Digraphical eccentric sets

Behzad and Simpson [1] characterized eccentric sets of graphs.

Theorem 3.1 ([1]). *A finite nonempty set $S = \{a_1, a_2, \ldots, a_m\}$ of positive integers, listed in increasing order, is a graphical eccentric set if and only if $m \leq a_1 + 1$ and $a_{i+1} = a_i + 1$ for all i $(1 \leq i \leq m-1)$.*

Next, we present the corresponding characterization for the directed case.

Theorem 3.2. *A finite nonempty set $S = \{a_1, a_2, \ldots, a_m\}$ of positive integers, listed in increasing order, is a digraphical eccentric set if and only if $a_{i+1} = a_i + 1$ for all i, $1 \leq i \leq m-1$.*

Proof. From Proposition 2.1, we know that given a digraph G the distinct numbers of its eccentricity sequence, listed in increasing order, are consecutive integers.

To prove the converse, given a set $S = \{a_1, a_1+1, \ldots, a_1+m-1\}$, where a_1 and m are positive integers, we construct a digraph whose eccentricity set is S. If $m = 1$ then $S = \{a_1\}$ is the eccentricity set of the directed cycle of order $a_1 + 1$. When $m > 1$ and $a_1 = 1$, we define G_m as the digraph with vertex set $V(G_m) = \{1, 2, \ldots, m+1\}$ and arc set

$$E(G_m) = \{(1,j),\ j = 2, \ldots, m+1\} \bigcup \{(j+1, j),\ j = 1, \ldots, m\}.$$

The eccentricity sequence of G_m is $1, 2, \ldots, m-1, m, m$. So, its eccentricity set is $S = \{1, 2, \ldots, m\}$. In the general case, where $m > 1$ and $a_1 > 1$, we consider the 'concatenation' of the digraph G_{m-1} with a directed cycle of order $a_1 + 1$ at the vertex 1. Thus, we obtain a digraph G with $V(G) = \{1, 2, \ldots, a_1 + m\}$ and

$$\begin{aligned} E(G) &= E(G_{m-1}) \bigcup \{(j, j+1),\ j = m+1, \ldots, a_1 + m - 1\} \\ &\quad \bigcup \{(1, m+1), (a_1 + m, 1)\} \end{aligned}$$

(see Figure 3), whose eccentricity sequence is

$$a_1, \overset{a_1}{\ldots}, a_1, a_1 + 1, a_1 + 1, a_1 + 2, \ldots, a_1 + m - 1.$$

Therefore, the eccentricity set of G is $S = \{a_1, a_1 + 1, \ldots, a_1 + m - 1\}$. □

Given a digraphical [resp., graphical] eccentric set S, $S = \{a_1, a_1 + 1, \ldots, a_1 + m - 1\}$, let $\gamma(S)$ be the minimum order among all digraphs [resp., graphs] whose eccentricity set is S. In the undirected case, Behzad and Simpson [1] proved that

$$\gamma(S) = \begin{cases} 2a_1 + m - 1, & \text{if } m \leq a_1 - 1, \\ a_1 + m, & \text{if } m = a_1 \text{ or } m = a_1 + 1. \end{cases}$$

In the directed case, we have the following result:

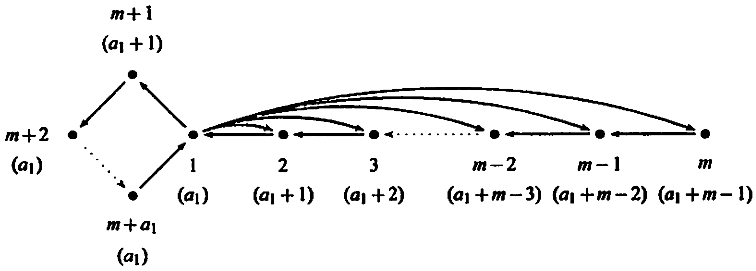

Figure 3: A digraph G with eccentricity set $S = \{a_1, a_1+1, \dots, a_1+m-1\}$.

Corollary 3.1. *If $S = \{a_1, a_1 + 1, \dots, a_1 + m - 1\}$, where a_1 and m are positive integers, then $\gamma(S) = a_1 + m$.*

Proof. From the constructive proof of Theorem 3.2, we know that $\gamma(S) \leq a_1 + m$. Taking into account that a digraph with maximum eccentricity $a_1 + m - 1$ must have order at least $a_1 + m$ the proof is concluded. □

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