The Classification of Regular Graphs on f-colorings [1]

Xia Zhang, Jihui Wang and Guizhen Liu [2]

School of Mathematics and System Science
Shandong University
Jinan, Shandong 250100, P.R.China
xzhang@math.sdu.edu.cn, gzliu@sdu.edu.cn

Abstract: An f-coloring of a graph G is a coloring of edges of $E(G)$ such that each color appears at each vertex $v \in V(G)$ at most $f(v)$ times. The minimum number of colors needed to f-color G is called the f-chromatic index of G and denoted by $\chi'_f(G)$. Any simple graph G has the f-chromatic index equal to $\Delta_f(G)$ or $\Delta_f(G)+1$, where $\Delta_f(G) = \max_{v \in V}\{\lceil \frac{d(v)}{f(v)} \rceil\}$. If $\chi'_f(G) = \Delta_f(G)$, then G is of C_f 1; otherwise G is of C_f 2. In this paper two sufficient conditions for a regular graph to be of C_f 1 or C_f 2 are obtained and two necessary and sufficient conditions for a regular graph to be of C_f 1 are also presented.

Keywords: Edge-coloring; f-coloring; Classification of graphs; Regular graphs; Factorization.

1. INTRODUCTION

Our terminology and notation will be standard. Readers are referred to [1] for undefined terms. Throughout this paper, the *graph* refers to a simple graph. A *multigraph* may have multiple edges but no loops. Let G be a multigraph with a finite vertex set V and a finite and nonempty edge set E. In the proper edge-coloring, each vertex has at most one edge colored with a given color. Hakimi and Kariv [5] generalized the proper edge-coloring and obtained many interesting results. The *degree* $d(v)$ of vertex v is the number of edges incident with v in the multigraph G. Let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V$ in G. An *f-coloring* of G is a coloring of edges such that each vertex v

[1] This research is supported by NSFC(10471078,60673047), RSDP(20040422004) and NSF of Hebei(A2007000002) of China.

[2] Corresponding author: Guizhen Liu

has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f-color G is called the *f-chromatic index* of G, and denoted by $\chi'_f(G)$.

The f-coloring problem is to find $\chi'_f(G)$ of a given multigraph G, which arises in many applications, including the network design problem, the file transfer problem in a computer network, scheduling problems, and so on [3,4,7]. The file transfer problem is modelled as follows. Assume that each computer v has a limited number of communication ports $f(v)$ and it takes an equal amount of time to transfer each file. Under these assumptions, the scheduling to minimize the total time for the overall transfer process corresponds to an f-coloring of a multigraph with minimum number of colors. If $f(v) = 1$ for all $v \in V$, then the f-coloring problem is reduced to the proper edge-coloring problem. Since the proper edge-coloring problem is NP-complete [6], the f-coloring problem is also NP-complete. Hakimi and Kariv [5] studied the f-coloring problem and obtained some upper bounds on $\chi'_f(G)$. Nakano et al. [8] obtained another upper bound on $\chi'_f(G)$. Zhang and Liu [10,11] studied the classification of graphs on f-colorings.

The remainder of the paper is organized as follows. In Section 2 we give some lemmas which will be used in the proofs of main results. In Section 3 we present two sufficient conditions for a regular graph to be of C_f 1 or C_f 2 and two necessary and sufficient conditions for a regular graph to be of C_f 1. Furthermore, an open problem is presented.

2. SOME LEMMAS

Let G be a multigraph and let f be a positive integer function defined on V. Set

$$\Delta = \max_{v \in V}\{d(v)\} \quad \text{and} \quad \Delta_f = \max_{v \in V}\{\lceil \frac{d(v)}{f(v)} \rceil\},$$

in where $\lceil \frac{d(v)}{f(v)} \rceil$ is the minimum integer not less than $\frac{d(v)}{f(v)}$. It is easy to verify that $\chi'_f(G) \geq \Delta_f$. The *multiplicity* $\mu(u, v)$ of a pair of u and v of distinct vertices is the number of edges of G joining u and v. Let $\mu(v) = \max_{u \in V}\{\mu(v, u)\}$. Vizing [9] proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for a graph G. Graphs for which $\Delta(G) = \chi'(G)$ are said to be *Class* 1, and otherwise they are said to be *Class* 2. In [10] and [11] we considered the classification of graphs on f-colorings.

The following lemma was given by Hakimi and Kariv [5].

Lemma 2.1. *Let G be a multigraph. Then*

$$\Delta_f \le \chi'_f(G) \le \max_{v\in V}\{\lceil \frac{d(v)+\mu(v)}{f(v)} \rceil\}.$$

If G is a graph, we have $\mu(v) = 1$ for all $v \in V$. Therefore the following corollary holds.

Corollary 2.2. *Let G be a graph. Then*

$$\Delta_f \le \chi'_f(G) \le \max_{v\in V}\{\lceil \frac{d(v)+1}{f(v)} \rceil\} \le \Delta_f + 1.$$

From the above corollary we can see that the f-chromatic index of any graph G must be Δ_f or $\Delta_f + 1$. This result naturally partitions all graphs into two classes for f-colorings, and we say that G is of C_f 1 if $\chi'_f(G) = \Delta_f$; and that G is of C_f 2 if $\chi'_f(G) = \Delta_f + 1$.

Hakimi and Kariv [5] obtained the following two results.

Lemma 2.3. *Let G be a bipartite graph. Then $\chi'_f(G) = \Delta_f$.*

Lemma 2.4. *Let G be a graph and $f(v)$ be even for all $v \in V$. Then $\chi'_f(G) = \Delta_f$.*

Let

$$f^* = \min_{v\in V}\{f(v)\},$$

$$V^* = \{v | \Delta_f = \lceil \frac{d(v)}{f(v)} \rceil, v \in V\}, \quad \text{and}$$

$$V_0^* = \{v | \Delta_f = \frac{d(v)}{f(v)}, v \in V\}.$$

In [10] and [11], we obtained the following results.

Lemma 2.5. *Let G be a graph. If $f(v) \nmid d(v)$ for all $v \in V^*$, then G is of C_f 1.*

Lemma 2.6. *Let G be a regular graph of degree $d(G) = \Delta$. If there exist positive integers k and p such that G has a k-factorization, $f^* \mid \Delta$ and $f^* = pk$, then G is of C_f 1.*

Lemma 2.7. *Let G be a complete graph K_n. If k and n are odd integers, $f(v) = k$ and $k \mid (n-1)$, then G is of C_f 2. Otherwise, G is of C_f 1.*

Lemma 2.8. *Let G be a graph and let G_0^* be the subgraph of G induced by the vertices of V_0^*. Then G is of C_f 1 if G_0^* is a forest.*

If $f(v) = 1$ for all $v \in V$, our problem of classification on f-colorings is reduced to the problem of classification on proper edge-colorings.

3. MAIN RESULTS

Throughout this section, G always denotes a simple graph. In [10] we solved the problem of classification for complete graphs on f-colorings completely (see Lemma 2.7). Therefore, in this paper we only consider the regular graphs which are not complete graphs. Note that for any subgraph G' of G we have $f_{G'}(v) = f_G(v)$ for all $v \in V(G')$.

The following theorem is very important for our classification of regular graphs on f-colorings.

Theorem 3.1. *Let G be a regular graph of degree $d(G) = \Delta$. Then G is of C_f 1 if $f^* \nmid \Delta$ or f^* is even.*

Proof. For a regular graph G, we have $\Delta_f = \lceil \frac{\Delta}{f^*} \rceil$. Thus $V^* = \{v | \Delta_f = \lceil \frac{\Delta}{f(v)} \rceil, v \in V\} \supseteq \{v | f(v) = f^*, v \in V\}$. If $f^* \nmid \Delta$, it is easy to verify that $f(v) \nmid \Delta$ for any $v \in V^*$. Then, by Lemma 2.5, G is of C_f 1.

If f^* is even, we consider two cases. If $f(v) = f^*$ for all $v \in V$, by Lemma 2.4, there always exists an f-coloring C_{f^*} of G with Δ_f colors. If $f(v) > f^*$ for some vertices $v \in V$, the coloring C_{f^*} is still an f-coloring of G with Δ_f colors, as desired. So, G is of C_f 1 when f^* is even. □

A *k-factor* of a graph G is a k-regular spanning subgraph. Let G be an mk-regular graph. If $H_1, H_2, \ldots, H_m$ are edge-disjoint k-factors of G, then $\{H_1, H_2, \ldots, H_m\}$ is called a *k-factorization* of G. By Theorem 3.1, we have the following result.

Corolary 3.2. *Let G be an even regular graph of degree $d(G) = \Delta$. If k is even and $k \mid \Delta$, then G has a k-factorization.*

We denote by C the set of colors available to color the edges of a graph G. An edge colored with color $c \in C$ is called a *c-edge*. Denote by $d(v, c)$ the number of c-edges of G incident with the vertex v. Now, we present a sufficient condition for a regular graph to be of C_f 2.

Theorem 3.3. *Let $n \geq 1$. Let G be a regular graph of order $2n+1$ and degree $d(G) = \Delta$. Then G is of C_f 2 if $f(v) = f^*$ is odd for all $v \in V$ and $f^* \mid \Delta$.*

Proof. By contradiction. We suppose that G is of C_f 1 and now G is f-colored with Δ_f colors. Thus $d(v,c) = f^*$ for each $c \in C$ and each $v \in V$. The number of edges colored with a color in C is $\frac{(2n+1)f^*}{2}$. Obviously, $\frac{(2n+1)f^*}{2}$ is not an integer. This contradicts our assumption. So G is of C_f 2 when $f(v) = f^*$ is odd for all $v \in V$ and $f^* \mid \Delta$. □

Next, we will give two necessary and sufficient conditions for a regular graph to be of C_f 1. Before that, we need some preliminary concepts in [8].

Define $m(v,c) = f(v) - d(v,c)$. G is *f-colored* if and only if $d(v,c) \leq f(v)$ for every $v \in V$ and $c \in C$. Color c is *available* at vertex v if $m(v,c) \geq 1$. Define $M(v) = \{c \in C | m(v,c) \geq 1\}$. A *walk* W is a sequence of distinct edges $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$, where the vertices $v_0, v_1, \ldots, v_k$ are not necessarily distinct. Walk W is often denoted simply by $v_0v_1 \ldots v_k$. We call v_0 the *start vertex* of W and v_k the *end vertex*. The *length* of W is k, the number of edges in W, and denoted by $|W|$. If $v_0 = v_k$, then W is called a *closed walk*. For two distinct colors $a, b \in C$, a walk W of length one or more is called an *ab-alternating walk* if W satisfies the following conditions:

(a) The edges of W are colored alternately with a and b, and the first edge of W is colored with b;

(b) $m(v_0, a) \geq 1$ if $v_0 \neq v_k$,
$m(v_0, a) \geq 2$ if $v_0 = v_k$ and $|W|$ is odd;

(c) $m(v_k, b) \geq 1$ if $v_0 \neq v_k$ and $|W|$ is even,
$m(v_k, a) \geq 1$ if $v_0 \neq v_k$ and $|W|$ is odd.

Note that any closed walk W of even length whose edges are colored with a and b alternately is an ab-alternating walk. If G is f-colored and W is an ab-alternating walk, then interchanging the colors a and b of the edges in walk W preserves an f-coloring of G. This operation is called *switching* W. When W is switched, $m(v_i, a)$ and $m(v_i, b)$ remain as they were if $i \neq 0, k$, while $m(v_0, b) \geq 1$ if W is not a closed walk of even length. Obviously, W is switched into a ba-alternating walk. We denote by $W(a, b; v_0)$ an ab-alternating walk which starts with vertex v_0. If confusions may occur, we write $d(v, c; G)$, $m(v, c; G)$ and $M(v; G)$ for $d(v, c)$, $m(v, c)$ and $M(v)$ of graph G, respectively.

Theorem 3.4. *Let G be a regular graph of order n and degree $d(G) = \Delta$, and let $G \neq K_n$. Let $w \in V$ be the only vertex such that $f(w) > f^*$. Then G is of C_f 1 if and only if $G \setminus w$ is of C_f 1.*

Proof. Since $G \neq K_n$, we have $\Delta_f(G \setminus w) = \Delta_f(G)$. Obviously, $G \setminus w$ is of C_f 1 if G is of C_f 1.

Conversely, suppose that $G \setminus w$ is of C_f 1, where $w \in V$ is the only vertex such that $f(w) > f^*$. We will show that G is of C_f 1. Clearly, we only need to prove that if $f(w) = f^* + 1$, then G is of C_f 1. We consider two cases.

Case 1. If $f^* \nmid \Delta$ or f^* is even, by Theorem 3.1, G is of C_f 1.

Case 2. If $f^* \mid \Delta$ and f^* is odd, we will give an f-coloring of G with $\Delta_f(G)$ colors as follows.

First, we f-color $G \setminus w$ with Δ_f colors. Note that $\Delta_f(G \setminus w) = \Delta_f(G) = \Delta_f$. Denote by $N_G(w)$ the set of all vertices adjacent to vertex w in G. Now $|M(v)| = 1$ for each $v \in N_G(w)$ and $|M(v)| = 0$ for each $v \in V \setminus (N_G(w) \cup w)$. Next, we color the edge $e = wv$ with the color in $M(v)$ for each $v \in N_G(w)$. Then $|M(v)| = 0$ for every $v \in V \setminus w$. If $d(w, c; G) \leq f^* + 1$ for all $c \in C$, we obtain an f-coloring of G with Δ_f colors. Otherwise, we can find a color $\alpha \in C$ with $d(w, \alpha; G) \geq f^* + 2$. Furthermore, there exists at least one color $\beta \in C$ with $d(w, \beta; G) \leq f^* - 1$. Finding a $\beta\alpha$-alternating walk $P = W(\beta, \alpha; w)$, we say that P must end at w (because $|M(v)| = 0$ for each $v \in V \setminus w$). In particular, when the walk P returns to w with a β-edge, continue to extend P by choosing an α-edge incident with w. Since $d(w, \alpha; G) - d(w, \beta; G) \geq 3$, there exists such an α-edge which has not been included in P so far. So we can say that P must end at w with an α-edge. Switching P makes $d(w, \alpha; G)$ decrease two and $d(w, \beta; G)$ increase two. Repeat this operation until $d(w, c; G) \leq f^* + 1$ for all $c \in C$.

This completes the proof. □

For regular graphs with even order, we present another necessary and sufficient condition slightly better than Theorem 3.4 as follows.

Theorem 3.5. *Let $n \geq 1$. Let G be a regular graph of order $2n$ and degree $d(G) = \Delta$, where $G \neq K_{2n}$. Let $f(v) = f^*$ for all $v \in V$. G is of C_f 1 if and only if $G \setminus w$ is of C_f 1, where $w \in V$.*

Proof. Since $G \neq K_{2n}$, we have $\Delta_f(G \setminus w) = \Delta_f(G)$. So, $G \setminus w$ is of C_f 1 if G is of C_f 1.

Conversely, suppose that $G \setminus w$ is of C_f 1. If $f^* \nmid \Delta$, then G is of C_f 1. Otherwise, $\Delta_f = \frac{\Delta}{f^*}$, where $\Delta_f = \Delta_f(G \setminus w) = \Delta_f(G)$. First, we f-color

$G \setminus w$ with Δ_f colors. Now $|M(v)| = 1$ for each $v \in N_G(w)$ and $|M(v)| = 0$ for each $v \in V \setminus (N_G(w) \cup w)$. Next, color the edge $e = wv$ with the color in $M(v)$ for each $v \in N_G(w)$. Define $d = \max_{1 \le i < j \le \Delta_f} \{|d(w, c_i; G) - d(w, c_j; G)|\}$. If $d = 0$, we obtain an f-coloring of G with Δ_f colors. Otherwise, $d \ge 2$. (Since $\Delta_f = \frac{\Delta}{f^*}$, $d \ne 1$.) Let α and β be two colors in C such that $d(w, \alpha; G) - d(w, \beta; G) = d$. If $d(w, \alpha; G) - d(w, \beta; G) \ge 3$, we can find a $\beta\alpha$-alternating walk $P = W(\beta, \alpha; w)$ which ends at w with an α-edge. Switching P makes $d(w, \alpha; G)$ decrease two and $d(w, \beta; G)$ increase two. Thus $|d(w, \alpha; G) - d(w, \beta; G)|$ decreases four or two. Repeat this operation until $d \le 2$. If d=2, without loss of generality, we assume that $d(w, \alpha; G) - d(w, \beta; G) = 2$. It is easy to see that $d(w, \alpha; G) = f^* + 1$ and $d(w, \beta; G) = f^* - 1$ since $\Delta_f = \frac{\Delta}{f^*}$. We have known that $d(v, \alpha; G) = f^*$ for all $v \in V \setminus w$. So the number of α-edges in G is $\frac{(2n-1)f^* + (f^*+1)}{2}$. Clearly, it is not an integer. Therefore, $d = 0$. We obtain an f-coloring of G with Δ_f colors. □

If $f(v)$=1 for all $v \in V$, we have the following corollary which is due to Chetwynd and Hilton [2].

Corollary 3.6. *Let $n \ge 1$. Let G be a regular graph of order $2n$, and $G \ne K_{2n}$. Let $w \in V$. Then G is Class 1 if and only if $G \setminus w$ is Class 1.*

Combining Lemma 2.7 and Lemma 2.8 with Theorem 3.4 or Theorem 3.5, readers can get some regular graphs of C_f 1 . In particular, we have the following results.

Corollary 3.7. *Let G be a regular graph of order $2n + 1$ such that the degree $d(G)$ is equal to $2n$ or $2n - 2$. Let $w \in V$ be the only vertex such that $f(w) > f^*$. Then G is of C_f 1.*

Corollary 3.8. *Let G be a regular graph of order $2n$ such that the degree $d(G)$ is equal to $2n-1$, $2n-2$ or $2n-3$. Then G has a 1-factorization.*

It is easy to see that the f-coloring problem has close relationships with the equitable edge-coloring problem or the factorization problem. From above two aspects, we probably find some general methods to classify graphs for f-colorings. Finally, we present the following open problem.

Question: *Find the sufficient conditions for a regular graph of degree $d(G) = \Delta$ to be of C_f 1 or C_f 2 when $f^* \mid \Delta$ and f^* is odd.*

ACKNOWLEDGMENT

The authors would like to thank the referee for carefully reading manuscript and for his (her) valuable suggestions on the presentation of the paper.

REFERENCES

1. J. A. Bondy and U.S.R.Murty, *Graph Theory with Applications*, MacMillan, London (1976)
2. A. G. Chetwynd and A. J. W. Hilton, Regular graphs of high degree are 1-factorizable, *Proc. London Math. Soc.*, 50 (3) 193-206 (1985)
3. H. Choi and S. L. Hakimi, Scheduling file transfers for trees and odd cycles,*SIAM J. Comput.*, 16(1) 162-168 (1987)
4. E. G. Coffman, Jr, M. R. Garey, D. S. Johnson and A.S.LaPaugh, Scheduling file transfers, *SIAM J. Comput.*,14(3) 744-780 (1985)
5. S. L.Hakimi and O. Kariv, A generalization of edge-coloring in graphs, *Journal of Graph Theory*, 10 139-154 (1986)
6. I. J. Holyer, The NP-completeness of edge-coloring, *SIAM J. Comput.*, 10 718-720 (1981)
7. H. Krawczyk and M. Kubale, An approximation algorithm for diagnostic test scheduling in multicomputer systems, *IEEE Trans. Comput.*, C- 34 869-872 (1985)
8. S. Nakano, T. Nishizeki and N. Saito, On the f-coloring of multigraphs, *IEEE Trans. Circuit and Syst.*, 35(3) 345-353 (1988)
9. V. G. Vizing, On an estimate of the chromatic class of a p-graph. *Diskret. Analiz*, 3 25-30 (1964)
10. X. Zhang and G. Liu, The classification of complete graphs K_n on f-coloring. *Journal of Applied Mathematics & Computing*, 19(1-2) 127-133 (2005)
11. X. Zhang and G. Liu, Some sufficient conditions for a graph to be of C_f 1. *Applied Mathematics Letters*, 19 38-44 (2006)