On Friendly Index Sets of Bipartite Graphs

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Dedicated to Professor B. Alspach

Abstract Let G be a graph with vertex set V(G) and edge set E(G), and let A be an abelian group. A labeling $f : V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x) + f(y)$, for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_f(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. Let $c(f) = \{| e_f(i) - e_f(j)| : (i,j) \in A \times A\}$. A labeling f of a graph G is said to be *A-friendly* if $| v_f(i) - v_f(j) | \leq 1$ for all $(i,j) \in A \times A$. If c(f) is a (0,1)-matrix for an A-friendly labeling f, then f is said to be A-cordial. When $A = Z_2$, the ***friendly index set*** of the graph G, FI(G), is defined as $\{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is } Z_2\text{-friendly}\}$. In this paper, we determine the friendly index set of cycles, complete graphs and some bipartite graphs.

Key words: vertex labeling, friendly labeling, cordiality, complete bipartite graph

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1. Introduction.

Let G be a graph with vertex set V(G) and edge set E(G), and let A be an abelian group. A labeling $f: V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x) + f(y)$, for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_f(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. Let $c(f)= \{| e_f(i)$

$- e_f(j) |: (i,j) \in A \times A\}$. A labeling f of a graph G is said to be A-*friendly* if $| v_f(i) - v_f(j) | \leq 1$ for all $(i,j) \in A \times A$. If c(f) is a (0,1)-matrix for an A-friendly labeling f, then f is said to be A-cordial.

The notion of A-cordial labelings was first introduced by Hovey [10], who generalized the concept of cordial graphs of Cahit [2]. Cahit considered $A = \mathbf{Z}_2$ and he proved the following: every tree is cordial; K_n is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all m and n; the wheel W_n is cordial if and only if $n \neq 3 \pmod 4$; C_n is cordial if and only if $n \neq 2 \pmod 4$; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Benson and Lee [1] showed a large class of cordial regular windmill graphs. Lee and Liu [12] investigated cordial complete k-partite graphs. Kuo, Chang and Kwong [11] determined all m and n for which mK_n is cordial. Cordial generalized Petersen graphs are completely characterized in [9]. Ho, Lee and Shee [8] investigated the construction of cordial graphs by Cartesian product and composition. Seoud and Abdel [15] proved certain cylinder graphs are cordial. Several constructions of cordial graphs were proposed in [14, 16, 17, 18, 19, 20]. For more details of known results and open problems on cordial graphs, see [4 and 7].

In this paper, we will exclusively focus on $A = Z_2$, and drop the reference to the group. In [6] the friendly index set FI(G) of the graph G is introduced. The set FI(G) is defined as $\{|e_f(0) - e_f(1)|$: the vertex labeling f is friendly$\}$. When the context is clear, we will drop the subscript f.

Note that if 0 or 1 is in FI(G), then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality. Cairnie and Edwards [5] have determined the computational complexity of cordial labeling and Z_k-cordial labeling. They proved that to decide whether a graph admits a cordial labeling is NP-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is NP-complete. Thus it is difficult to determine the friendly index sets of graphs. This paper could be regarded as a first focused effort at this difficult problem. We will determine the friendly index sets of a few classes of graphs, in particular, complete bipartite graphs and cycles.

We begin with a useful result on the friendly index set of a general graph.

Theorem 1. For any graph with q edges, the friendly index set $FI(G) \subseteq \{0, 2, 4, \ldots, q\}$ if q is even and $FI(G) \subseteq \{1, 3, \ldots, q\}$ if q is odd.

Proof. Since $e(0) + e(1) = q$, we see that $e(0) - e(1) = q - 2e(1)$ is odd or even according to whether q is odd or even. Obviously $|e(0) - e(1)| \leq q$, the number of edges in the graph.

2. Friendly index sets of complete bipartite graphs.

Theorem 2. Consider $K_{m,n}$, where $m \leq n$. If m + n is even and m is even, then $FI(K_{m,n}) = \{m^2, (m-2)^2, (m-4)^2, \ldots, 2^2, 0^2\}$. If m + n is even and m is odd, then $FI(K_{m,n}) = \{m^2, (m-2)^2, (m-4)^2, \ldots, 3^2, 1^2\}$. If m + n is odd, then $FI(K_{m,$

$_n) = \{(m + 1)m, m(m - 1), (m - 1)(m - 2), \ldots, 2.1, 1.0\}$.

Proof. The set of vertices of $K_{m,n}$ can be partitioned into 2 subsets, one with m vertices, and the other with n vertices. Two vertices are adjacent if and only if they come from different subsets.

Case 1: $m + n = 2k$, where k is a positive integer.

Consider any friendly labeling. Assume that i vertices from the first subset are labeled 0. Then the remaining $(m - i)$ vertices in this subset must be labeled 1. Since $v(0) = v(1) = k$, in the second subset, $(k - i)$ vertices must be labeled 0, and the remaining $(k - m + i)$ vertices must be labeled 1. It follows that $e(0) = i(k - i) + (m - i)(k - m + i)$ and $e(1) = i(k - m + i) + (k - i)(m - i)$. Then $e(1) - e(0) = i(2i - m) + (m - i)(m - 2i) = (m - 2i)^2$. Since $i = 0, 1, 2, \ldots, m$, by exhausting all the values of i, we obtain the friendly index sets stated in the theorem.

Case 2: $m + n = 2k + 1$, where k is a positive integer.

We consider two subcases:

Case 2.1: $v(0) = k$ and $v(1) = k + 1$.

We use reasoning similar to that in Case 1. Consider any friendly labeling. If the first subset has i vertices labeled 0, then the remaining $(m - i)$ vertices will have label 1. Since $v(0) = k$ and $v(1) = k + 1$, the second subset must have $(k - i)$ vertices labeled 0, and the remaining $(k + 1 - m + i)$ vertices labeled 1. Then $e(0) = i(k - i) + (m - i)(k + 1 - m + i)$ and $e(1) = i(k + 1 - m + i) + (k - i)(m - i)$, giving $e(1) - e(0) = i(2i - m + 1) + (m - i)(m - 2i - 1) = (m - 2i - 1)(m - 2i)$. Since $i = 0, 1, 2, \ldots, m$, by exhausting all the values of i, we verify that the absolute difference $|e(0) - e(1)|$ can take the values $(m + 1)m, m(m - 1), (m - 1)(m - 2), \ldots, 2.1, 1.0$.

Case 2.2: $v(0) = k + 1$ and $v(1) = k$.

Again consider any friendly labeling. If the first subset has i vertices labeled 0, then the remaining $(m - i)$ vertices will be labeled 1. Since $v(0) = k + 1$ and $v(1) = k$, the other subset must have $(k + 1 - i)$ vertices labeled 0, and the remaining $(k - m + i)$ vertices labeled 1. Then $e(0) = i(k + 1 - i) + (m - i)(k - m + i)$ and $e(1) = i(k - m + i) + (k + 1 - i)(m - i)$, giving $e(1) - e(0) = i(2i - m - 1) + (m - i)(m - 2i + 1) = (m - 2i + 1)(m - 2i)$. By exhausting all the values of $i = 0, 1, 2, \ldots, m$, we verify that the absolute difference $|e(0) - e(1)|$ can take the values $(m + 1)m, m(m - 1), (m - 1)(m - 2), \ldots, 2.1, 1.0$.

The following examples show that the above result is not necessarily valid for a bipartite graph that is not complete.

<u>**Example 1.**</u> Figure 1 shows a bipartite graph with $FI(2K_2) = \{2\}$.

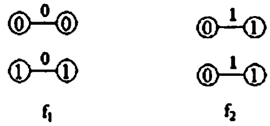


Figure 1.

Example 2. Figure 2 is bipartite and has $FI(K_2 \cup K_{1,2}) = \{1, 3\}$.

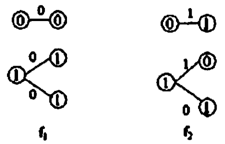


Figure 2.

3. Friendly index sets of trees.

The following Corollary follows from Theorem 2.

Corollary 3. The star $K_{1,n}$ has the following friendly index set:
$FI(K_{1,n}) = \{1\}$ if n is odd, and $FI(K_{1,n}) = \{0, 2\}$ if n is even.

Notation: For any $n > 2$, we will consider the following tree L_n, where $V(L_n) = \{u_1, u_2, \ldots, u_{n-1}\} \cup \{v_1, v_2, \ldots, v_n\}$ and $E(L_n) = \{(u_i, v_i): i = 1, \ldots, n-1\} \cup \{(v_i, v_{i+1}): i = 1, \ldots, n-1\}$.

Theorem 4. The friendly index set of $FI(L_n) = \{0, 2, 4, \ldots, 2(n-1)\}$.
Proof. The graph L_n has $2(n-1)$ edges. From Theorem 1, we know that $FI(L_n)$ can only contain even numbers not exceeding $2(n-1)$.

For $0 \le k \le n-1$, we label all the v_i by 0 and u_i by 1, where $i = 1, 2, \ldots, k$. For $i \ge k+1$, the remaining v_i vertices are alternately labeled by 0, 1, 0, 1, 0, ... and the remaining u_i vertices are alternately labeled by 1, 0, 1, 0, ... We see that $e(0) = k$ and $e(1) = 2n-2-k$, and $|e(0) - e(1)| = 2n-2-2k$. Letting k range from 0 to $n-1$, we obtain the friendly index set of $\{0, 2, 4, \ldots, 2n-2\}$.

Example 3. $FI(L_4) = \{0, 2, 4, 6\}$.

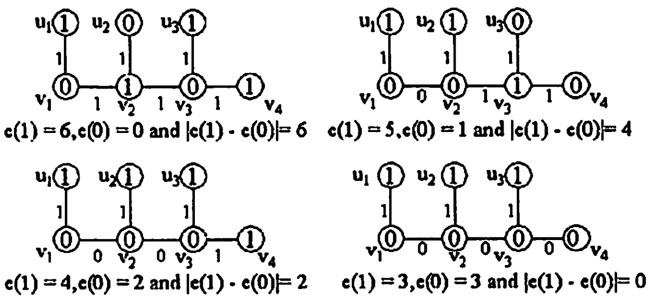


Figure 3.

Given two graphs G and H, the coronation of G with H, denoted by G © H, is the graph with base graph G, and for each vertex u in G, we disjoint union a copy of H and join u with each vertex in H. This construction of graphs was introduced by Frucht and Harary [7].

Theorem 5. The friendly index set of P_n © K_1 is $\{1, 3, \ldots, 2n-1\}$.

Proof. The set of vertices of P_n © K_1 can be partitioned into two subsets, one with the n vertices $\{v_1, v_2, \ldots, v_n\}$ of P_n, and the other consisting of the pendant vertices $\{u_1, u_2, \ldots, u_n\}$. The graph has $(2n-1)$ edges. From Theorem 1, we know that its friendly index set can only contain odd numbers not exceeding $2n-1$.

For $0 \le k \le n-1$, we label all the v_i by 1 and u_i by 0, where $i = 1, 2, \ldots, k$. For $i \ge k+1$, the remaining v_i vertices are alternately labeled by 1, 0, 1, 0, ... and the remaining u_i vertices are alternately labeled by 0, 1, 0, 1, ... We see that $e(0)= k$ and $e(1)=2n-1-k$, and $|e(0)-e(1)| = 2n-1-2k$. Letting k range from 0 to $n-1$, we obtain the friendly index set of $\{1, 3, \ldots, 2n-1\}$.

Example 4. Figure 4 shows that FI(P_5 © K_1) = $\{1, 3, 5, 7, 9\}$.

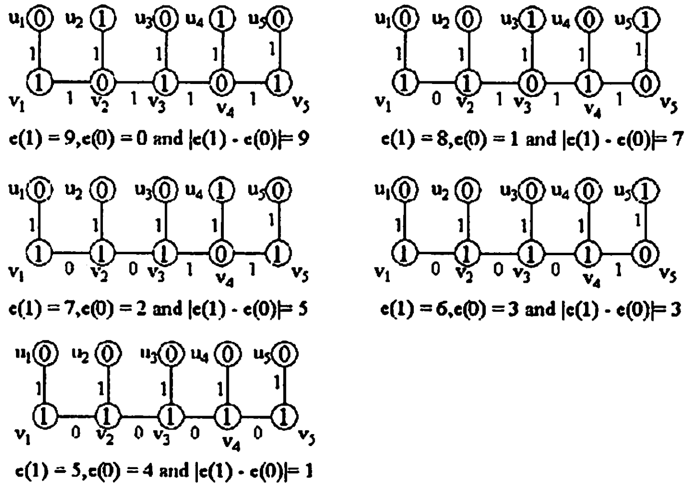


Figure 4.

For the remainder of this section we determine the friendly index set of full binary trees.

Notation: Note that a full binary tree must have an even number of edges. We say that such a tree with a friendly vertex labeling is (v, x, g), where v = v+ or v−, depending on whether $v(1) = v(0) + 1$ or $v(0) - 1$, x = 0 or 1, depending on whether the root is labeled 0 or 1, and $g = e(1) - e(0)$. If the vertex labeling is not friendly, we will use vx as the first coordinate in the triple.

A full binary tree with depth d + 1 can be constructed by adding a vertex (the root) above two full binary trees with depth d. We obtain equations like (v-, 0, g_1) + (v+, 1, g_2) + 0 = (v-, 0, $g_1 + g_2$), meaning that the newly added root vertex is labeled 0. The validity of this equation can be easily verified. Since one of the two subtrees has one more 0-labeled vertices than 1-labeled vertices, and the other subtree has one more 1-labeled vertices than 0-labeled vertices, and the new vertex is labeled 0, we must have v(1) = v(0) – 1 in the new tree. Since the two newly added edges have induced labels 0 and 1, the value of e(1) – e(0) for the new tree must be the sum of the g values for the two subtrees.

Lemma 3.1. There is a vertex labeling (not necessarily friendly) for a full binary tree with depth d so that $g = 2^{d+1} - 2$.

Proof. Label the root 0, the vertices at the next level 1, the vertices at the following level 0, etc. Obviously all induced edge labels, altogether $2^{d+1} - 2$ of them, are labeled 1. Alternatively, we can label the root 1 and the other levels alternately by 0 and 1 to produce the same result. We will call these trees (vx, 0, $2^{d+1} - 2$) and (vx, 1, $2^{d+1} - 2$) respectively. Note that the v(0) and v(1) values of one of these two trees are exactly the v(1) and v(0) values respectively of the other tree.

Theorem 6. A full binary tree with depth 1 has FI = {0, 2}. A full binary tree with depth d > 1 has FI = {0, 2, 4, ... , $2^{d+1} - 4$}.

Proof. For a full binary tree with depth 1, labeling the root 0 and the leaves 0 and 1, and then labeling the root 0 and the leaves 1 and 1, give the desired result.

Now consider depth d > 1. For the value e(1) – e(0) to equal $2^{d+1} - 2$, the vertex labeling must be as in the proof of Lemma 3.1 above. Such a vertex labeling is obviously not friendly. By Theorem 1, the only possible values in the friendly index set are 0, 2, 4, ... , $2^{d+1} - 4$. We will show that all of them are attainable.

We will use induction to show that for any even value of g between 0 and $2^{d+1} - 4$ inclusive, (v-, 0, g) and (v+, 1, g) are possible if g is divisible by 4, and (v-, 1, g) and (v+, 0, g) are possible if g is not divisible by 4.

For d = 2, the following trees verify the desired result.

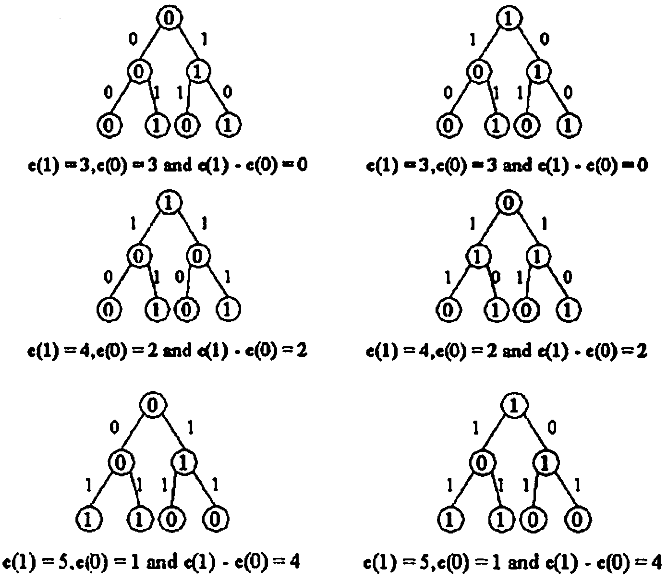


Figure 5.

Assume that the result is true for full binary trees with depth d ≥ 2. Now consider a full binary tree with depth d + 1.

Case 1: Consider any even value of g between 0 and $2^{d+1} - 4$ inclusive.
If g is divisible by 4, (v-, 0, 0) + (v+, 1, g) + 0 = (v-, 0, g), and (v+, 1, 0) + (v-, 0, g) + 1 = (v+, 1, g) give the desired result.
If g is not divisible by 4, (v-, 0, 0) + (v-, 1, g) + 1 = (v-, 1, g), and (v+, 1, 0) + (v+, 0, g) + 0 = (v+, 0, g) give the desired result.

Case 2: Consider any even value of g between $2^{d+1} - 2$ and $2^{d+2} - 8$ inclusive.
If g is divisible by 4, (v-, 0, g – ($2^{d+1} - 4$)) + (v+, 1, $2^{d+1} - 4$) + 0 = (v-, 0, g), and (v+, 1, g – ($2^{d+1} - 4$)) + (v-, 0, $2^{d+1} - 4$) + 1 = (v+, 1, g) give the desired result.
If g is not divisible by 4, (v-, 1, g – ($2^{d+1} - 4$)) + (v-, 0, $2^{d+1} - 4$) + 1 = (v-, 1, g), and (v+, 0, g – ($2^{d+1} - 4$)) + (v+, 1, $2^{d+1} - 4$) + 0 = (v+, 0, g) give the desired result.

Case 3: Consider g = $2^{d+2} - 6$, which is not divisible by 4.
(v-, 0, $2^{d+1} - 4$) + (v-, 0, $2^{d+1} - 4$) + 1 = (v-, 1, $2^{d+2} - 6$), and (v+, 1, $2^{d+1} - 4$) + (v+, 1, $2^{d+1} - 4$) + 0 = (v+, 0, $2^{d+2} - 6$) give the desired result.

Case 4: Consider g = $2^{d+2} - 4$, which is divisible by 4.
Use the trees in Lemma 3.1 above. Then (vx, 0, $2^{d+1} - 2$) + (vx, 1, $2^{d+1} - 2$) + 0 = (v-, 0, $2^{d+2} - 4$), and (vx, 0, $2^{d+1} - 2$) + (vx, 1, $2^{d+1} - 2$) + 1 = (v+, 1, $2^{d+2} - 4$) give the desired result.€.

4. Friendly index sets of complete graphs and cycles.

Theorem 7. The friendly index set of the complete graph K_n is $FI(K_n) = \{\lfloor n/2 \rfloor\}$ for all $n > 2$.

Proof. Assume $n = 2k$. We label k vertices with 0 and the other k vertices with 1. We see that

$e(0) = 2k(k-1)/2 = k^2 - k$ and $e(1) = k^2$. Thus $|e(0) - e(1)|$ has the value k.

Assume $n = 2k + 1$. We label $k + 1$ vertices with 0 and the other k vertices with 1. We see that $e(0) = k(k + 1)/2 + k(k - 1)/2 = k^2$ and $e(1) = k(k + 1) = k^2 + k$. Thus $|e(0) - e(1)|$ has the value k.

Hence $FI(K_n) = \{\lfloor n/2 \rfloor\}$ for all $n > 2$.

For the remainder of this section we will consider the friendly index set of cycles.

Lemma 4.1. Any vertex labeling (not necessarily friendly) of a cycle must have e(1) equal to an even number.

Proof. Traverse the cycle, starting at an arbitrary vertex. At any vertex labeled x, where x = 0 or 1, an edge labeled 0 leads to another vertex labeled x, while an edge labeled 1 leads to another vertex labeled $1 - x$. Since we need the same label when we reach the starting vertex again, the number of edges labeled 1 must be even, i.e., e(1) must be even.

Lemma 4.2. Any friendly vertex labeling of a cycle must have $e(1) \geq 2$.

Proof. If not, e(1) = 0, i.e., all edge labels are 0, meaning that all vertex labels are the same, contradicting the definition of a friendly vertex labeling.

Theorem 8. The friendly index set of a cycle is given as follows:

(i) $FI(C_{2n}) = \{0, 4, 8, \ldots, 2n\}$ if n is even.

$FI(C_{2n}) = \{2, 6, 10, \ldots, 2n\}$ if n is odd.

(ii) $FI(C_{2n+1}) = \{1, 3, 5, \ldots, 2n - 1\}$.

Proof:

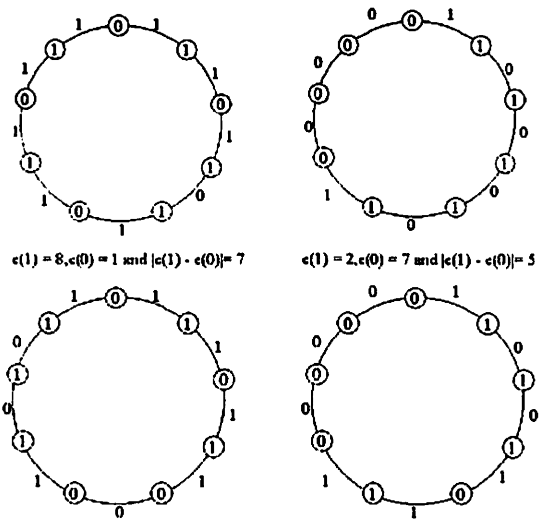
(i) The possible values of e(1) are 2, 4, 6, ... , 2n, and the corresponding values of e(0) are $2n - 2, 2n - 4, 2n - 6, \ldots, 0$ respectively. Thus the possible values of $|e(0) - e(1)|$ are 0, 4, 8, ... , 2n if n is even, and 2, 6, 10,... , 2n if n is odd. We will show that all these values are attainable.

Let the vertices of C_{2n} be $v_1, v_2, \ldots, v_{2n}$. For each $i = 0, 1, \ldots, n - 1$, let $f(v_1) = f(v_3) = \ldots = f(v_{2i-1}) = 0$, $f(v_2) = f(v_4) = \ldots = f(v_{2i}) = 1$, $f(v_{2i+1}) = f(v_{2i+2}) = \ldots = f(v_{i+n}) = 0$, $f(v_{i+n+1}) = f(v_{i+n+2}) = \ldots = f(v_{2n}) = 1$. Then $v(0) = v(1) = n$, i.e., the vertex labeling f is friendly. Counting the edge labels, we readily see that $e(1) = 2i + 2$, and $e(0) = 2n - 2i - 2$, with an absolute difference of $|2n - 4i - 4|$. Letting i run through all the values from 0 to $n - 1$ inclusive, we get the desired result.

(ii) The possible values of $e(1)$ are 2, 4, 6, ... , 2n, and the corresponding values of $e(0)$ are $2n - 1, 2n - 3, 2n - 5, \ldots, 1$ respectively. Thus the possible values of $|e(0) - e(1)|$ are $1, 3, 5, \ldots, 2n - 1$. We will show that all these values are attainable.

Let the vertices of C_{2n+1} be $v_1, v_2, \ldots, v_{2n}, v_{2n+1}$. For each $i = 0, 1, \ldots, n - 1$, let $f(v_1) = f(v_3) = \ldots = f(v_{2i-1}) = 0$, $f(v_2) = f(v_4) = \ldots = f(v_{2i}) = 1$, $f(v_{2i+1}) = f(v_{2i+2}) = \ldots = f(v_{i+n}) = 0$, $f(v_{i+n+1}) = f(v_{i+n+2}) = \ldots = f(v_{2n}) = f(v_{2n+1}) = 1$. Then $v(0) = n$ and $v(1) = n + 1$, i.e., the vertex labeling f is friendly. Counting the edge labels, we readily see that $e(1) = 2i + 2$, and $e(0) = 2n - 2i - 1$, with an absolute difference of $|2n - 4i - 3|$. Letting i run through all the values from 0 to $n - 1$ inclusive, we get the desired result.

Example 5. $FI(C_9) = \{1, 3, 5, 7\}$.



$e(1) = 8, e(0) = 1$ and $|e(1) - e(0)| = 7$

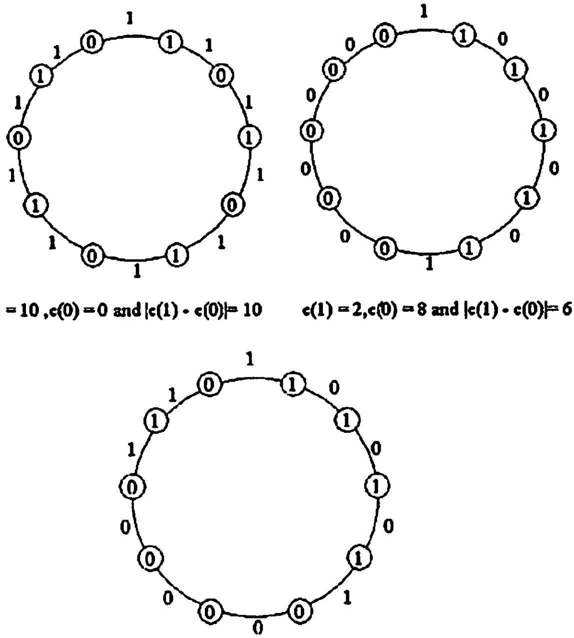
$e(1) = 2, e(0) = 7$ and $|e(1) - e(0)| = 5$

$e(1) = 6, e(0) = 3$ and $|e(1) - e(0)| = 3$

$e(1) = 4, e(0) = 5$ and $|e(1) - e(0)| = 1$

Figure 6.

Example 6. $FI(C_{10}) = \{2, 6, 10\}$.



$e(1) = 10, e(0) = 0$ and $|e(1) - e(0)| = 10$ $e(1) = 2, e(0) = 8$ and $|e(1) - e(0)| = 6$

$e(1) = 4, e(0) = 6$ and $|e(1) - e(0)| = 2$

Figure 7.

5. Friendly index sets of $P_2 \times P_n$.

In this section we will consider the friendly index set $FI(P_2 \times P_n)$. Note that $P_2 \times P_n$ has an even number of vertices. For a vertex labeling to be friendly, we must have $v(0) = v(1)$. Let $g = e(1) - e(0)$.

Lemma 5.1. If each vertex label x is changed to $1 - x$, the vertex labeling remains friendly, and the difference $g = e(1) - e(0)$ remains the same.

Lemma 5.2. The friendly index set of $P_2 \times P_n$ can only contain even numbers if n is even, and can only contain odd numbers if n is odd.

Proof. The number of edges is even or odd according to whether n is even or odd. The result follows from Theorem 1.

Lemma 5.3. If $P_2 \times P_n$ has q edges, then $q - 2$ cannot be in the friendly index set.

Proof. If $q - 2$ is in the friendly index set, then $e(1) = 1$ or $e(0) = 1$. Thus there is a 4-cycle with exactly one edge labeled 1 or 0. This contradicts Lemma 4.1, which states that a cycle must have an even number of edges labeled 1.

We consider two cases.

Theorem 9. $FI(P_2 \times P_{2n+1}) = \{1, 3, 5, \ldots, q-6, q-4, q\}$, i.e., all odd numbers between 1 and q inclusive, except $q-2$. (Note that $q = 6n + 1$. Thus $FI(P_2 \times P_{2n+1}) = \{1, 3, 5, \ldots, 6n-5, 6n-3, 6n+1\}$.)

Proof. By Lemmas 5.2 and 5.3, it suffices to show that all these numbers are attainable.

For n = 0, the following diagram verifies the result.

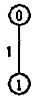


Figure 8.

For n = 1, the following diagrams verify the result.

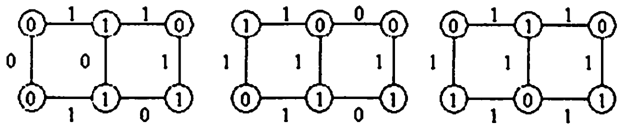


Figure 9.

For n = 2, the following diagrams verify the result.

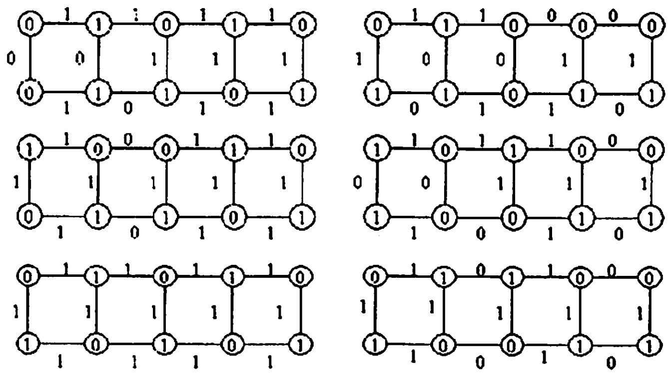


Figure 10.

We will now use induction to show that if $P_2 \times P_{2n+1}$ satisfies the theorem with vertices of the rightmost edge labeled 0 at the top and 1 at the bottom, then $P_2 \times P_{2n+5}$ also satisfies the theorem with vertices of the rightmost edge labeled 0 at the top and 1 at the bottom. The cases n = 1 and 2 serve as base cases in the induction.

For each difference $g = e(1) - e(0)$ in $FI(P_2 \times P_{2n+1})$, the same value of g can be attained in $FI(P_2 \times P_{2n+5})$ by adding the following.

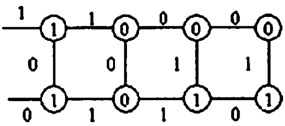


Figure 11.

For $g = q - 2 = 6n - 1$ that does not exist in $FI(P_2 \times P_{2n+1})$, this value can be realized in $P_2 \times P_{2n+5}$ by adding the following to the labeling of $P_2 \times P_{2n+1}$ that gives $q - 6 = 6n - 5$.

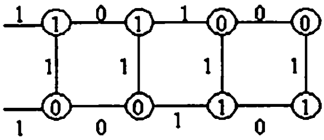


Figure 12.

For $P_2 \times P_{2n+5}$, we need to provide labelings that can give $g = e(1) - e(0) = 6n + 3, 6n + 5, 6n + 7, 6n + 9$, and $6n + 13$.

Start with $g = 6n - 5$ in $P_2 \times P_{2n+1}$, the following additions realize $6n + 3$ and $6n + 7$ in $P_2 \times P_{2n+5}$.

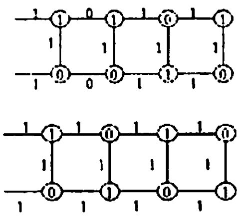


Figure 13.

Note that after the first construction, Lemma 5.1 should be applied to restore the rightmost edge to the desired orientation.

Start with $g = 6n - 3$ in $P_2 \times P_{2n+1}$, the above two additions realize $6n + 5$ and $6n + 9$ in $P_2 \times P_{2n+5}$.

Finally, start with $g = 6n + 1$ in $P_2 \times P_{2n+1}$, the second addition above realizes $6n + 13$ in $P_2 \times P_{2n+5}$.

__**Theorem 10.**__ $FI(P_2 \times P_{2n}) = \{0, 2, 4, \ldots, q - 6, q - 4, q\}$, i.e., all even numbers between 0 and q inclusive, except $q - 2$. (Note that $q = 6n - 2$. Thus $FI(P_2 \times P_{2n})$

= {0, 2, 4, ... , 6n – 8, 6n – 6, 6n – 2}.)

Proof. By Lemmas 5.2 and 5.3, it suffices to show that all these numbers are attainable.

For n = 1, the following diagrams verify the result.

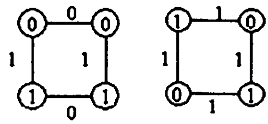


Figure 14.

For n = 2, the following diagrams verify the result.

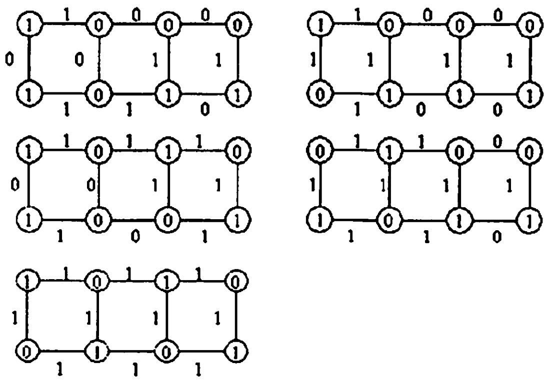
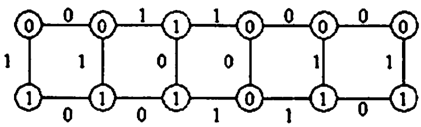
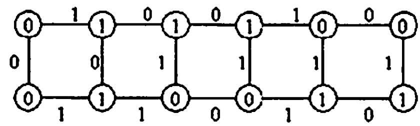


Figure 15.

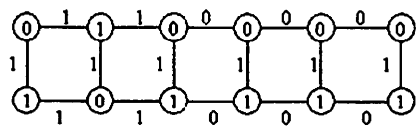
For n = 3, the following diagrams verify the result.



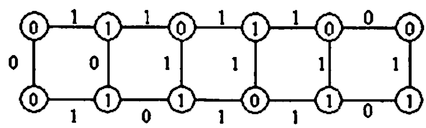
e(1)=8, e(0) = 8 and |e(1) - e(0)| = 0



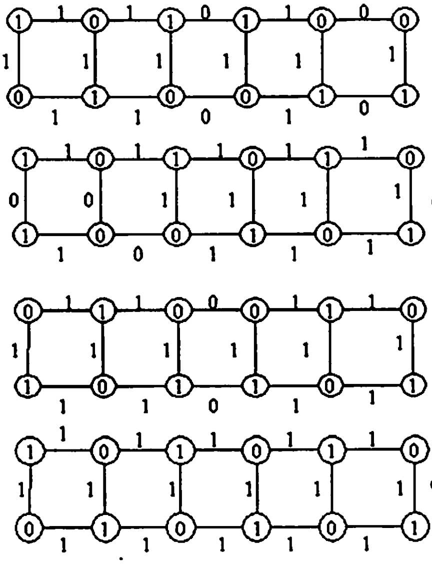
e(1)=9, e(0) = 7 and |e(1) - e(0)| = 2



e(1)=10, e(0) = 6 and |e(1) - e(0)| = 4



e(1)=11, e(0) = 5 and |e(1) - e(0)| = 6



$e(1)=12, e(0)=4$ and $|e(1)-e(0)|=8$

$e(1)=13, e(0)=3$ and $|e(1)-e(0)|=10$

$e(1)=14, e(0)=2$ and $|e(1)-e(0)|=12$

$e(1)=16, e(0)=0$ and $|e(1)-e(0)|=16$

Figure 16.

We will now use induction to show that if $P_2 \times P_{2n}$ satisfies the theorem with vertices of the rightmost edge labeled 0 at the top and 1 at the bottom, then $P_2 \times P_{2n+1}$ also satisfies the theorem with vertices of the rightmost edge labeled 0 at the top and 1 at the bottom. The cases n = 2 and 3 serve as base cases in the induction. The inductive proof is essentially the same as that of Theorem 9.

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